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Optimal Debt Ratio and Investment-Consumption Strategies with Taxation in the Presence of Jump Risks

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Abstract

This paper derives an optimal debt ratio, consumption rate and investment strategies with taxation for an investor who invest under four background risks: investment, taxation, income and jump risks. The underlying assets considered in this paper are a riskless and risky asset. The risky asset is assumed to follow a jump-diffusion process. We also assume that the income growth rate and tax payment of the investor follow a jump-diffusion process. The aim of the investor is to derive the wealth-after-tax process. The wealth-after-tax process of the investor is taken to be the difference between the wealth-before-tax and the tax payment processes of the investor. The resulting wealth-after-tax process of the investor was solved using dynamic programming approach. As a result, we derive the optimal investment strategies, optimal debt ratio and optimal consumption rate for the investor over time by assuming that the investor chooses a power utility function. The optimal investment strategies were found to involve four components: a speculative portfolio, a tax risks hedging portfolio strategy, an income growth rate risks hedging portfolio strategy and a risk-free fund that holds only the riskless asset. Interestingly, we found that before loan is taken or given, the following must be considered: interest rate on loan to be taken or given, the nominal interest rate, income growth rate, coefficient of the investor willingness to bear the risk of taking debt. We also found that as the income growth rate of the investor increases, the debt profile of the investor decreases. We observe that as the coefficient of risk aversion with respect to debt ratio tends to unity, the amount of debt will be unbearable. It was also observed that the higher an investor willingness to bear the risk of taking debt, the smaller the optimal debt ratio of the investor over time. We further found that when tax rate increases, consumption rate decreases and vice versa. To ascertain the validity of our models, data were collected from six companies in Nigerian Stocks Exchange, and SPSS package was used to analyze the data. Some empirical results were obtained in this paper, using MATLAB software.

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1 Introduction

Managing investments and debt in the presence of investment, taxation, income and jump risks can be challenging. Many investments today are funded by loans and these loans are collected with interest. For effective and efficient investment management strategy, the investment must be managed to generate returns that can offset the liability in the investment portfolio at all time. Failure to manage the investment effectively may lead to a crisis. Hence, there is the need to study and determine the optimal debt in an investment portfolio. The debt process of the investor is defined as expenditure less income. In this paper, we assume that the more the investor borrow to finance his/her investment, the higher the return. Hence, the higher the debt, the higher the return of the investor's investment portfolio. The income process of the investor is expected to grow at a certain rate, and this rate is assumed to follow a jump-diffusion process. Again, the investor must pay tax to government over time. The tax base of the investor is assume to follow a jump-diffusion process. The amount paid by the investor as tax is taken to be the product of the tax base and the tax rate over time. Since the general market is full of jumps, it therefore follows that the dynamics of tax payment of the investor follows a jump-diffusion process. The wealth-before-tax process of the investor was obtained and is taken to be difference between assets value and debt of the investor. The wealth-after-tax process of the investor is the difference between wealth-before-tax process and the tax payment of the investor over time.

It is imperative to note that poor debt management may lead to an investment and financial crisis. In fact, one of the causes of the investment and financial crises of 2008 was the in ability of firms with leveraged portfolio to pay their debt when the market started experiencing shocks. Jin (2014) studied the optimal debt ratio and consumption plan for an investor during a financial crisis. The impact of labor market condition was also studied. He assumed that the production rate function of the investor is stochastic and being influenced by the government policy, employment and unanticipated risks. Nkeki (2018a) considered backup security as a buffer for loans in a fnancial crisis. Nkeki and Modugu (2020) studied an optimal investment portfolio, net debt ratio with collateralized security and consumption plan for an investor that faces jump risks in the presence of intangiable assets. Stein (2003) derived the optimal debt of an investor under a recession using stochastic optimal control and Krouglov (2013) applied the mathematical model of economic system to study the development of a financial crisis. Lui and Jin (2014) proposed a stochastic optimal control debt management model for the estimation of optimal debt ratio for both the public and private sectors under two different market regimes. Jin et al. (2015) considered the surplus process of an insurer and obtained an optimal policy for debt and dividend payment for an insurer. Zhao et al. (2018) considered the optimal debt ratio, investment and dividend payment strategies for an insurance company in a finite time horizon by maximizing the total expected discounted utility of dividend payment, Qian et al. (2018) considered the optimal liability management strategies and dividend payment for an insurance company that experiences catastrophic risks.

Significant movements or jumps in the prices of financial assets caused by sudden large shocks like the subprime crisis, corporate scandal, the COVID-19 pandemic e.t.c., have been empirically shown to exist, see Lee and Mykland (2007), Bakshi et al. (1997) and Eraker(2004). Lui et al. (2003) solved the optimal portfolio choice problem in closed form for a model where there is a risky asset with jumps in both stock price and volatility, and suggested that jumps that results in event risk affects the optimal strategy of the investor. Das and Uppal (2004) investigated the effect of systemic jumps in stock prices on international portfolio selection. Zuo et al. (2009) studied a dividend optimization problem for an insurer who faces a jump-diffusion risk and obtained the optimal policies under the risk neutral assumption. Ait-Sahalia (2009) provided a new analytical framework for studying the optimal portfolio choice problem in the presence of both Brownian and jump risks. Nyeyn (2012) considered optimal consumption with bounded downside risk for an investor with power utility functions in the presence of jump diffusion process. Ait-Sahalia (2015) proposed a model that captured the dynamics of asset returns during a financial crisis which are generalized by contagion in the presence of jump processes. Nkeki (2018b) investigated empirically and theoretically the optimal portfolio strategies of a defined contributory (DC) pension under inflation risk in a jump-diffusion environment and Zhu et al (2024) considered the optimal debt and dividend payment policies for an insurer concerned with model misspecification. They showed that with ambiguity in value functions, the optimal debt can be affected.

