Earthline Journal of Mathematical Sciences
E-ISSN: 2581-8147
Volume 15, Number 2, 2025, Pages 169-180
https://doi.org/10.34198/ejms.15225.169180

Chatterjea-Type Contraction Mapping Theorem for Four Self-Mappings in Cone Pentagonal Metric Space

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Abstract

In this paper we obtain a common fixed point theorem for four self mappings under Chatterjea contractive conditions in cone pentagonal metric space. We present an example in support of the main result. Some Corollaries conclude the paper.

1 Introduction and Preliminaries

Definition 1.1. *[1] Let E be a real Banach space and P be a subset of E. Then P is called a cone if and only if*

(a) *P is closed, nonempty, and $P \neq \{0\}$*

(b) *$a, b \in \mathbb{R}$, $a, b \geq 0$ and $x, y \in P \Longrightarrow ax + by \in P$*

(c) *$x \in P$ and $-x \in P \Longrightarrow x = 0$.*

Notation 1.2. *[2] Given a cone $P \subseteq E$ we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in int(P)$, where $int(P)$ denotes the interior of P.*

Remark 1.3. *In this paper, we always suppose that E is a real Banach space and P is a cone in E with $int(P) \neq \emptyset$ and $\leq$ is a partial ordering with respect to P.*

Definition 1.4. *[1] Let X be a nonempty set. Suppose the mapping $d : X \times X \mapsto E$ satisfies*

(a) *$0 < d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$*

Received: November 12, 2024; Accepted: December 19, 2024; Published: December 26, 2024
2020 Mathematics Subject Classification: 47H10, 54H25.
Keywords and phrases: common fixed point theorem, Chatterjea contraction, cone pentagonal metric space.

(b) $d(x,y) = d(y,x)$ *for all* $x, y \in X$

(c) $d(x,y) \leq d(x,z) + d(z,y)$ *for all* $x, y, z \in X$.

Then d *is called a cone metric on* X, *and* (X,d) *is called a cone metric space.*

Remark 1.5. *The concept of a cone metric space is more general than that of a metric space, because each metric space is a cone metric space where* $E = \mathbb{R}$ *and* $P = [0, \infty)$ *(e.g. see [1]).*

Definition 1.6. *[3] Let* X *be a nonempty set. Suppose the mapping* $d : X \times X \mapsto E$ *satisfies*

(a) $0 < d(x,y)$ *for all* $x, y \in X$ *and* $d(x,y) = 0$ *if and only if* $x = y$

(b) $d(x,y) = d(y,x)$ *for all* $x, y \in X$

(c) $d(x,y) \leq d(x,w) + d(w,z) + d(z,y)$ *for all* $x, y \in X$ *and for all distinct points* $w, z \in X - \{x, y\}$ *[Rectangular property].*

Then d *is called a cone rectangular metric on* X *and* (X,d) *is called a cone rectangular metric space.*

Remark 1.7. *Every cone metric space is a cone rectangular metric space. The converse is not necessarily true (e.g. see [3]).*

Definition 1.8. *[4] Let* X *be a nonempty set. Suppose the mapping* $d : X \times X \mapsto E$ *satisfies*

(a) $0 < d(x,y)$ *for all* $x, y \in X$ *and* $d(x,y) = 0$ *if and only if* $x = y$

(b) $d(x,y) = d(y,x)$ *for all* $x, y \in X$

(c) $d(x,y) \leq d(x,z) + d(z,w) + d(w,u) + d(u,y)$ *for all* $x, y, z, w, u \in X$ *and for all distinct points* $z, w, u \in X - \{x, y\}$ *[Pentagonal property].*

Then d *is called a cone pentagonal metric on* X *and* (X,d) *is called a cone pentagonal metric space.*

Remark 1.9. *Every cone metric space and cone rectangular metric space is a cone pentagonal metric space. The converse is not necessarily true (e.g. see [4]).*

Definition 1.10. *[2] Let* (X,d) *be a cone pentagonal metric space. Let* $\{x_n\}$ *be a sequence in* (X,d) *and* $x \in X$.

