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Coefficient Estimates for Two New Subclasses of Bi-univalent Functions Involving Laguerre Polynomials

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Abstract

In this paper, we introduce two new subclasses of regular and bi-univalent functions using Laguerre polynomials. Then, we define some upper limits for the Taylor Maclaurin coefficients. In addition, the Fekete-Szegö problem for the functions of the new subclasses. Finally, we provide some corollaries for certain values of parameters.

1 Introduction

One of the most attractive sub branch of the complex analysis in mathematics is univalent function theory. Determining geometric properties of complex valued functions is scope of this theory. Also, this field interest in finding some bounds for the coefficients of functions belonging to some subclasses of regular and one-to-one functions. If a complex valued function $\mathfrak{f} : \mathcal{D} \subset \mathbb{C} \to \mathbb{C}$ does not take the same value twice, then this function is called univalent or schlicht on $\mathcal{D}$. After understanding the importance of Riemann mapping theorem [8], researches on the univalent functions has become very attractive. One of the most important problem of the 20th century is known as Bieberbach conjecture. This conjecture estimates an upper bound for the n-th coefficient of analytic and univalent function. Bieberbach conjecture attracted attention of numerious mathematicians in the mentioned century. During this period, a number of papers were published on the solution of the mentioned problem for a number of subclasses of analytic and univalent functions. Since then, lots of subclasses of regular and univalent functions were introduced and certain properties of these function classes were investigated. Also, it is worthy to mention here that these function subclasses were defined by using some special polynomial due to their coefficient properties

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of generating functions. In general, upper bounds for the coefficients $|a_2|$ and $|a_3|$, Fekete-Szegö and Hankel determinant problems for the mentioned subclasses were handled in the recent papers. Recently, De Branges proved Bieberbach s conjecture for the class of analytic and univalent function on the unit disk $\mathbb{E} = \{\mathfrak{z} \in \mathbb{C} : |\mathfrak{z}| < 1\}$ normalized by the conditions $\mathfrak{f}(0) = \mathfrak{f}'(0) - 1 = 0$. An important subclass of analytic and univalent function class on the unit disk $\mathbb{E}$ is the bi-univalent function class and is denoted by $\mathfrak{S}$. In the literature, an analytic and univalent function f is called bi-univalent function in $\mathbb{E}$ if both $\mathfrak{f}$ and $\mathfrak{f}^{-1}$ are univalent in $\mathbb{E}$. We would like to emphasize here that the problem finding an upper bound for the coefficient $|a_n|$ of the functions belonging to class $\mathfrak{S}$ is still an open problem. A wide range of coefficient estimates for the functions in the class $\mathfrak{S}$ can be found in the literature. For instance, Brannan and Clunie, and Lewin presented very interesting upper bounds on $|a_2|$ in [3] and [11], respectively. Also, in [4], Brannan and Taha studied on some subclasses of bi-univalent functions and proved certain coefficient estimates. As mentioned above, one of the most attractive open problems in univalent function theory is to find an upper bound on $|a_n|$ $(n \in \mathbb{N}, n \geq 3)$ for the functions in the class $\mathfrak{S}$. Since this attraction, motivated by the works [3, 4, 11] and [13], in [1, 5–7, 9, 10, 14–17, 19] and references therein, the authors introduced numerous subclasses of bi-univalent functions and obtained non-sharp estimates on the initial coefficients of functions in these subclasses.

As a result of the literature review we did not find any papers dealing with the coefficient estimations for the subclasses of analytic and bi-univalent function class $\mathfrak{S}$ defined by Laguerre polynomials. In this paper, the main objective is to obtain some upper bounds for the second and third coefficients, and Fekete-Szegö functional of the functions in the subclasses defined. A rich history for the class $\mathfrak{S}$ can be found in the pionnering work [25] published by Srivastava et al.