Tax is an important component of government revenue and is also taken into consideration by investors in the determination of optimal investment policies and consumption plan. The amount which is paid as tax by the investor can deter investment if it is prohibitive. Hence, understanding the dynamic impact of taxes on investment is critical for an investor. Ma et al. (2018) investigated an optimal investment problem in the presence of jump risk and capital gains tax. They also considered tax evasion in the investment portfolio, see Levaggi and Menoncin (2011). Dammon et al. (2004) investigated optimal intertemporal asset allocation and location decisions for investors making taxable and tax-deferred investments. Further, Marekwica (2012) considered asset allocation and optimal tax-timing when tax rebates on capital losses are limited and Dai et al. (2015) proposed a continuous-time model to investigate the impact of asymmetric tax structure and limited tax return on the behaviour of investors.

In this paper, the underlying assets considered are a risk-free asset and multiple risky assets. The income, debt and tax payment processes of the investor are also considered. The dynamics of the risky assets, income growth rate and taxation follow a jump-diffusion process. The after-tax wealth process is obtained as the difference between the wealth-before-tax process and taxation. The resulthing wealth-after-tax process of the investor is solved using stochastic dynamic programming techniques and the resulting Hamilton-Jacobi-Bellman (HJB) equation was obtained by transversility condition. As a result, the optimal investment strategies, optimal debt ratio and optimal consumption processes were obtained by assuming that the investor is risk averse and chooses the constant relative risk aversion (CRRA) utility function. Also, we provide a verification theorem for our problem. Numerical results for our model were obtained using data collected from the Nigerian Stock Exchange, see NSE (2021) and analysed using SPSS package. The empirical results obtained in the paper are determine using MATLAB.

The rest of the paper is structured as follows: In Section 2, the probability space, underlying financial assets, the debt, income growth rate and wealth process, the tax and wealth-after-tax process of the investor are presented. Section 3 presents the optimal controls and value function, optimal debt ratio, optimal consumption plan and the optimal portfolio strategies of the investor. Also, in this section, we presents the optimal debt and optimal tax payment of the investor over time. The numerical results for our models are presented in Section 4 and a verification theorem for our model is given in Section 5. We conclude the paper, in Section 6.

2 The Models

In this section, we present dynamics of the underlying financial assets, the debt process, the dynamics of wealth-before-tax process, the tax payment process and wealth-after-tax dynamics of the investor at time t.

2.1 The financial asset models

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\leq t\leq T}, \mathbf{P})$ be a filtered probability space equipped with the filtration $\{\mathcal{F}_t\}_{0\leq t\leq T}$ which satisfies the usual conditions i.e $\{\mathcal{F}_t\}_{0\leq t\leq T}$ is right-continuous and $\mathbf{P}$ denotes real world probability measure, $\mathcal{F}_t$ is a filtration that represents information avaliable at time t. We assume that all stochastic processes used below are well-defined on the given probability space and adapted to the filtration $\{\mathcal{F}_t\}_{0\leq t\leq T}$.

We assume that the investor trade on two classes of assets. The first class of asset is riskless $S_0(t)$ which pays interest rate r(t) at time t and the dynamics is given by

$$dS_0(t) = r(t)S_0(t)dt, S_0(0) = s_0 \in \mathbb{R}. \tag{1}$$

We assume that $r(t)$ is bounded. That is,

$$\forall r(t) \in [0, T] \times \mathcal{R}, |r(t)| \leq \Theta_r. \tag{2}$$

The second class of asset is $n-$dimentional risky assets (stocks) $S(t_{-}) = (S_1(t_{-}), \ldots, S_n(t_{-}))$ and are assume to following a jump-diffusion process

$$\frac{d\mathrm{S}(t_{-})}{\mathrm{S}(t_{-})} = \mu(t)dt + \Sigma(t)\mathrm{dW}_S(t) + \mathrm{J}(t)d\mathrm{N}_S(t), S(0) = s \in \mathbb{R}^n, \tag{3}$$

where the vector $\mu(t) = (\mu_1(t), \mu_2(t), ..., \mu_n(t))'$ is the expected return of stock prices at time t, the sign $'$ represents transpose, $\Sigma(t) \in [0,T] \times \mathbb{R}^{n\times n}$ is the volatility matrix of stock prices at time t, $\mathrm{W}_S(t) = (\mathrm{W}_1(t), \mathrm{W}_2(t), ..., \mathrm{W}_n(t))'$ is an n-dimensional Brownian motion with respect to stocks at time t, $\mathrm{J}(t) \in [0,T] \times \mathbb{R}^{\mathrm{M}_S\times \mathrm{M}_S}$ is an $\mathrm{M}_S \times \mathrm{M}_S$ diagonal matrix of jumps scaling factors with respect to stocks at time t, the vector $\mathrm{N}_S(t) = (\mathrm{N}_{S_1}(t), \ldots, \mathrm{N}_{S_{M_S}}(t))'$ is a pure Lévy jump process that explains jumps in stock prices with Lévy measure $\lambda_{b,S}\nu_{b,S}(dz_{b,S}), b = 1, 2, ..., \mathrm{M}_S$ such that $z_{b,S}$ are the jump sizes with respect to stock b and $\lambda_{b,S} \geq 0$ are the jump intensities. The measure $\nu_{b,S}$ satisfies $\int_{\mathbb{R}} \min(1, |z_{b,S}|)\nu_{b,S}(dz_{b,S}) < \infty$, which guarantees that variations from stock prices are finite. We assume that $\mathrm{W}_S(t)$ and $\mathrm{N}_S(t)$ are independent of each other. We also assume that $\mu(t)$ is bounded. That is,

$$\forall \mu(t) \in [0,T] \times \mathbb{R}^n, \|\mu(t)\| \leq \Theta. \tag{4}$$

2.2 Investment portfolio of the investor

Let $\mathrm{G}(t_{\text{-}})$ be the investment portfolio process of the investor at time t, $\Delta(t) = (\Delta_{S1}(t), \Delta_{S2}(t), ..., \Delta_{Sn}(t))$ is amount of fund invested in risky assets and the remainder $\Delta_0(t) = G(t_{\text{-}}) - \Delta(t)\kappa$ is invested in riskless asset at time t, where $\kappa = [1, 1, ..., 1]' \in \mathbb{R}^n$. We now have the following definition.