(a) *If for every* $c \in E$ *with* $0 \ll c$ *there exists* $n_0 \in \mathbb{N}$ *such that for all* $n > n_0$, $d(x_n, x) \ll c$, *then* $\{x_n\}$ *is said to be convergent and* $\{x_n\}$ *converges to* x. *We denote this by* $\lim_{n\to\infty} x_n = x$ *or* $x_n \to x$ *as* $n \to \infty$.

(b) *If for every* $c \in E$ *with* $0 \ll c$ *there exists* $n_0 \in \mathbb{N}$ *such that for all* $n, m > n_0$, $d(x_n, x_m) \ll c$, *then* $\{x_n\}$ *is called a Cauchy sequence in* (X, d).

(c) *If every Cauchy sequence is convergent in* (X, d), *then* X *is called a complete cone pentagonal metric space.*

Definition 1.11. *[2] Let* T *and* S *be self-maps of a nonempty set* X.

(a) *If* $w = Tx = Sx$ *for some* $x \in X$, *then* x *is called a coincidence point of* T *and* S *and* w *is called a point of coincidence of* T *and* S.

(b) T *and* S *are said to be weakly compatible if they commute at their coincidence points, that is,* $Tx = Sx$ *implies that* $TSx = STx$.

Lemma 1.12. *[5] Let* T *and* S *be weakly compatible self mappings of a nonempty set* X. *If* T *and* S *have a unique point of coincidence* $w = Tx = Sx$, *then* w *is the unique common fixed point of* T *and* S.

Lemma 1.13. *[6] Let* (X, d) *be a cone metric space with cone* P *not necessarily to be normal. Then for* $a, c, u, v, w \in E$, *we have*

(a) *If* $a \leq ha$ *and* $h \in [0, 1)$, *then* $a = 0$.

(b) *If* $0 \leq u \ll c$ *for each* $0 \ll c$, *then* $u = 0$.

(c) *If* $u \leq v$ *and* $v \ll w$, *then* $u \ll w$.

(d) *If* $c \in int(P)$ *and* $a_n \to 0$, *then* $\exists n_0 \in \mathbb{N} \ni \forall n > n_0$, $a_n \ll c$.

Lemma 1.14. *[2] Let* (X, d) *be a complete cone pentagonal metric space. Let* $\{x_n\}$ *be a Cauchy sequence in* X *and suppose there is a natural number* N *such that*

(a) $x_n \neq x_m$ *for all* $n, m > N$.

(b) x_n, x *are distinct points in* X *for all* $n > N$.

(c) x_n, y *are distinct points in* X *for all* $n > N$.

(d) $x_n \to x$, $x_n \to y$ *as* $n \to \infty$.

Then $x = y$.

2 Main Result

In this section, we prove Chatterjea-type theorem for four self mappings in cone pentagonal metric space.

Theorem 2.1. *Let (X, d) be a cone pentagonal metric space. Suppose the mappings $f, g, U, V : X \mapsto X$ satisfy the following contractive conditions*

(a) $d(fx, gy) \leq \lambda[d(Ux, gy) + d(Vy, fx)]$

(b) $d(fx, fy) \leq \lambda[d(Ux, fy) + d(Uy, fx)]$

(c) $d(gx, gy) \leq \lambda[d(Vx, gy) + d(Vy, gx)]$

for all $x, y \in X$, where $\lambda \in [0, \frac{1}{2})$. Suppose that $f(X) \subseteq V(X)$, $g(X) \subseteq U(X)$, and one of $f(X), g(X), U(X), V(X)$ is a complete subspace of X, then the pairs (f, U) and (g, V) have a unique point of coincidence in X. Moreover, if (f, U) and (g, V) are weakly compatible pairs then f, g, U, V have a unique common fixed point in X.