1.1 Some basic concepts in Geometric Function Theory

Let $\mathcal{A}$ denote the class of all analytic functions of the form

$$\mathfrak{f}(\mathfrak{z}) = \mathfrak{z} + a_2\mathfrak{z}^2 + \cdots = \mathfrak{z} + \sum_{n=2}^{\infty} a_n\mathfrak{z}^n, \tag{1.1}$$

in the open unit disk $\mathbb{E} = \{\mathfrak{z} \in \mathbb{C} : |\mathfrak{z}| < 1\}$. It is clear that the functions in $\mathcal{A}$ satisfy the conditions $\mathfrak{f}(0) = \mathfrak{f}'(0) - 1 = 0$, known as normalization conditions. We show by $\mathfrak{S}$ the subclass of $\mathcal{A}$ consisting of functions univalent in $\mathcal{A}$. The Koebe one quarter theorem (see [8]) guarantees that if $\mathfrak{f} \in \mathfrak{S}$, then there exists the inverse function $\mathfrak{f}^{-1}$ satisfying

$$\mathfrak{f}^{-1}(\mathfrak{f}(\mathfrak{z})) = \mathfrak{z},\ (\mathfrak{z} \in \mathbb{E}) \quad \text{and} \quad \mathfrak{f}(\mathfrak{f}^{-1}(\omega)) = \omega, \left(|\omega| < r_0(\mathfrak{f}),\ r_0(\mathfrak{f}) \geq \frac{1}{4}\right)$$

where

$$\mathfrak{g}(\omega) = \mathfrak{f}^{-1}(\omega) = \omega - a_2\omega^2 + (2a_2^2 - a_3)\omega^3 - (5a_2^3 - 5a_2a_3 + a_4)\omega^4 + \dots. \tag{1.2}$$

In the univalent function theory, one of the most important notions is subordination principle. Let the functions $\mathfrak{f} \in \mathcal{A}$ and $F \in \mathcal{A}$. Then, $\mathfrak{f}$ is called to be subordinate to F if there exists a Schwarz function ω such that

$$\omega(0) = 0,\ |\omega(\mathfrak{z}) < 1| \quad \text{and} \quad \mathfrak{f}(\mathfrak{z}) = F(\omega(\mathfrak{z})) \qquad (\mathfrak{z} \in \mathbb{E}).$$

This subordination is shown by

$$\mathfrak{f} \prec F \quad \text{or} \quad \mathfrak{f}(\mathfrak{z}) \prec F(\mathfrak{z}) \qquad (\mathfrak{z} \in \mathbb{E}).$$

Especially, if the function F is univalent in $\mathbb{E}$, then this subordination is equivalent to

$$\mathfrak{f}(0) = F(0), \qquad f(\mathbb{E}) \subset F(\mathbb{E}).$$

A comprehensive information about the subordination concept can be found in [19].

1.2 Laguerre polynomial and its some properties

The Laguerre equation is a second-order linear differential equation that arises in physics, particularly in the context of quantum mechanics and the study of certain physical systems. It is named after Edmond Laguerre, the same mathematician after whom Laguerre polynomials are named. Consider the differential equation [4]

$$ry'' + (1 + \delta - r)y' + ty = 0, \tag{1.3}$$

where $\delta + 1 > 0$, $\delta \in \mathbb{R}$ and t is non negative. The polynomial solution $y(r)$ to this differential equation is said to be the generalized Laguerre polynomial or associated Laguerre polynomial and it is denoted by $\mathfrak{L}_t^\delta(r)$. It has several applications in Mathematical physics and quantum mechanics. For example in integration of Helmholtz's equation in paraboloidal coordinated and also in theory of propagation of electromagnetic oscillations. These polynomials satisfy given recurrence relations, such as

$$\mathfrak{L}_{t+1}^\delta(r) = \frac{2t + 1 + \delta - r}{t + 1}\mathfrak{L}_t^\delta(r) - \frac{t + \delta}{t + 1}\mathfrak{L}_{t-1}^\delta(r) \quad (t \geq 1) \tag{1.4}$$

with the initial values

$$\mathfrak{L}_0^\delta(r) = 1,\ \mathfrak{L}_1^\delta(r) = 1 + \delta - r,\ \mathfrak{L}_2^\delta(r) = \frac{r^2}{2} - (\delta + 2)r + \frac{(\delta + 1)(\delta + 2)}{2}. \tag{1.5}$$

We obtain this equation from (1.4)

$$\mathfrak{L}_3^\delta(r) = \frac{-r^3}{6} + \frac{(\delta + 3)r^2}{2} - \frac{(\delta + 2)(\delta + 3)r}{2} + \frac{(\delta + 1)(\delta + 2)(\delta + 3)}{6}, \tag{1.6}$$

and so on.