Definition 1. *The investment portfolio process, $G(t)$ of an investor at time t is define as*

$$dG(t_{\text{-}}) = \Delta_0(t)\frac{dS_0(t)}{S_0(t)} + \Delta(t)\frac{dS(t_{\text{-}})}{S(t_{\text{-}})}, \;\; G(0) = G_0 \in \mathbb{R}_+. \tag{5}$$

Using (1) and (3) on (5), we have the following dynamics:

$$\begin{aligned} \mathrm{dG}(t_{\text{-}}) = &\big(r(t)\mathrm{G}(t_{\text{-}}) + \Delta(t)(\mu(t) - r(t)\kappa)\big)dt + \Delta(t)\Sigma(t)\mathrm{dW}_S(t) \\ &+ \Delta(t)\mathrm{J}(t)\mathrm{dN}_S(t), \mathrm{G}(0) = \mathrm{G}_0 \in \mathbb{R}_+. \end{aligned} \tag{6}$$

(6) represents the dynamics of the investment portfolio process of the investor at time t.

2.3 The debt, income growth rate and wealth process

In this subsection, we present the debt, income growth rate and wealth-before-tax process of the investor at time t. We describe the wealth-before-tax process of the investor $\mathrm{X}(t_{\text{-}})$ as the difference between the investment portfolio value of the investor $\mathrm{G}(t_{\text{-}})$ and debt (or liability) $\mathrm{K}(t)$ taken by the investor and pays an interest rate, $r_k(t)$ on the liability at time t. We assume that $r_k(t)$ is bounded. That is,

$$\forall r_k(t) \in [0,T] \times \mathbb{R}, |r_k(t)| \leq \Theta_k. \tag{7}$$

In this paper, we describe debt as expenditure less income. The expenditure component include the debt payable by the investor at time t, $r_k(t)\mathrm{K}(t)$ and consumption process of the investor at time t, $\mathrm{C}(t)$ which evolves as follows:

$$\mathrm{dC}(t) = \alpha(t)\mathrm{X}(t_{-})dt, \tag{8}$$

where $\alpha(t)$ is the consumption rate at time t.

Remark 1. *Note that consumption and debt processes are taken as expenditure of the investor.*

The income component involves the product of income growth rate given as $g(t)$ and the investment portfolio value process $\mathrm{G}(t)$ of the investor at time t. Let $\mathrm{Z}(t)$ be the income generated by the portfolio of the investor at time t. Hence, the product of $g(t)$ and $\mathrm{G}(t_{-})$ of the investor at time t is given by the following equation:

$$\mathrm{dZ}(t) = g(t)\mathrm{G}(t_{-})dt, \mathrm{Z}(0) = \mathrm{Z}_0. \tag{9}$$

Hence, the net change in debt is given as

$$\mathrm{dK}(t) = r_k(t)\mathrm{K}(t)dt + \alpha(t)\mathrm{X}(t_{-})dt - g(t)\mathrm{G}(t_{-})dt. \tag{10}$$

Remark 2. *We assume that the more the investor borrow to finance its investment, the higher the return. Hence, the higher the debt, the higher the return of the investor's investment.*

The income growth rate $g(t_{-})$ may experience jumps which might result from changes in government regulations, economic policies, natural disaster, the Corona virus global pandemic, e.t.c. Hence, the dynamics of $g(t)$ is assumed to follow a jump-diffusion process given by

$$dg(t_{-}) = (\Lambda(g(t_{-})) + g(t_{-})\tilde{\eta}(\omega))dt + \sigma_g(t)\mathrm{dW}_g(t) + \mathrm{J}_g(t)\mathrm{dN}_g(t), g(0) = g_0, \tag{11}$$

where $\Lambda(g(t_{-})) \in [0, T] \times \mathbb{R}_{+}$ is part of the the drift term of the income growth rate at time t and it represents the increment factor of the income growth of the investor, $\tilde{\eta}(\omega) \in \mathbb{R}$ represents the impact of labour force on the income growth rate. For an expanding investment $\tilde{\eta}(\omega) > 0$, if $\tilde{\eta}(\omega) = 0$, it implies that the income growth rate of the investor is not affected by labour force and if $\tilde{\eta}(\omega) < 0$ it implies that the investment is experiencing crisis, $\sigma_g(t) \in [0, T] \times \mathbb{R}^q$ is the volatility of the income growth rate, $\mathrm{W}_g(t) = (\mathrm{W}_{g_1}(t), \mathrm{W}_{g_2}(t), ..., \mathrm{W}_{g_q}(t))'$ is a q-dimensional Brownian motion, $\mathrm{J}_g(t) \in [0, T] \times \mathbb{R}^{\mathrm{M}_g}$ is the jump process arising from income growth rate and $\mathrm{N}_g(t) \in [0, T] \times \mathbb{R}^{\mathrm{M}_g}$ is a column vector of pure Lévy jump process that explains jumps in the income growth rate with Lévy measure $\lambda_{j,g}\nu_{j,g}(dz_{j,g}), j = 1, 2, ..., \mathrm{M}_g$ such that $z_{j,g}$ are the jump sizes with respect to income growth rate q and $\lambda_{j,g} \geq 0$ are the jump intensities. The measure $\nu_{j,g}$ satisfies $\int_{\mathbb{R}} \min(1, |z_{j,g}|)\nu_{j,g}(dz_{j,g}) < \infty$ and guarantee that variations from $g(t_{-})$ is finite.