Proof. Let $x_0 \in X$. Since $f(X) \subseteq V(X)$ and $g(X) \subseteq U(X)$, starting with x_0 we define a sequence $\{y_n\}$ in X such that

$$y_{2n} = fx_{2n} = Vx_{2n+1} \text{ and } y_{2n+1} = gx_{2n+1} = Ux_{2n+2} \text{ for all } n = 0, 1, 2, \cdots .$$

Suppose that $y_k = y_{k+1}$ for some $k \in \mathbb{N}$. If $k = 2m$, then $y_{2m} = y_{2m+1}$ for some $m \in \mathbb{N}$, then from (a) we obtain

$$\begin{aligned} d(y_{2m+2}, y_{2m+1}) &= d(fx_{2m+2}, gx_{2m+1}) \\ &\leq \lambda(d(Ux_{2m+2}, gx_{2m+1}) + d(Vx_{2m+1}, fx_{2m+2})) \\ &\leq \lambda(d(y_{2m+1}, y_{2m+1}) + d(y_{2m}, y_{2m+2})) \\ &= \lambda d(y_{2m}, y_{2m+2}) \\ &\leq \lambda(d(y_{2m}, y_{2m+1}) + d(y_{2m+1}, y_{2m+2})) \\ &\leq \lambda d(y_{2m+1}, y_{2m+2}). \end{aligned}$$

From the above we have $d(y_{2m+1}, y_{2m+2}) = 0$. That is $y_{2m+1} = y_{2m+2}$. In a similar way we can deduce that $y_{2m+2} = y_{2m+3} = y_{2m+4} = \cdots$. Hence $y_n = y_k$, for all $n \geq k$. Therefore, $\{y_n\}$ is a Cauchy sequence in (X, d). Now assume that $y_n \neq y_{n+1}$ for all $n \in \mathbb{N}$. Then from (a) we have

$$\begin{aligned} d(y_{2m}, y_{2m+1}) &= d(fx_{2m}, gx_{2m+1}) \\ &\leq \lambda(d(Ux_{2m}, gx_{2m+1}) + d(Vx_{2m+1}, fx_{2m})) \\ &\leq \lambda(d(y_{2m-1}, y_{2m+1}) + d(y_{2m}, y_{2m})) \\ &\leq \lambda d(y_{2m-1}, y_{2m+1}) \\ &\leq \lambda(d(y_{2m-1}, y_{2m}) + d(y_{2m}, y_{2m+1})). \end{aligned}$$

From the above we have

$$d(y_{2m}, y_{2m+1}) \leq \frac{\lambda}{1-\lambda} d(y_{2m-1}, y_{2m}) = \alpha d(y_{2m-1}, y_{2m}) \tag{1}$$

where $\alpha = \frac{\lambda}{1-\lambda} \in [0, 1)$. Also

$$\begin{aligned} d(y_{2m+1}, y_{2m+2}) &= d(fx_{2m+1}, gx_{2m+2}) \\ &\leq \lambda(d(Ux_{2m+1}, gx_{2m+2}) + d(Vx_{2m+2}, fx_{2m+1})) \\ &\leq \lambda(d(y_{2m}, y_{2m+2}) + d(y_{2m+1}, y_{2m+1})) \\ &\leq \lambda d(y_{2m}, y_{2m+2}) \\ &\leq \lambda(d(y_{2m}, y_{2m+1}) + d(y_{2m+1}, y_{2m+2})). \end{aligned}$$

From the above we obtain

$$d(y_{2m+1}, y_{2m+2}) \leq \frac{\lambda}{1-\lambda} d(y_{2m}, y_{2m+1}) = \alpha d(y_{2m}, y_{2m+1}). \tag{2}$$

From (2) and (3) it follows that

$$\begin{aligned} d(y_{2m}, y_{2m+1}) &\leq \alpha d(y_{2m-1}, y_{2m}) \\ &\leq \alpha^2 d(y_{2m-2}, y_{2m-1}) \\ &\vdots \\ &\leq \alpha^{2m} d(y_0, y_1) \ \forall m \geq 1 \end{aligned} \tag{3}$$

and

$$\begin{aligned} d(y_{2m+1}, y_{2m+2}) &\leq \alpha d(y_{2m}, y_{2m+1}) \\ &\leq \alpha^2 d(y_{2m-1}, y_{2m}) \\ &\vdots \\ &\leq \alpha^{2m+1} d(y_0, y_1) \ \forall m \geq 1. \end{aligned} \tag{4}$$