We can see that by putting $\delta = 0$, in generalized Laguerre polynomial we get Laguerre polynomials such as

$$\mathfrak{L}_t^0(r) = \mathfrak{L}_t(r).$$

Lemma 1.1. *Let $\mathfrak{P}(r,\mathfrak{z})$ be the generating function of the generalized Laguerre polynomial*

$$\mathfrak{P}(r,\mathfrak{z})=\sum_{t=0}^{\infty}\mathcal{L}_t^{\delta}(r)\mathfrak{z}^t=\frac{e^{-\frac{rz}{1-\mathfrak{z}}}}{(1-\mathfrak{z})^{\delta+1}},\quad (r\in\mathbb{R},\ \mathfrak{z}\in\mathcal{U}). \tag{1.7}$$

1.3 New subclasses of bi-univalent functions

In this subsection, we introduce some new function subclasses of analytic and bi-univalent function class Σ which is subordinate to Laguerre polynomials.

Definition 1.1. A function $\mathfrak{f}(\mathfrak{z})\in\Sigma$ of the form (1.1) is said to be in the class $\mathcal{C}_{\Sigma}(\mathfrak{P}(r,\mathfrak{z}))$ if the following conditions hold true:

$$1+\frac{\mathfrak{z}\mathfrak{f}''(\mathfrak{z})}{\mathfrak{f}'(\mathfrak{z})}\prec\frac{e^{-\frac{r\mathfrak{z}}{1-\mathfrak{z}}}}{(1-\mathfrak{z})^{\delta+1}}=\mathfrak{P}(r,\mathfrak{z}) \tag{1.8}$$

and

$$1+\frac{\omega\mathfrak{g}''(\omega)}{\mathfrak{g}'(\omega)}\prec\frac{e^{-\frac{r\omega}{1-\omega}}}{(1-\omega)^{\delta+1}}=\mathfrak{P}(r,\omega) \tag{1.9}$$

where $z,\omega\in\mathbb{E}$, $\mathfrak{g}$ is inverse of $\mathfrak{f}$ and it is of the form (1.2).

Definition 1.2. A function $\mathfrak{f}(\mathfrak{z})\in\Sigma$ of the form (1.1) is said to be in the class $\mathfrak{S}^*_{\Sigma}(\mathfrak{P}(r,\mathfrak{z}))$ if the following conditions hold true:

$$\frac{z\mathfrak{f}'(\mathfrak{z})}{\mathfrak{f}(\mathfrak{z})}\prec\frac{e^{-\frac{rz}{1-z}}}{(1-\mathfrak{z})^{\delta+1}}=\mathfrak{P}(r,\mathfrak{z}) \tag{1.10}$$

and

$$\frac{\omega\mathfrak{g}'(\omega)}{\mathfrak{g}(\omega)}\prec\frac{e^{-\frac{r\omega}{1-\omega}}}{(1-\omega)^{\delta+1}}=\mathfrak{P}(r,\omega) \tag{1.11}$$

where $\mathfrak{z},\omega\in\mathbb{E}$, $\mathfrak{g}$ is inverse of $\mathfrak{f}$ and it is of the form (1.2).

2 Coefficient Estimates for the Classes $\mathcal{C}_{\Sigma}(\mathfrak{P}(r,\mathfrak{z}))$ and $\mathfrak{S}^*_{\Sigma}(\mathfrak{P}(r,\mathfrak{z}))$

In this section, we present initial coefficients estimates for the functions belonging to the subclasses $\mathcal{C}_{\Sigma}(\mathfrak{P}(r,\mathfrak{z}))$ and $\mathfrak{S}^*_{\Sigma}(\mathfrak{P}(r,\mathfrak{z}))$, respectively.

Theorem 2.1. *If the function $\mathfrak{f}(\mathfrak{z})\in\mathcal{C}_{\Sigma}(\mathfrak{P}(r,\mathfrak{z}))$, then*

$$|a_2|\le\frac{|1+\delta-r|\sqrt{|1+\delta-r|}}{\sqrt{2\left|(1+\delta-r)^2-4\left(\frac{r^2}{2}-(\delta+2)r+\frac{(\delta+1)(\delta+2)}{2}\right)\right|}} \tag{2.1}$$

and

$$|a_3|\le\frac{|1+\delta-r|}{4}\left(\frac{|1+\delta-r|}{3}+\frac{1}{2}\right). \tag{2.2}$$