We assume that $\Lambda(g(t_{-}))$ and $\mathrm{J}_g(t)\kappa$ are bounded. That is,

$$\forall\Lambda(g(t_{-})) \in [0, T] \times \mathbb{R}, \mathrm{J}_g(t)\kappa \in [0, T] \times \mathbb{R}, |\Lambda(g(t_{-}))| \leq L; |\mathrm{J}_g(t)\kappa| \leq \Theta_g. \tag{12}$$

The dynamics of wealth-before-tax is defined as the difference between assets value and debt process of the investor at time t. Hence, $\mathrm{X}(t_{-}) = \mathrm{G}(t_{-}) - \mathrm{K}(t)$. We now state the following proposition.

Proposition 1. *Let $X(t_{-})$ be the wealth before-tax process of the investor at time t, then the dynamics of $X(t_{-})$ is*

$$\begin{aligned} dX(t_{-}) = \big[&\Delta(t)(\mu(t) - r(t)\kappa) + K(t)(r(t) + g(t_{-}) - r_k(t)) \\ &+ X(t_{-})(r(t) + g(t_{-}) - \alpha(t))\big]dt + \Delta(t)\Sigma(t)dW_S(t) \\ &+ \Delta(t)J(t)dN_S(t), X(0) = x_0. \end{aligned} \tag{13}$$

Proof. By definition, we have that

$$\mathrm{X}(t_{-}) = \mathrm{G}(t_{-}) - \mathrm{K}(t). \tag{14}$$

Taking the differential of both sides of (14) and using equations (6) and (10), we have

$$\begin{aligned} \mathrm{dX}(t_{-}) = &\big(r(t)\mathrm{G}(t_{-}) + \Delta(t)(\mu(t) - r(t)\kappa)\big)dt + \Delta(t)\Sigma(t)\mathrm{dW}_S(t) \\ &+ \Delta(t)\mathrm{J}(t)\mathrm{dN}_S(t) - r_k(t)\mathrm{K}(t)dt - \alpha(t)\mathrm{X}(t_{-})dt + g(t_{-})\mathrm{G}(t_{-})dt. \end{aligned} \tag{15}$$

Using equation (14), we rewrite (15) as follows

$$\begin{aligned} \mathrm{dX}(t_{-}) = \big[&\Delta(t)(\mu(t) - r(t)\kappa) + \mathrm{K}(t)(r(t) + g(t_{-}) - r_k(t)) \\ &+ \mathrm{X}(t_{-})(r(t) + g(t_{-}) - \alpha(t))\big]dt + \Delta(t)\Sigma(t)\mathrm{dW}_S(t) \\ &+ \Delta(t)\mathrm{J}(t)\mathrm{dN}_S(t), \mathrm{X}(0) = x_0. \end{aligned} \tag{16}$$

□

(16) is the wealth-before-tax process of the investor at time t. In the next subsection, we consider the tax dynamics and wealth-after-tax process of the investor at time t.

2.4 The tax and wealth-after-tax process of the investor

Here, we consider the taxation of the investor's investment portfolio at time t. The model assumes the existence of tax payment by the investor and the amount paid by the investor as tax is a product of the tax base and tax rate. The tax base is usually coomposed of the total amount of income, asset value, transactions or other economic activities subject to taxation by the tax authority. It follows that changes in economic, political, environmental factors etc., can cause fluctuations in the tax base. We can therefore model the tax base as a jump-diffusion process. Let $Y_B(t_{-})$ be the tax base at time t. We describe $Y_B(t_{-})$ by the following dynamics

$$dY_B(t_{-}) = Y_B(t_{-})\big(\mu_B(t) + \sigma_B(t)dW_B(t) + J_B(t)dN_B(t)\big), Y_B(0) = y_B, \tag{17}$$

where $\mu_B(t) \in [0,T] \times \mathbb{R}_+$ is the expected growth rate of the tax base at time t, $\sigma_B(t) \in [0,T] \times \mathbb{R}^l$ is the volatility vector associated with tax base at time t, $\mathrm{W}_B(t) = (\mathrm{W}_{B_1}(t), \mathrm{W}_{B_2}(t), \ldots, \mathrm{W}_{B_l}(t))'$ is an l-dimensional Brownian motion, $\mathrm{J}_B(t) \in [0,T] \times \mathbb{R}^{\mathrm{M}_l}$ is the jump scaling vector of tax base, the vector $\mathrm{N}_B(t) = (\mathrm{N}_{B_1}(t), \ldots, \mathrm{N}_{B_l}(t))'$ is a pure Lévy jump process that explains jumps in taxation with Lévy measure $\lambda_{h,Y_B}\nu_{h,Y_B}(dz_{h,Y_B})$, $h = 1, 2, ..., M_l$, such that z_{h,Y_B} is the jump sizes with respect to tax payment, $\lambda_{h,Y_B} \geq 0, h = 1, ..., M_l$ is the jump intensities of tax base. The measure ν_{h,Y_B} which satisfy $\int_{\mathbb{R}} \min(1, |z_{h,Y_B}|)\nu_{h,Y_B}(dz_{h,Y_B}) < \infty$ implies that the jump variation in the tax base is finite.