Hence, from (4) and (5) we deduce that

$$d(y_n, y_{n+1}) \leq \alpha^n d(y_0, y_1) \ \forall n \geq 1. \tag{5}$$

From (b), (c), (6) and the fact that $0 \leq \lambda \leq \alpha < 1$, we obtain

$$\begin{aligned}
d(y_{2m}, y_{2m+2}) &= d(fx_{2m}, fx_{2m+2}) \\
&\leq \lambda(d(Ux_{2m}, fx_{2m+2}) + d(Ux_{2m+2}, fx_{2m})) \\
&\leq \lambda(d(y_{2m-1}, y_{2m+2}) + d(y_{2m+1}, y_{2m})) \\
&\leq \lambda(d(y_{2m-1}, y_{2m}) + d(y_{2m}, y_{2m+1}) + d(y_{2m+1}, y_{2m+2}) + d(y_{2m+1}, y_{2m})) \\
&\leq \lambda(d(y_{2m-1}, y_{2m}) + 2d(y_{2m}, y_{2m+1}) + d(y_{2m+1}, y_{2m+2})) \\
&\leq \lambda(\alpha^{2m-1} d(y_0, y_1) + 2\alpha^{2m} d(y_0, y_1) + \alpha^{2m+1} d(y_0, y_1)) \\
&\leq \alpha^{2m} d(y_0, y_1) + 2\alpha^{2m+1} d(y_0, y_1) + \alpha^{2m+2} d(y_0, y_1) \\
&\leq \alpha^{2m} d(y_0, y_1)(1 + 2\alpha + \alpha^2) \\
&\leq \alpha^{2m} d(y_0, y_1)(1 + 2 + \alpha) \\
&\leq \alpha^{2m} d(y_0, y_1)(3 + \alpha) \ \forall m \geq 1
\end{aligned} \tag{6}$$

and

$$\begin{aligned}
d(y_{2m+1}, y_{2m+3}) &= d(gx_{2m+1}, gx_{2m+3}) \\
&\leq \lambda(d(Vx_{2m+1}, gx_{2m+3}) + d(Vx_{2m+3}, gx_{2m+1})) \\
&\leq \lambda(d(y_{2m}, y_{2m+3}) + d(y_{2m+2}, y_{2m+1})) \\
&\leq \lambda(d(y_{2m}, y_{2m+1}) + d(y_{2m+1}, y_{2m+2}) + d(y_{2m+2}, y_{2m+3}) + d(y_{2m+2}, y_{2m+1})) \\
&\leq \lambda(d(y_{2m}, y_{2m+1}) + 2d(y_{2m+1}, y_{2m+2}) + d(y_{2m+2}, y_{2m+3})) \\
&\leq \lambda(\alpha^{2m} d(y_0, y_1) + 2\alpha^{2m+1} d(y_0, y_1) + \alpha^{2m+2} d(y_0, y_1)) \\
&\leq \alpha^{2m+1} d(y_0, y_1) + 2\alpha^{2m+2} d(y_0, y_1) + \alpha^{2m+3} d(y_0, y_1) \\
&\leq \alpha^{2m+1} d(y_0, y_1)(1 + 2\alpha + \alpha^2) \\
&\leq \alpha^{2m} d(y_0, y_1)(1 + 2 + \alpha) \\
&\leq \alpha^{2m} d(y_0, y_1)(3 + \alpha).
\end{aligned} \tag{7}$$

Hence from (7) and (8) we have

$$d(y_n, y_{n+2}) \leq (3 + \alpha)\alpha^n d(y_0, y_1) \ \forall n \geq 1. \tag{8}$$