Proof. Assume that $\mathfrak{f}(\mathfrak{z}) \in \mathcal{C}_\Sigma(\mathfrak{P}(r,\mathfrak{z}))$ and $\mathfrak{g} \in \mathfrak{f}^{-1}$ given by (1.2). By virtue of Definition 1.2, from the relations (1.8) and (1.9) we can write that

$$1 + \frac{z\mathfrak{f}''(\mathfrak{z})}{\mathfrak{f}'(\mathfrak{z})} = \mathfrak{P}(r, \rho(\mathfrak{z})) \tag{2.3}$$

and

$$1 + \frac{\omega\mathfrak{g}''(\omega)}{\mathfrak{g}'(\omega)} = \mathfrak{P}(r, \xi(\omega)), \tag{2.4}$$

where $\rho, \xi : \mathbb{E} \to \mathbb{E}$, $\rho(\mathfrak{z}) = \rho_1\mathfrak{z} + \rho_2\mathfrak{z}^2 + \rho_3\mathfrak{z}^3 + \ldots$ and $\xi(\omega) = \xi_1\omega + \xi_2\omega^2 + \xi_3\omega^3 + \ldots$ are Schwarz functions such that $\rho(0) = \xi(0) = 0$ and $|\rho(\mathfrak{z})| < 1$, $|\xi(\omega)| < 1$ for all $\mathfrak{z}, \omega \in \mathbb{E}$. On the other hand, it is known that the conditions $|\rho(\mathfrak{z})| < 1$ and $|\xi(\omega)| < 1$ imply

$$|\rho_j| < 1, \tag{2.5}$$

and

$$|\xi_j| < 1, \tag{2.6}$$

for all $j \in \mathbb{N}$. Some basic calculations yield that

$$1 + \frac{z\mathfrak{f}''(\mathfrak{z})}{\mathfrak{f}'(\mathfrak{z})} = 1 + 2a_2\mathfrak{z} + (6a_3 - 4a_2^2)\mathfrak{z}^2 + (12a_4 - 18a_2a_3 + 8a_2^3)\mathfrak{z}^3 + \ldots, \tag{2.7}$$

$$1 + \frac{\omega\mathfrak{g}''(\omega)}{\mathfrak{g}'(\omega)} = 1 - 2a_2\omega + (8a_2^2 - 6a_3)\omega^2 + (42a_2a_3 - 32a_2^3 - 12a_4)\omega^3 + \ldots, \tag{2.8}$$

$$\mathfrak{P}(r, \rho(\mathfrak{z})) = [\mathfrak{L}_1^\delta(r)\rho_1]\mathfrak{z} + [\mathfrak{L}_1^\delta(r)\rho_2 + \mathfrak{L}_2^\delta(r)\rho_1^2]\mathfrak{z}^2 + \ldots, \tag{2.9}$$

$$\mathfrak{P}(r, \xi(\omega)) = [\mathfrak{L}_1^\delta(r)\xi_1]\omega + [\mathfrak{L}_1^\delta(r)\xi_2 + \mathfrak{L}_2^\delta(r)\xi_1^2]\omega^2 + \ldots \tag{2.10}$$

Now, using equation (2.3) and comparing the coefficients of (2.7) and (2.9), we get

$$2a_2 = \mathfrak{L}_1^\delta(r)\rho_1, \tag{2.11}$$

$$6a_3 - 4a_2^2 = \mathfrak{L}_1^\delta(r)\rho_2 + \mathfrak{L}_2^\delta(r)\rho_1^2. \tag{2.12}$$

Similarly, using equation (2.4) and comparing the coefficients of (2.8) and (2.10), we get

$$-2a_2 = \mathfrak{L}_1^\delta(r)\xi_1, \tag{2.13}$$

$$8a_2^2 - 6a_3 = \mathfrak{L}_1^\delta(r)\xi_2 + \mathfrak{L}_2^\delta(r)\xi_1^2. \tag{2.14}$$

Now, from equations (2.11) and (2.13) we have

$$\rho_1 = -\xi, \tag{2.15}$$

$$\frac{8a_2^2}{(\mathfrak{L}_1^\delta(r))^2} = \rho_1^2 + \xi_1^2. \tag{2.16}$$

Also, summing of the equations (2.12) and (2.14), we easily obtain that

$$4a_2^2 = \mathfrak{L}_1^{\delta}(r)(\rho_2 + \xi_2) + \mathfrak{L}_2^{\delta}(r)(\rho_1^2 + \xi_1^2). \tag{2.17}$$

Substituting equation (2.16) in equation (2.17) we deduce

$$a_2^2 = \frac{(\mathfrak{L}_1^{\delta}(r))^3(\rho_2 + \xi_2)}{4(\mathfrak{L}_1^{\delta}(r))^2 - 8\mathfrak{L}_2^{\delta}(r)}. \tag{2.18}$$