Hence, tax payment is given by

$$B(t_{-}) = \tau Y_B(t_{-}), \tag{18}$$

where $B(t_-)$ is the amount paid by the investor as tax at time t and τ is the tax rate which we assume to be constant. This implies that the dynamics of tax payment at time t satisfies the following dynamics

$$\frac{dB(t_-)}{B(t_-)} = \mu_B(t)dt + \sigma_B(t)\mathrm{dW}_B(t) + \mathrm{J}_B(t)\mathrm{dN}_B(t), B(0) = B_0. \tag{19}$$

We assume that $\mu_B(t)$ and $\mathrm{J}_B(t)\kappa$ are bounded. That is,

$$\forall \mu_B(t) \in [0,T] \times \mathbb{R}, \mathrm{J}_B(t)\kappa \in [0,T] \times \mathbb{R}, |\mu_B(t)| \leq \Theta_{\mu_B}; |\mathrm{J}_B(t)\kappa| \leq \Theta_{JB}. \tag{20}$$

Let $\bar{X}(t_-)$ be the wealth-after-tax process of the investor at time t. We define $\bar{X}(t)$ as the difference between wealth-before-tax process $\mathrm{X}(t)$ and tax payment $\mathrm{B}(t)$ of the investor at tme t. The following proposition gives the dynamics of the wealth-after-tax process of the investor at time t. From now on we assume that $n = q = l = M_S = M_g = M_l$. Before the proposition, we give the following definition:

Definition 2. *The wealth-after-tax of the investor at time t is defined mathematically as*

$$\bar{X}(t_-) = X(t_-) - B(t_-).$$

Proposition 2. *Let $\bar{X}(t_-)$ be the wealth-after-tax process of the investor at time t, then the dynamics of $\bar{X}(t_-)$ is*

$$\begin{aligned}
&\tfrac{d\bar{X}(t_-)}{\bar{X}(t_-)} = \big(\xi(t)(r(t) + g(t_-) - \mu_B(t) - \alpha(t)) + \pi(t)(\mu(t) - r(t)\kappa) \\
&+\varphi(t)(r(t) + g(t_-) - r_k(t)) + (r(t) + g(t_-) - \alpha(t))\big)dt + \pi(t)\Sigma(t)dW_S(t) \\
&-\xi(t)\sigma_B(t)dW_B(t) + \pi(t)J(t)dN_S(t) - \xi(t)J_B(t)dN_B(t),\ \bar{X}(0) = \bar{x}_0 \in \mathbb{R}_+.
\end{aligned} \tag{21}$$

Proof. By definition, we have that

$$\bar{X}(t_-) = X(t_-) - B(t_-). \tag{22}$$

Taking the differential of bothsides of (22), and using (16) and (19), we have

$$\begin{aligned}
d\bar{X}(t_-) = &\big[\Delta(t)(\mu(t) - r(t)\kappa) + \mathrm{K}(t)(r(t) + g(t_-) - r_k(t)) + \mathrm{X}(t_-)(r(t) + g(t_-) - \alpha(t))\big]dt \\
&+ \Delta(t)\Sigma(t)\mathrm{dW}_S(t) + \Delta(t)\mathrm{J}(t)\mathrm{dN}_S(t) - B(t_-)\mu_B(t)dt - B(t_-)\sigma_B(t)\mathrm{dW}_B(t) \\
&- B(t_-)\mathrm{J}_B(t)\mathrm{dN}_B(t),\ \bar{X}(0) = \bar{x}_0 \in \mathbb{R}_+.
\end{aligned} \tag{23}$$

From (22), we have that $X(t_-) = \bar{X}(t_-) + B(t_-)$ and using it in (23), we have that

$$\begin{aligned} d\bar{X}(t_-) &= \big(B(t_-)(r(t) + g(t_-) - \mu_B(t) - \alpha(t)) + \Delta(t)(\mu(t) - r(t)\kappa) \\ &\quad + K(t)(r(t) + g(t_-) - r_k(t)) + \bar{X}(t_-)(r(t) + g(t_-) - \alpha(t))\big)dt + \Delta(t)\Sigma(t)dW_S(t) \\ &\quad - B(t_-)\sigma_B(t)dW_B(t) + \Delta(t)J(t)dN_S(t) - B(t_-)J_B(t)dN_B(t),\ \bar{X}(0) = \bar{x}_0 \in \mathbb{R}_+. \end{aligned} \tag{24}$$

Let
$\pi(t) = \frac{\Delta(t)}{\bar{X}(t_-)}$ be the portfolio strategy of the investor at time t,
$\varphi(t) = \frac{\mathrm{K}(t)}{\bar{X}(t_-)}$ be the debt ratio of the investor at time t and
$\xi(t) = \frac{\mathrm{B}(t_-)}{\bar{X}(t_-)}$ be the tax ratio of the investor at time t.

It therefore follows that

$$\begin{aligned} \frac{d\bar{X}(t_-)}{\bar{X}(t_-)} &= \big(\xi(t)(r(t) + g(t_-) - \mu_B(t) - \alpha(t)) + \pi(t)(\mu(t) - r(t)\kappa) \\ &\quad +\varphi(t)(r(t) + g(t_-) - r_k(t)) + (r(t) + g(t_-) - \alpha(t))\big)dt + \pi(t)\Sigma(t)dW_S(t) \\ &\quad -\xi(t)\sigma_B(t)dW_B(t) + \pi(t)J(t)dN_S(t) - \xi(t)J_B(t)dN_B(t),\ \bar{X}(0) = \bar{x}_0 \in \mathbb{R}_+. \end{aligned} \tag{25}$$

□

We now define our admissible strategies of the investor. For admissible portfolio startegies $\pi(t)$, $t \in (0, \infty)$, we have

$$E \int_0^\infty \pi(t)\pi'(t)dt < \infty. \tag{26}$$

For the debt ratio $\varphi(t)$ and the consumption rate $\alpha(t)$, we assume that they are non-negatives and are bounded above with upper bounds H_1 and H_2, respectively such that $0 \le \varphi(t) \le H_1 < \infty$ and $0 \le \alpha(t) \le H_2 < \infty$.

