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On Some Relationships of Symmetric Sums: $u^n + v^n + w^n + (u+v+w)^n = k(u+v+w)(x^{n-1} + y^{n-1} + z^{n-1})$

Lao Hussein Mude

Department of Pure and Applied Sciences, Kirinyaga University, P. O. Box 143-10300, Kerugoya, Kenya
e-mail: hlao@kyu.ac.ke

Abstract

Let u, v, w, x, y, k and z be any integers and suppose that n is a given exponent. This study focuses on the interplay between sums of four powers and product of symmetric sums. In particular, the Diophantine equation $u^n + v^n + w^n + (u+v+w)^n = k(u+v+w)(x^{n-1} + y^{n-1} + z^{n-1})$ is introduced and partially characterized within the set of integers for exponent $n = 3$. Moreover, this research formulates a conjecture for the equation presented in the title.

1 Introduction

The concept of symmetric sums has its roots in the early development of algebra, particularly in the work of mathematicians who were studying the properties of polynomial equations. Although the idea of symmetric sums is not attributed to a single individual, it emerged gradually as mathematicians began to explore the relationships between the roots of polynomials and their coefficients.

One of the earliest significant contributors to this area was François Viète, a French mathematician who lived in the 16th century. Viète's work laid the foundation for what would later become Vieta's formulas, which express the coefficients of a polynomial in terms of the elementary symmetric sums of its roots. This was a major step forward in the algebraic understanding of equations.

The study of integer decomposition into sums of powers are classical and has been a subject of considerable attention in recent years. Perhaps this is because the study of integer decomposition has direct applications in cryptography. Most researchers seem to have devoted their attention to Ramanujan-Nagell Equation and sums of powers. For recent work on polynomial equations of sums of powers the reader may survey [9,10,11,12,13,14,15,16,17,18,19,20] and for detailed recap on Ramanujan-Nagell Equation the reader may refer to [1,2,3,4,5,6,7,8]. In most of these studies, the literature involving mixed polynomial and sums of

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powers is still hardily available. Moreover, documented results on Diophantine equation $u^n+v^n+w^n+(u+v+w)^n = k(u+v+w)(x^{n-1}+y^{n-1}+z^{n-1})$ proposed in this study is not known. This study is therefore, set to introduce and develop the relationships $u^n+v^n+w^n+(u+v+w)^n = k(u+v+w)(x^{n-1}+y^{n-1}+z^{n-1})$ which involves the relationships between sums of four powers and product of symmetric sums.

2 Main Results

In the following sections, we present our findings in the form of conjecture and proceed to solve particular cases. It is important to note that, in this research, the condition $w > v > u$ is maintained.

Conjecture 2.1. *For any integer $n > 3$, the Diophantine equation*

$$u^n + v^n + w^n + (u + v + w)^n = k(u + v + w)(x^{n-1} + y^{n-1} + z^{n-1})$$

has no solution in integers for all u, v, w, x, y, k and z.

In the sequel, this research partially determines the specific cases of conjecture 2.1 as follows:

Theorem 2.2. *For any integers u, v and w, the Diophantine equation*

$$u^3 + v^3 + w^3 + (u + v + w)^3 = 2(u + v + w)(d^2 + v^2 + (2v)^2)$$

admits solutions within the set of integers if $w - v = v - u = d$.

Proof. Consider the equation

$$u^3 + v^3 + w^3 + (u + v + w)^3 = 2(u + v + w)(d^2 + v^2 + (2v)^2) \cdots (*)$$

and suppose that $v = u + d$ and $w = u + 2d$. The L.H.S of equation $(*)$, expressed as

$$u^3 + v^3 + w^3 + (u + v + w)^3 = u^3 + (u + d)^3 + (u + 2d)^3 + (3u + 2d)^3$$

simplifies to

$$30u^3 + 90u^2d + 96ud^2 + 36d^3 \cdots (**).$$

Decomposing equation $(**)$ into product of sums of symmetric sums, we have

$$30u^3 + 90u^2d + 96ud^2 + 36d^3 = 2(15u^3 + 45u^2d + 48ud^2 + 18d^3)$$

$$= 2(3u + 2d)(5u^2 + 10ud + 6d^2) = 2(u + (u + d) + (u + 2d))(d^2 + (u^2 + 2ud + d^2) + (4u^2 + 8ud + 4d^2))$$

$$= 2(u + (u + d) + (u + 2d))(d^2 + (u + d)^2 + (2u + 2d)^2).$$

Since $v = u + d$ and $w = u + 2d$, we have

$$u^3 + v^3 + w^3 + (u + v + w)^3 = 2(u + v + w)(d^2 + v^2 + (2v)^2).$$

This concludes proof.

□

Theorem 2.3. *For any integers u, v and w, the Diophantine equation*

$$u^3 + v^3 + w^3 + (u + v + w)^3 = 2(u + v + w)(d^2 + v^2 + (u + w)^2)$$

admits solution within the set of integers if $w - v = v - u = d$.

Proof. The proof easily follows from Theorem 2.2 with some slight modification.

□

2.1 Examples

In this subsection, we provide some examples to support our results in Theorem 2.2.

Table 1: $u^3 + v^3 + w^3 + (u + v + w)^3 = 2(u + v + w)(d^2 + v^2 + (2v)^2)$.

u^3	v^3	w^3	$(u+v+w)^3$	I	$2(u+v+w)$	d^2	v^2	$(2v)^2$
1	8	27	216	252	12	1	4	16
1	27	125	729	882	18	4	9	36
8	125	512	3375	4020	30	9	25	100
27	216	729	5832	6804	36	9	36	144
64	216	512	5832	6624	36	4	36	144
8	216	1000	5832	7056	36	16	36	144
1	125	729	3375	4230	30	16	25	100
27	512	2197	13824	16560	48	25	64	256
125	1000	3375	27000	31500	60	25	100	400

In this subsection, we provide some examples to argument our results in Theorem 2.3.

Table 2: $u^3 + v^3 + w^3 + (u + v + w)^3 = 2(u + v + w)(d^2 + v^2 + (u + v)^2)$.

u^3	v^3	w^3	$(u+v+w)^3$	I	$2(u+v+w)$	d^2	v^2	$(u+v)^2$
1	8	27	216	252	12	1	4	16
1	27	125	729	882	18	4	9	36
8	125	512	3375	4020	30	9	25	100
27	216	729	5832	6804	36	9	36	144
64	216	512	5832	6624	36	4	36	144
8	216	1000	5832	7056	36	16	36	144
1	125	729	3375	4230	30	16	25	100
27	512	2197	13824	16560	48	25	64	256
125	1000	3375	27000	31500	60	25	100	400

3 Conclusion

In summary, this research has provided some relationships between sums of four powers and product of symmetric sums. Although this research has made significant progress in this area, there is still room for further investigation. Future studies may also delve into extending these results to other classes of Diophantine equations with higher degrees.

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