Taking into account (1.4) and (1.5) in (2.18) we have

$$a_2^2 = \frac{(1+\delta-r)^3(\rho_2+\xi_2)}{4(1+\delta-r)^2 - 8\left(\frac{r^2}{2} - (\delta+2)r + \frac{(\delta+1)(\delta+2)}{2}\right)}. \tag{2.19}$$

Now, using the well-known triangular inequality with the inequalities (2.5) and (2.6), we get

$$|a_2^2| \le \frac{|1+\delta-r|^3}{\left|2(1+\delta-r)^2 - 8\left(\frac{r^2}{2} - (\delta+2)r + \frac{(\delta+1)(\delta+2)}{2}\right)\right|}. \tag{2.20}$$

Taking square root both sides of the inequality (2.20), we deduce

$$|a_2| \le \frac{|1+\delta-r|\sqrt{|1+\delta-r|}}{\sqrt{2\left|(1+\delta-r)^2 - 4\left(\frac{r^2}{2} - (\delta+2)r + \frac{(\delta+1)(\delta+2)}{2}\right)\right|}}.$$

On the other hand, if we subtract the equation (2.12) from the equation (2.14) and consider equation (2.15), then we obtain

$$a_3 = \frac{\mathfrak{L}_1^{\delta}(r)(\rho_2 - \xi_2)}{12} + a_2^2. \tag{2.21}$$

Considering the equation (2.19) and (2.21) and a straightforward calculation yield that

$$a_3 = \frac{\mathfrak{L}_1^{\delta}(r)(\rho_2 - \xi_2)}{12} + \frac{(\mathfrak{L}_1^{\delta}(r))^2(\rho_1^2 + \xi_1^2)}{8}. \tag{2.22}$$

By making use of the equation (1.4) and triangle inequality with the inequalities (2.5) and (2.6) we can write that

$$|a_3| = \left|\frac{(1+\delta-r)(\rho_2-\xi_2)}{12} + \frac{(1+\delta-r)^2(\rho_1^2+\xi_1^2)}{8}\right| \le \frac{|1+\delta-r|}{4}\left(\frac{|1+\delta-r|}{3} + \frac{1}{2}\right)$$

which is desired. □

Taking $\delta = 0$ in Theorem 2.1 we get:

Corollary 2.2. *If the function* $\mathfrak{f}(\mathfrak{z}) \in \mathcal{C}_\Sigma(\mathfrak{P}(x,\mathfrak{z}))$, *then*

$$|a_2| \leq \frac{|1-r|\sqrt{|1-r|}}{\sqrt{2\,|(r^2-12r+3)|}} \tag{2.23}$$

and

$$|a_3| \leq \frac{|1-r|}{4}\left(\frac{|1-r|}{3}+\frac{1}{2}\right). \tag{2.24}$$

Theorem 2.3. *If the function* $\mathfrak{f}(\mathfrak{z}) \in \mathfrak{S}^*_\Sigma(\mathfrak{P}(r,\mathfrak{z}))$, *then*

$$|a_2| \leq \frac{|1+\delta-r|\sqrt{|1+\delta-r|}}{\sqrt{\left|(1+\delta-r)^2-\left(\frac{r^2}{2}-(\delta+2)r+\frac{(\delta+1)(\delta+2)}{2}\right)\right|}} \tag{2.25}$$

and

$$|a_3| \leq |1+\delta-r|\left(|1+\delta-r|+\frac{1}{2}\right). \tag{2.26}$$

Proof. Assume that $\mathfrak{f}(\mathfrak{z}) \in \mathfrak{S}^*_\Sigma(\mathfrak{P}(x,\mathfrak{z}))$ and $\mathfrak{g} \in \mathfrak{f}^{-1}$ given by (1.2). By virtue of Definition 1.2, from the relations (1.10) and (1.11) we can write that

$$\frac{z\mathfrak{f}'(\mathfrak{z})}{\mathfrak{f}(\mathfrak{z})} = \mathfrak{P}(r,\eta(\mathfrak{z})) \tag{2.27}$$

and

$$\frac{\omega\mathfrak{g}'(\omega)}{\mathfrak{g}(\omega)} = \mathfrak{P}(r,\mu(\omega)), \tag{2.28}$$