A strategy $u(\cdot) = \{(\pi(t), \varphi(t), \alpha(t)) : t \ge 0\}$ that is progressively measurable with respect to $\{W_S(t), W_B(t), W_g(t) : t \ge 0\}$ is called an admissible strategy. Let $\mathcal{A}$ be the collection of all admissible control strategies, then we have that $\mathcal{A}$ can be defined as $\{u(\cdot) = (\pi(t), \varphi(t), \alpha(t)) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} : E\int_0^\infty \pi(t)\pi(t)'dt < \infty; 0 \le \varphi(t) \le H_1 < \infty; 0 \le \alpha(t) \le H_2 < \infty, t \ge 0\}$.

3 Optimal Policies

In this section, we consider an investor whose motive is to choose optimal portfolio strategies, debt ratio and consumption rate that will maximize the expected discounted utility of consumption and debt in an infinite time horizon. For an arbirtary admissible strategy $u(\cdot) = (\pi(t), \varphi(t), \alpha(t)) : t \ge 0)$, the objective function $\mathrm{F}(t, \bar{x}, g)$ follows:

$$\mathrm{F}(t, \bar{x}, g) = \mathrm{E}_{\bar{x},g} \int_t^{\infty} e^{-\delta s} \Big(\mathrm{V}(\alpha(s)\bar{x}(s)) + \mathrm{V}^k(\varphi(s)\bar{x}(s)) \Big) ds, \tag{27}$$

where $\mathrm{V}(\alpha(s)x(s))$ is utility of consumption rate, $\mathrm{V}^k(\varphi(s)\bar{x}(s))$ is utility of debt ratio, $0 \leq \delta < 1$ is the discount rate and $\bar{X}(t_-) = \bar{x}, g(t_-) = g$.

We define the value function as

$$\mathrm{H}(t, \bar{x}, g) := \sup_{u \in \mathcal{A}} \Big[\mathrm{F}(t, \bar{x}, g) | \bar{X}(t_-) = \bar{x}, g(t_-) = g \Big]. \tag{28}$$

The stochastic control problem is solved by adopting the dynamic programming approach to obtain the Hamilton-Jacobi-Bellman (HJB) equation of our problem. We shall now omit the variable t in the functionals for convenience. The HJB equation characterizing the optimal solution is given as follows:

$$\begin{aligned}
0 &= \mathrm{H}_t + \bar{x}(\xi(r + g - \alpha - \mu_B) + \pi(\mu - r\kappa) + \varphi(r + g - r_k) \\
&+(r + g - \alpha))\mathrm{H}_{\bar{x}} + (\Lambda(g) + g\tilde{\eta}(\omega))\mathrm{H}_g + \tfrac{1}{2}\bar{x}^2(\pi\Sigma)(\pi\Sigma)'\mathrm{H}_{\bar{x}\bar{x}} + \tfrac{1}{2}\bar{x}^2\xi^2\sigma_B\sigma_B'\mathrm{H}_{\bar{x}\bar{x}} \\
&+\tfrac{1}{2}\sigma_g\sigma_g'\mathrm{H}_{gg} - \bar{x}^2\rho\pi\Sigma\sigma_B'\xi\mathrm{H}_{\bar{x}\bar{x}} + \rho_1\bar{x}\pi\Sigma\sigma_g'\mathrm{H}_{\bar{x}g} \\
&+e^{-\delta}V^k(\varphi\bar{x}) + e^{-\delta}V(\alpha\bar{x}) + \int_{\mathbb{R}} [\mathrm{H}(t, \bar{x} + \bar{x}\pi J\kappa, g) - \mathrm{H}(t, \bar{x}, g)]\lambda_S\nu_S(dz_S) \\
&+\int_{\mathbb{R}} [\mathrm{H}(t, \bar{x} - \bar{x}\xi J_B\kappa, g) - \mathrm{H}(\bar{x}, g)]\lambda_g\nu_g(dz_g) + \int_{\mathbb{R}} [\mathrm{H}(t, \bar{x}, g + J_g\kappa) - \mathrm{H}(\bar{x}, g)]\lambda_g\nu_g(dz_g)
\end{aligned} \tag{29}$$

with transvasility condition $\lim_{t\to\infty} \mathrm{E}[\mathrm{H}(t, \bar{x}, \varphi, g)] = 0$, see [15], where ρ is the correlation coefficient between stock and taxation and ρ_1 is the correlation between stock market and income growth rate.

The infinite time-horizon problem is given as

$$\begin{aligned}
\mathrm{H}(t, \bar{x}, g) &= \sup_{\{\pi(s), \varphi(s), \alpha(s)\}_{t \leq s \leq \infty}} \mathrm{E}_t \int_t^{\infty} \Big[e^{-\delta(s-t)} \Big(V(\alpha(s)\bar{x}(s)) + \mathrm{V}^k(\varphi(s)\bar{x}(s)) \Big) \Big] ds \\
&= \mathrm{U}(t, \bar{x}, g)e^{-\delta t}.
\end{aligned} \tag{30}$$