For the sequence $\{y_n\}$ we consider $d(y_n, y_{n+p})$ in two cases as follows: If p is odd say $p = 2k + 1$ where

$k \geq 1$, then by the pentagonal property and (6), we have

$$\begin{aligned}
d(y_n, y_{n+2k+1}) &\leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + d(y_{n+2}, y_{n+3}) + d(y_{n+3}, y_{n+2k+1}) \\
&\leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + d(y_{n+2}, y_{n+3}) + \cdots \\
&+ d(y_{n+2k-1}, y_{n+2k}) + d(y_{n+2k}, y_{n+2k+1}) \\
&\leq \alpha^n d(y_0, y_1) + \alpha^{n+1} d(y_0, y_1) + \alpha^{n+2} d(y_0, y_1) + \cdots \\
&+ \alpha^{n+2k-1} d(y_0, y_1) + \alpha^{n+2k} d(y_0, y_1) \\
&\leq \frac{\alpha^n}{1-\alpha} d(y_0, y_1) \ \forall n \geq 1.
\end{aligned}$$

If p is even say $p = 2k$ where $k \geq 1$, then by the pentagonal property, (6) and (9), we have

$$\begin{aligned}
d(y_n, y_{n+2k}) &\leq d(y_n, y_{n+2}) + d(y_{n+2}, y_{n+3}) + d(y_{n+3}, y_{n+4}) + d(y_{n+4}, y_{n+2k}) \\
&\leq d(y_n, y_{n+2}) + d(y_{n+2}, y_{n+3}) + d(y_{n+3}, y_{n+4}) + \cdots \\
&+ d(y_{n+2k-2}, y_{n+2k-1}) + d(y_{n+2k-1}, y_{n+2k}) \\
&\leq (3+\alpha)\alpha^n d(y_0, y_1) + \alpha^{n+2} d(y_0, y_1) + \alpha^{n+3} d(y_0, y_1) + \cdots \\
&+ \alpha^{n+2k-2} d(y_0, y_1) + \alpha^{n+2k-1} d(y_0, y_1) \\
&\leq \frac{\alpha^n}{1-\alpha} d(y_0, y_1) \ \forall n \geq 1.
\end{aligned}$$

Therefore, combining the above two cases, we get

$$d(y_n, y_{n+p}) \leq \frac{\alpha^n}{1-\alpha} d(y_0, y_1) \ \forall n, p \in \mathbb{N}.$$

Since $\alpha \in [0, 1)$, we get, as $n \to \infty$, $\frac{\alpha^n}{1-\alpha} \to 0$. Hence, for every $c \in E$ with $c \gg 0$ $\exists n_0 \in \mathbb{N}$ such that

$$d(y_n, y_{n+p}) \ll c \ \forall n \geq n_0.$$

Therefore, $\{y_n\}$ is a Cauchy sequence in (X, d). Suppose $U(X)$ is a complete subspace of X, then there exists points $p, q \in U(X)$ such that $\lim_{n\to\infty} y_{2n+1} = \lim_{n\to\infty} Ux_{2n+2} = q = Up$. Now we show that $Up = fp$. Given $c \gg 0$, we choose natural numbers T_1, T_2, T_3, T_4, T_5 such that $d(q, y_{2n}) \ll \frac{c(1-\lambda)}{5\lambda}$ $\forall n \geq T_1$, $d(q, y_{2n-1}) \ll \frac{c(1-\lambda)}{5\lambda}$ $\forall n \geq T_2$, $d(y_{2n+1}, y_{2n}) \ll \frac{c(1-\lambda)}{5}$ $\forall n \geq T_3$, $d(y_{2n+1}, y_{2n+2}) \ll \frac{c(1-\lambda)}{5}$ $\forall n \geq T_4$, $d(y_{2n+2}, q) \ll \frac{c(1-\lambda)}{5}$ $\forall n \geq T_5$. Since $y_n \neq y_m$ for $n \neq m$, by pentagonal property and (b) we have