where $\eta,\mu : \mathbb{E} \to \mathbb{E}$, $\eta(\mathfrak{z}) = \eta_1 z + \eta_2 z^2 + \eta_3 z^3 + \dots$ and $\mu(\omega) = \mu_1\omega + \mu_2\omega^2 + \mu_3\omega^3 + \dots$ are Schwarz functions such that $\eta(0) = \mu(0) = 0$ and $|\eta(\mathfrak{z})| < 1$, $|\mu(\omega)| < 1$ for all $z,\omega \in \mathbb{E}$. On the other hand, it is known that the conditions $|\eta(\mathfrak{z})| < 1$ and $|\mu(\omega)| < 1$ imply

$$|\eta_j| < 1, \tag{2.29}$$

and

$$|\mu_j| < 1, \tag{2.30}$$

for all $j \in \mathbb{N}$. Some basic calculations yield that

$$\frac{z\mathfrak{f}'(\mathfrak{z})}{\mathfrak{f}(\mathfrak{z})} = 1 + a_2 z + (2a_3 - a_2^2)z^2 + (4a_4 - 3a_2a_3 + a_2^3)z^3 + \dots, \tag{2.31}$$

$$\frac{\omega\mathfrak{g}'(\omega)}{\mathfrak{g}(\omega)} = 1 - a_2\omega + (3a_2^2 - 2a_3)\omega^2 + (12a_2a_3 - 10a_2^3 - 3a_4)\omega^3 + \dots, \tag{2.32}$$

$$\mathfrak{P}(r,\eta(\mathfrak{z})) = \mathfrak{L}_0^\delta(r) + [\mathfrak{L}_1^\delta(r)\eta_1]z + [\mathfrak{L}_1^\delta(r)\eta_2 + \mathfrak{L}_2^\delta(r)\eta_1^2]z^2 + \dots, \tag{2.33}$$

$$\mathfrak{P}(r,\mu(\omega)) = \mathfrak{L}_0^{\delta}(r) + [\mathfrak{L}_1^{\delta}(r)\mu_1]\omega + [\mathfrak{L}_1^{\delta}(r)\mu_2 + \mathfrak{L}_2^{\delta}(r)\mu_1^2]\omega^2 + \dots . \tag{2.34}$$

Now, using equation (2.25) and comparing the coefficients of (2.29) and (2.31), we get

$$a_2 = \mathfrak{L}_1^{\delta}(r)\eta_1, \tag{2.35}$$

$$2a_3 - a_2^2 = \mathfrak{L}_1^{\delta}(r)\eta_2 + \mathfrak{L}_2^{\delta}(r)\eta_1^2. \tag{2.36}$$

Similarly, using equation (2.26) and comparing the coefficients of (2.30) and (2.32), we get

$$-a_2 = \mathfrak{L}_1^{\delta}(r)\mu_1, \tag{2.37}$$

$$3a_2^2 - 2a_3 = \mathfrak{L}_1^{\delta}(r)\mu_2 + \mathfrak{L}_2^{\delta}(r)\mu_1^2. \tag{2.38}$$

Now, from equations (2.33) and (2.35) we have

$$\eta_1 = -\mu, \tag{2.39}$$

$$\frac{2a_2^2}{(\mathfrak{L}_1^{\delta}(r))^2} = \eta_1^2 + \mu_1^2. \tag{2.40}$$

Also, summing of the equations (2.34) and (2.36), we easily obtain that

$$2a_2^2 = \mathfrak{L}_1^{\delta}(r)(\eta_2 + \mu_2) + \mathfrak{L}_2^{\delta}(r)(\eta_1^2 + \mu_1^2). \tag{2.41}$$

Substituting equation (2.38) in equation (2.39) we deduce

$$a_2^2 = \frac{(\mathfrak{L}_1^{\delta}(r))^3(\eta_2 + \mu_2)}{2(\mathfrak{L}_1^{\delta}(r))^2 - 2\mathfrak{L}_2^{\delta}(r)}. \tag{2.42}$$

Taking into account (1.4) and (1.5) in (2.40) we have

$$a_2^2 = \frac{(1+\delta-r)^3(\eta_2+\mu_2)}{2(1+\delta-r)^2 - 2\left(\frac{r^2}{2} - (\delta+2)r + \frac{(\delta+1)(\delta+2)}{2}\right)}. \tag{2.43}$$

Now, using the well-known triangular inequality with the inequalities (2.27) and (2.28), we get

$$|a_2^2| = \frac{|1+\delta-r|^3}{\left|(1+\delta-r)^2 - \left(\frac{r^2}{2} - (\delta+2)r + \frac{(\delta+1)(\delta+2)}{2}\right)\right|}. \tag{2.44}$$