Hence, we have

$$\begin{aligned}
0 &= \mathrm{U}_t + \bar{x}(\xi(r + g - \alpha - \mu_B) + \pi(\mu - r\kappa) + \varphi(r + g - r_k) \\
&+(r + g - \alpha))\mathrm{U}_{\bar{x}} + (\Lambda(g) + g\tilde{\eta}(\omega))\mathrm{U}_g + \tfrac{1}{2}\bar{x}^2(\pi\Sigma)(\pi\Sigma)'\mathrm{U}_{\bar{x}\bar{x}} + \tfrac{1}{2}\bar{x}^2\xi^2\sigma_B\sigma_B'\mathrm{U}_{\bar{x}\bar{x}} \\
&+\tfrac{1}{2}\sigma_g\sigma_g'\mathrm{U}_{gg} - \bar{x}^2\rho\pi\Sigma\sigma_B'\xi\mathrm{U}_{\bar{x}\bar{x}} + \rho_1\bar{x}\pi\Sigma\sigma_g'\mathrm{U}_{\bar{x}g} \\
&+V^k(\varphi\bar{x}) + V(\alpha\bar{x}) - 2\delta U + \int_{\mathbb{R}} [\mathrm{U}(t, \bar{x} + \bar{x}\pi J\kappa, g) - \mathrm{U}(t, \bar{x}, g)]\lambda_S\nu_S(dz_S) \\
&+\int_{\mathbb{R}} [\mathrm{U}(t, \bar{x} - \bar{x}\xi J_B\kappa, g) - \mathrm{U}(\bar{x}, g)]\lambda_g\nu_g(dz_g) + \int_{\mathbb{R}} [\mathrm{U}(t, \bar{x}, g + J_g\kappa) - \mathrm{U}(\bar{x}, g)]\lambda_g\nu_g(dz_g).
\end{aligned} \tag{31}$$

Let $\mathrm{V}(\alpha\bar{x}) = \frac{(\alpha\bar{x})^{1-\gamma}}{1-\gamma}$ and $\mathrm{V}^k(\varphi\bar{x}) = \frac{(\varphi\bar{x})^{\phi}}{\phi}$ be the utility functions of the investor with respect to consumption rate and debt ratio for $\gamma > 0$, $\phi > 1$ such that $\gamma \in (0, 1) \cup (1, \infty)$ and $\phi \in (1, \infty)$, where γ is the constant relative risk aversion parameter with respect to consumption and ϕ is the coefficient of the investor willingness to bear the risk of taking debt.

To solve (31), we conjecture a solution $\bar{\mathrm{U}}(t,\bar{x},g)$ of the form

$$\mathrm{U}(t,\bar{x},g) := \frac{\bar{x}^{1-\gamma}}{1-\gamma}e^{h(t,g)}. \tag{32}$$

It then follows that by differentiating (32) with respect to $t, \bar{x}, \bar{x}\bar{x}, g, gg$ and $\bar{x}g$, we have the following:

$$\begin{aligned}
&\mathrm{U}_{\bar{x}} = \bar{x}^{-1}(1-\gamma)\mathrm{U}(t,\bar{x},g), \mathrm{U}_{\bar{x}\bar{x}} = \bar{x}^{-2}\gamma(1-\gamma)\mathrm{U}(t,\bar{x},g),\\
&\mathrm{U}_g = h_g\mathrm{U}(t,\bar{x},g), \mathrm{U}_{gg} = (h_g^2 + h_{gg})\mathrm{U}(t,\bar{x},g),\\
&\mathrm{U}_{\bar{x}g} = h_g(1-\gamma)\bar{x}^{-1}\mathrm{U}(t,\bar{x},g), \bar{\mathrm{U}}_t = h_t\mathrm{U}(t,\bar{x},g).
\end{aligned} \tag{33}$$

Substituting (33) into (31), and then divide through by $(1-\gamma)\mathrm{U}$, we have the following partial differential equation:

$$\begin{aligned}
0 = &\tfrac{h_t}{1-\gamma} + \xi(r+g-\alpha-\mu_B) + \pi(\mu - r\kappa) + \varphi(r+g-r_k) + (r+g-\alpha)\\
&+(\Lambda(g)+g\tilde{\eta}(\omega))\tfrac{h_g}{(1-\gamma)} - \tfrac{1}{2}(\pi\Sigma)(\pi\Sigma)'\gamma - \tfrac{1}{2}\xi^2\sigma_B\sigma_B'\gamma + \rho\pi\Sigma\sigma_B'\xi\gamma\\
&+\tfrac{1}{2(1-\gamma)}\sigma_g\sigma_g'(h_g^2+h_{gg}) - \rho_1\pi\Sigma\sigma_g' h_g + \tfrac{\lambda_S}{(1-\gamma)}\int_{\mathbb{R}}[(1+\pi J\kappa)^{1-\gamma}-1]\nu_S(dz_S)\\
&+\tfrac{\varphi^\phi}{\phi}\bar{x}^{\phi+\gamma-1}e^{-\delta-h(t,g)} + \tfrac{\lambda_g}{(1-\gamma)}\int_{\mathbb{R}}[(e^{J_g\kappa}-1]\nu_g(dz_g) + \frac{\alpha^{1-\gamma}}{1-\gamma}e^{-h(t,g)}\\
&-\tfrac{2\delta}{1-\gamma} + \tfrac{\lambda_B}{(1-\gamma)}\int_{\mathbb{R}}[(1-\xi J_B\kappa)^{1-\gamma}-1]\nu_B(dz_B).
\end{aligned} \tag{34}$$