$$\begin{aligned}
d(fp, q) &\leq d(fp, y_{2n}) + d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2n+2}) + d(y_{2n+2}, q) \\
&\leq d(fp, fx_{2n}) + d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2n+2}) + d(y_{2n+2}, q) \\
&\leq \lambda(d(Up, fx_{2n}) + d(Ux_{2n}, fp)) + d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2n+2}) + d(y_{2n+2}, q) \\
&\leq \lambda d(q, y_{2n}) + \lambda d(y_{2n-1}, fp) + d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2n+2}) + d(y_{2n+2}, q) \\
&\leq \lambda d(q, y_{2n}) + \lambda(d(y_{2n-1}, q) + d(q, fp)) + d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2n+2}) + d(y_{2n+2}, q).
\end{aligned}$$

From the above we have

$$\begin{aligned}
d(fp,q) &\leq \frac{1}{1-\lambda}(\lambda d(q,y_{2n}) + \lambda d(y_{2n-1},q) + d(y_{2n},y_{2n+1}) + d(y_{2n+1},y_{2n+2}) + d(y_{2n+2},q)) \\
&\ll \frac{c}{5} + \frac{c}{5} + \frac{c}{5} + \frac{c}{5} + \frac{c}{5} \\
&= c \ \forall n \geq L_1
\end{aligned}$$

where $L_1 = \max\{T_1, T_2, T_3, T_4, T_5\}$. Since c is arbitrary we have $d(fp,q) \ll \frac{c}{m}$, $\forall m \in \mathbb{N}$. Since $\frac{c}{m} \to 0$ as $m \to \infty$, we conclude $\frac{c}{m} - d(fp,q) \to -d(fp,q)$ as $m \to \infty$. Since P is closed, $-d(fp,q) \in P$. Hence $d(fp,q) \in P \cap -P$. By definition of cone we get that $d(fp,q) = 0$, and so $Up = fp = q$. Hence q is a point of coincidence of f and U. Since $q = fp \in f(X)$ and $f(X) \subseteq V(X)$, there exists $r \in X$ such that $q = Vr$. Now, we show that $Vr = gr$. Given $c \gg 0$, we can choose natural numbers $T_6, T_7, T_8, T_9, T_{10}$ such that $d(q, y_{2n-1}) \ll \frac{c(1-\lambda)}{5\lambda}$ $\forall n \geq T_6$, $d(q, y_{2n}) \ll \frac{c(1-\lambda)}{5\lambda}$ $\forall n \geq T_7$, $d(y_{2n+1}, y_{2n}) \ll \frac{c(1-\lambda)}{5}$ $\forall n \geq T_8$, $d(y_{2n+1}, y_{2n+2}) \ll \frac{c(1-\lambda)}{5}$ $\forall n \geq T_9$, $d(y_{2n+2}, q) \ll \frac{c(1-\lambda)}{5}$ $\forall n \geq T_{10}$. Since $y_n \neq y_m$ for $n \neq m$, by pentagonal property and (a) we have that

$$\begin{aligned}
d(gr,q) &\leq d(gr,y_{2n}) + d(y_{2n},y_{2n+1}) + d(y_{2n+1},y_{2n+2}) + d(y_{2n+2},q) \\
&\leq d(gr,fx_{2n}) + d(y_{2n},y_{2n+1}) + d(y_{2n+1},y_{2n+2}) + d(y_{2n+2},q) \\
&\leq \lambda(d(Ux_{2n},gr) + d(Vr,fx_{2n})) + d(y_{2n},y_{2n+1}) + d(y_{2n+1},y_{2n+2}) + d(y_{2n+2},q) \\
&\leq \lambda d(y_{2n-1},gr) + \lambda d(q,y_{2n}) + d(y_{2n},y_{2n+1}) + d(y_{2n+1},y_{2n+2}) + d(y_{2n+2},q) \\
&\leq \lambda(d(y_{2n-1},q) + d(q,gr)) + \lambda d(q,y_{2n}) + d(y_{2n},y_{2n+1}) + d(y_{2n+1},y_{2n+2}) + d(y_{2n+2},q).
\end{aligned}$$