Taking square root both sides of the inequality (2.42), we deduce

$$|a_2| \leq \frac{|1+\delta-r|\sqrt{|1+\delta-r|}}{\sqrt{\left|(1+\delta-r)^2 - \left(\frac{r^2}{2} - (\delta+2)r + \frac{(\delta+1)(\delta+2)}{2}\right)\right|}}.$$

On the other hand, if we subtract the equation (2.36) from the equation (2.34) and consider equation (2.37), then we obtain

$$a_3 = \frac{\mathfrak{L}_1^{\delta}(r)(\eta_2 - \mu_2)}{4} + a_2^2. \tag{2.45}$$

Considering the equation (2.44) and (2.43) and a straightforward calculation yield that

$$a_3 = \frac{\mathfrak{L}_1^{\delta}(r)(\eta_2 - \mu_2)}{4} + \frac{(\mathfrak{L}_1^{\delta}(r))^2(\eta_1^2 + \mu_1^2)}{2}. \tag{2.46}$$

By making use of the equation (1.4) and triangle inequality with the inequalities (2.27) and (2.28) we can write that

$$|a_3| = \left| \frac{(1+\delta-r)(\eta_2-\mu_2)}{4} + \frac{(1+\delta-r)^2(\eta_1^2+\mu_1^2)}{2} \right| \leq |1+\delta-r| \left(|1+\delta-r| + \frac{1}{2} \right)$$

which is desired. □

Taking $\delta = 0$ in Theorem 2.3 we get:

Corollary 2.4. *If the function $\mathfrak{f}(\mathfrak{z}) \in \mathfrak{S}^*_\Sigma(\mathfrak{P}(r,\mathfrak{z}))$, then*

$$|a_2| \leq \frac{|1-r|\sqrt{|1-r|}}{\sqrt{\left|(1-r)^2 - \left(\frac{r^2}{2} - 2r + 1\right)\right|}} \tag{2.47}$$

and

$$|a_3| \leq |1-r| \left(|1-r| + \frac{1}{2} \right). \tag{2.48}$$

3 Fekete-Szegö Inequalities for the Classes $\mathcal{C}_\Sigma(\mathfrak{P}(r,\mathfrak{z}))$ and $\mathfrak{S}^*_\Sigma(\mathfrak{P}(r,\mathfrak{z}))$

Theorem 3.1. *Let the function $\mathfrak{f}(\mathfrak{z}) \in \mathcal{C}_\Sigma(\mathfrak{P}(r,\mathfrak{z}))$ and $\delta \in \mathbb{R}$. Then,*

$$|a_3 - \tau a_2^2| \leq \begin{cases} \dfrac{|1+\delta-r|}{2}, & |1-\tau| \leq \frac{\Upsilon_1}{(1+\delta-r)^2} \\[2ex] \dfrac{(1+\delta-r)^3|1-\tau|}{2|\Upsilon_1|}, & |1-\delta| \geq \frac{\Upsilon_1}{(1+\delta-r)^2} \end{cases} \tag{3.1}$$

where

$$\Upsilon_1 = |(1+\delta-r)^2 - (r^2 - 2(\delta+2)r + (\delta+1)(\delta+2))|.$$

Proof. Let the function $\mathfrak{f}(\mathfrak{z}) \in \mathcal{C}_\Sigma(\mathfrak{P}(r,\mathfrak{z}))$ and $\delta \in \mathbb{R}$. Then, from the equations (2.18) and (2.21), we can write that

$$\begin{aligned} a_3 - \tau a_2^2 =& a_2^2 + \frac{\mathfrak{L}_1^\delta(r)(\rho_2 - \xi_2)}{12} - \delta a_2^2 \\ =&(1-\tau)a_2^2 + \frac{\mathfrak{L}_1^\delta(r)(\rho_2 - \xi_2)}{12} \\ =&(1-\tau)\frac{(\mathfrak{L}_1^\delta(r))^3(\rho_2+\xi_2)}{4(\mathfrak{L}_1^\delta(r))^2 - 8\mathfrak{L}_2^\delta(r)} + \frac{\mathfrak{L}_1^\delta(r)(\rho_2 - \xi_2)}{12} \\ =&\mathfrak{L}_1^\delta(r)\left\{\left(h_1(\delta) + \frac{1}{12}\right)\rho_2 + \left(h_1(\delta) - \frac{1}{12}\right)\xi_2\right\}, \end{aligned} \tag{3.2}$$

where $h_1(\delta) = \frac{(1-\delta)(\mathfrak{L}_1^\delta(r))^2}{4(\mathfrak{L}_1^\delta(r))^2 - 8\mathfrak{L}_2^\delta(r)}$. Now, taking modulus and using triangle inequality with the (2.5), (2.6), (1.4) and (1.5) in (3.2), we complete the proof. □