From the above we have

$$\begin{aligned}
d(gr,q) &\leq \frac{1}{1-\lambda}(\lambda d(y_{2n-1},q) + \lambda d(q,y_{2n}) + d(y_{2n},y_{2n+1}) + d(y_{2n+1},y_{2n+2}) + d(y_{2n+2},q)) \\
&\ll \frac{c}{5} + \frac{c}{5} + \frac{c}{5} + \frac{c}{5} + \frac{c}{5} \\
&= c \ \forall n \geq L_2
\end{aligned}$$

where $L_2 = \max\{T_6, T_7, T_8, T_9, T_{10}\}$. Since c is arbitrary we have $d(gr,q) \ll \frac{c}{m}$, $\forall m \in \mathbb{N}$. Since $\frac{c}{m} \to 0$ as $m \to \infty$, we conclude $\frac{c}{m} - d(gr,q) \to -d(gr,q)$ as $m \to \infty$. Since P is closed, $-d(gr,q) \in P$. Hence $d(gr,q) \in P \cap -P$. By definition of cone we get that $d(gr,q) = 0$, and so $Vr = gr = q$. Hence q is a point of coincidence of g and V. Thus, the pairs (f,U) and (g,V) have common point of coincidence q in X. Now, suppose the pairs (f,U) and (g,V) are weakly compatible mappings. Then

$$fq = fUp = Ufp = Uq = q_1 \text{ for some } q_1 \in X$$

and

$$gq = gVr = Vgr = Vq = q_2 \text{ for some } q_2 \in X.$$

Hence, from (a) we have

$$\begin{aligned} d(q_1, q_2) &= d(fq, gq) \\ &\leq \lambda(d(Uq, gq) + d(Vq, fq)) \\ &= \lambda(d(q_1, q_2) + d(q_2, q_1)) \\ &= 2\lambda d(q_1, q_2) \end{aligned}$$

which implies $d(q_1, q_2) = 0$. Hence, $q_1 = q_2$. Therefore

$$fq = gq = Uq = Vq.$$

Also

$$\begin{aligned} d(q, gq) &= d(fp, gq) \\ &\leq \lambda(d(Up, gq) + d(Vq, fp)) \\ &= \lambda(d(q, gq) + d(gq, q)) \\ &= 2\lambda d(q, gq) \end{aligned}$$

which implies $d(q, gq) = 0$. Hence $gq = q$ or $fq = gq = Uq = Vq = q$. Thus, q is the common fixed point of f, g, U, V. Next we show that q is unique. For suppose q' be another common fixed point of f, g, U, V. That is,

$$fq' = gq' = Uq' = Vq' = q'$$

for some $q' \in X$. Then from (a) we have

$$\begin{aligned} d(q, q') &= d(fq, gq') \\ &\leq \lambda(d(Uq, gq') + d(Vq', fq)) \\ &= \lambda(d(q, q') + d(q', q)) \\ &= 2\lambda d(q, q') \end{aligned}$$

which implies $d(q, q') = 0$. Hence, $q = q'$. Therefore the mappings $f, g, U.V$ have a unique common fixed point in X. Similarly if $f(X)$, $g(X)$, or $V(X)$ is a complete subspace of X, then we can easily prove that f, g, U, V have unique common fixed point in X. This completes the proof of the theorem. □

Example 2.2. *Let* $X = \{1, 2, 3, 4, 5\}$, $E = \mathbb{R}^2$ *and* $P = \{(x, y) : x, y \geq 0\}$ *is a cone in* E. *Define* $d : X \times X \mapsto E$ *as follows:*

$$d(x, x) = 0 \ \forall x \in X$$

$$d(1, 2) = d(2, 1) = (4, 16)$$

$$d(1, 3) = d(3, 1) = d(3, 4) = d(4, 3) = d(2, 3) = d(3, 2) = d(2, 4) = d(4, 2) = d(1, 4) = d(4, 1) = (1, 4)$$

$$d(1,5) = d(5,1) = d(2,5) = d(5,2) = d(3,5) = d(5,3) = d(4,5) = d(5,4) = (5,20).$$