Taking $\tau = 1$ in Theorem 3.1 we get:

Corollary 3.2. *If the function $f(\mathfrak{z}) \in \mathcal{C}_\Sigma(\mathfrak{P}(r,\mathfrak{z}))$. Then,*

$$|a_3 - a_2^2| \le \frac{|1+\delta-r|}{2}. \tag{3.3}$$

Theorem 3.3. *Let the function $\mathfrak{f}(\mathfrak{z}) \in \mathfrak{S}_\Sigma^*(\mathfrak{P}(r,\mathfrak{z}))$ and $\delta \in \mathbb{R}$. Then,*

$$|a_3 - \tau a_2^2| \le \begin{cases} \frac{3}{2}|1+\delta-r|, & |1-\tau| \le \frac{\Upsilon_2}{2(1+\delta-r)^2} \\ \frac{(1+\delta-r)^3|1-\tau|}{\Upsilon_2}, & |1-\tau| \ge \frac{\Upsilon_2}{2(1+\delta-r)^2} \end{cases} \tag{3.4}$$

where

$$\Upsilon_2 = |2(1+\delta-r)^2 - (r^2 - 2(\delta+2)r + (\delta+1)(\delta+2))|.$$

Proof. Let the function $\mathfrak{f}(\mathfrak{z}) \in \mathfrak{S}_\Sigma^*(\mathfrak{P}(r,\mathfrak{z}))$ and $\delta \in \mathbb{R}$. Then, from the equations (2.40) and (2.43), we can write that

$$\begin{aligned} a_3 - \tau a_2^2 =& a_2^2 + \frac{\mathfrak{L}_1^\delta(r)(\delta_2 - \mu_2)}{4} - \delta a_2^2 \\ =&(1-\tau)a_2^2 + \frac{\mathfrak{L}_1^\delta(r)(\delta_2 - \mu_2)}{4} \\ =&(1-\tau)\frac{(\mathfrak{L}_1^\delta(r))^3(\delta_2+\mu_2)}{2(\mathfrak{L}_1^\delta(r))^2 - 2\mathfrak{L}_2^\delta(r)} + \frac{\mathfrak{L}_1^\delta(r)(\delta_2 - \mu_2)}{4} \\ =&\mathfrak{L}_1^\delta(r)\left\{\left(h_2(\delta) + \frac{1}{4}\right)\delta_2 + \left(h_2(\delta) - \frac{1}{4}\right)\mu_2\right\}, \end{aligned} \tag{3.5}$$

where $h_2(\delta) = \frac{(1-\tau)(\mathfrak{L}_1^\delta(r))^2}{2(\mathfrak{L}_1^\delta(r))^2 - 2\mathfrak{L}_2^\delta(r)}$. Now, taking modulus and using triangle inequality with the (2.5), (2.6), (1.4) and (1.5) in (3.5), we complete the proof. □

Taking $\tau = 1$ in Theorem 3.3 we get:

Corollary 3.4. *If the function* $\mathfrak{f}(\mathfrak{z}) \in \mathfrak{S}^*_\Sigma(\mathfrak{P}(r,\mathfrak{z}))$. *Then,*

$$|a_3 - a_2^2| \leq \frac{3}{2}|1 + \delta - r|. \tag{3.6}$$

4 Conclusion

The fact that we can find many unique and effective uses of a large variety of specific polynomials in geometric function theory provided the primary inspiration for our analysis in this article. The purpose of our present work is to create a new subclasses $\mathcal{C}_\Sigma(\mathfrak{P}(r,\mathfrak{z}))$ and $\mathfrak{S}^*_\Sigma(\mathfrak{P}(r,\mathfrak{z}))$ of regular and bi-univalent functions by using the generalized Laguerre polynomials. We derived the initial Taylor-Maclaurin coefficient inequalities for functions in these newly introduced bi-univalent subclasses and viewed the famous Fekete-Szegö problem. As future research directions, the contents of the paper on a generalized Laguerre polynomials could inspire further research related to other subclasses.

Declarations

- Conflict of interest: Not Applicable

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