Then (X,d) is a complete cone pentagonal metric space, but (X,d) is not a complete cone rectangular metric space because it lacks the rectangular property:

$$\begin{aligned}(4,16) &= d(1,2)\\ &> d(1,3) + d(3,4) + d(4,2)\\ &= (1,4) + (1,4) + (1,4)\\ &= (3,12) \ \textit{as} \ (4,16) - (3,12) = (1,4) \in P.\end{aligned}$$

Define mappings $f, g, U.V : X \mapsto X$ as follows:

$$f(x) = 4 \ \forall x \in X$$

$$g(x) = \begin{cases} 4 & \textit{if } x \neq 5 \\ 2 & \textit{if } x = 5 \end{cases}$$

$$U(x) = \begin{cases} 3 & \textit{if } x = 1 \\ 1 & \textit{if } x = 2 \\ 2 & \textit{if } x = 3 \\ 4 & \textit{if } x = 4 \\ 5 & \textit{if } x = 5 \end{cases}$$

$$V(x) = x \ \forall x \in X.$$

Clearly $f(X) \subseteq V(X)$, $g(X) \subseteq U(X)$, and the pairs (f,U) and (g,V) are weakly compatible mappings. The condition of the above theorem holds for all $x, y \in X$, where $\lambda = \frac{1}{5}$, and 4 is the unique common fixed point of the mappings f, g, U, V.

3 Consequences of the Main Result

If $V = U$ in the above theorem, then we have the following

Corollary 3.1. *Let (X,d) be a cone pentagonal metric space. Suppose the mappings $f, g, U : X \mapsto X$ satisfy the following contractive conditions*

(a) $d(fx, gy) \leq \lambda[d(Ux, gy) + d(Uy, fx)]$

(b) $d(fx, fy) \leq \lambda[d(Ux, fy) + d(Uy, fx)]$

(c) $d(gx, gy) \leq \lambda[d(Ux, gy) + d(Uy, gx)]$

for all $x, y \in X$, *where* $\lambda \in [0, \frac{1}{2})$. *Suppose that* $f(X) \cup g(X) \subseteq U(X)$, *and if* $U(X)$ *or* $f(X) \cup g(X)$ *is a complete subspace of* X, *then the pairs* (f, U) *and* (g, U) *have a unique point of coincidence in* X. *Moreover, if* (f, U) *and* (g, U) *are weakly compatible pairs then* f, g, U *have a unique common fixed point in* X.

If $g = f$ and $V = U$ in the above theorem, then we have the following

Corollary 3.2. *Let* (X, d) *be a cone pentagonal metric space. Suppose the mappings* $f, U : X \mapsto X$ *satisfy the condition:*

$$d(fx, fy) \leq \lambda(d(Ux, fy) + d(Uy, fx))$$

for all $x, y \in X$, *where* $\lambda \in [0, \frac{1}{2})$. *Suppose that* $f(X) \subseteq U(X)$, *and, if* $U(X)$ *or* $f(X)$ *is a complete subspace of* X, *then the pair* (f, U) *have a unique point of coincidence in* X. *Moreover, if* f *and* U *are weakly compatible pairs, then* f *and* U *have a unique common fixed point in* X.

If $g = f$, $V = U = I$ (*identity mapping*), and P is a normal cone in the above theorem, then we have the following

Corollary 3.3. *Let* (X, d) *be a complete cone pentagonal metric space and* P *be a normal cone with normal constant* k. *Suppose the mapping* $f : X \mapsto X$ *satisfies the contractive condition:*

$$d(fx, fy) \leq \lambda(d(x, fy) + d(y, fx))$$

for all $x, y \in X$ *where* $\lambda \in [0, \frac{1}{2})$. *Then*

(a) f *has a unique fixed point in* X

(b) *For any* $x \in X$, *the iterative sequence* $\{f^n x\}$ *converges to the fixed point.*

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