(34) is characterized by the following differential equations:

$$(r+g-r_k)\varphi + \tfrac{\varphi^\phi}{\phi}\bar{x}^{\phi+\gamma-1}e^{-\delta-h(t,g)} = 0, \tag{35}$$

$$-\alpha(1+\xi) + \tfrac{\alpha^{1-\gamma}}{1-\gamma}e^{-h(t,g)} = 0, \tag{36}$$

$$h_t + (1-\gamma)h_g\beta_1 + \tfrac{1}{2}\sigma_g\sigma_g'(h_g^2+h_{gg}) + \beta_2(1-\gamma) = 0, \tag{37}$$

where $\beta_1 = \frac{\Lambda(g)+g\tilde{\eta}(\omega)}{1-\gamma} - \rho_1\pi\Sigma\sigma_g'$, $\beta_2 = \xi(r+g-\mu_B) + \pi(\mu-r\kappa) + (r+g-\alpha) - \frac{1}{2}(\pi\Sigma)(\pi\Sigma)'\gamma - \frac{1}{2}\xi^2\sigma_B\sigma_B'\gamma + \rho\pi\Sigma\sigma_B'\xi\gamma + \frac{\lambda_S}{(1-\gamma)}\int_{\mathbb{R}}[(1+\pi J\kappa)^{1-\gamma}-1]\nu_S(dz_S) + \dfrac{\lambda_g}{(1-\gamma)}\int_{\mathbb{R}}[(e^{J_g\kappa}-1]\nu_g(dz_g) - \dfrac{2\delta}{1-\gamma} + \dfrac{\lambda_B}{(1-\gamma)}\int_{\mathbb{R}}[(1-\xi J_B\kappa)^{1-\gamma}-1]\nu_B(dz_B)$.

3.1 Optimal debt ratio

In this subsection, we are interested in the optimal debt ratio of the investor. We now give the following definition.

Definition 3. *The optimal policy for the debt ratio φ^* of the investor is defined as*

$$\varphi^* = arg\max_{\varphi} f(\varphi),$$

where $f(\varphi) = (r + g - r_k)\varphi + \frac{\varphi^\phi}{\phi}\bar{x}^{\phi+\gamma-1}e^{-\delta-h(t,g)}$ is a concave function.

Proposition 3. *The optimal debt rate φ^* of the investor is*

$$\varphi^* = \left[(r_k - r - g)\bar{x}^{1-\phi-\gamma}e^{\delta+h(t,g)}\right]^{\frac{1}{\phi-1}}. \tag{38}$$

Proof. By first principle for optimal debt ratio, we have

$$\frac{\partial f}{\partial \varphi} = (r + g - r_k) + \varphi^{1-\phi}\bar{x}^{\phi+\gamma-1}e^{-\delta-h(t,g)} = 0. \tag{39}$$

Making φ^* the subject of formular, we have

$$\varphi^* = \left[(r_k - r - g)\bar{x}^{1-\phi-\gamma}e^{\delta+h(t,g)}\right]^{\frac{1}{\phi-1}}. \tag{40}$$

□

From (40), we found that the optimal debt ratio depend on interest rate on loan r_k, the norminal interest rate r, income growth rate g, coefficient of risk aversion with respect to debt ϕ, coefficient of risk aversion with respect to consumption γ, discount rate δ as well as the functional $h(t, g)$. Interestingly, this shows that before loan is taken or given the following must be considered: interest rate on loan to be taken or given, the norminal interest rate, income growth rate of the investor, discount rate, the functional $h(t, g)$, the coefficient of risk aversion with respect to debt and consumption of the investor.

It is also observed that for the optimal debt ratio to be positive, $r_k > (r + g)$. The optimal debt ratio will be negative, if $r_k < (r + g)$, and zero, if $r_k = (r + g)$, all other parameters remain fixed. It is interesting to see that, if the income growth rate of the investor increases, the debt profile of the investor decreases, all other parameters remain fixed.

We further observe from (40) that as the coefficient of risk aversion with respect to debt ratio tends to unity i.e $\phi \to 1$, for all parameters remain fixed, the optimal debt ratio, φ^* will tends to infinity. This simply implies that the amount of debt will be unbearable and no investor will be interested in taking loan. Hence, the condition that $\phi > 1$ must hold. If $\phi = 2$, then $\varphi^* = \left[(r_k - r - g)\bar{x}^{-(1+\gamma)}e^{\delta+h(t,g)}\right]$. If $\phi = 3$, then $\varphi^* = \left[(r_k - r - g)\bar{x}^{-(2+\gamma)}e^{\delta+h(t,g)}\right]^{\frac{1}{2}}$. Hence, if $\phi = N$, then $\varphi^* = \left[(r_k - r - g)\bar{x}^{-(N+\gamma)}e^{\delta+h(t,g)}\right]^{\frac{1}{N-1}}$. Therefore, if $\phi = \infty$, then φ^* will tends to zero. This shows that as the coefficient risk aversion with respect to debt becomes increasingly large, the optimal debt ratio of the investor will tend to zero over time.

Observe that the optimal debt ratio is independent of the tax ratio. This implies that the debt ratio of the investor is not affected by taxation. In other words, an increase or decrease in taxation will not impact the decision of the investor as to what levels of debt it should incur to grow his/her portfolio.

But, the change in optimal debt ratio with respect to interest rate on loan is obtain as follows

$$\frac{\partial\varphi^*}{\partial r_k} = \frac{\left[(r_k - r - g)\bar{x}^{1-\phi+\gamma}\right]^{\frac{2-\phi}{\phi-1}}}{\bar{x}^{\phi+\gamma-1}(1-\phi)} e^{\frac{1}{\phi-1}(\delta+h(t,g))}. \tag{41}$$

The change in optimal debt ratio with respect to nominal interest rate is obtain as follows

$$\frac{\partial\varphi^*}{\partial r} = \frac{\left[(r_k - r - g)\bar{x}^{1-\phi+\gamma}\right]^{\frac{2-\phi}{\phi-1}}}{\bar{x}^{\phi+\gamma-1}(\phi-1)} e^{\frac{(3-2\phi)}{\phi-1}(\delta+h(t,g))}. \tag{42}$$

The change in optimal debt ratio with respect to income growth rate is obtain as follows

$$\frac{\partial\varphi^*}{\partial g} = \frac{\left[(r_k - r - g)\bar{x}^{1-\phi+\gamma}\right]^{\frac{2-\phi}{\phi-1}}}{\bar{x}^{\phi+\gamma-1}(\phi-1)}(1 + gh_g(t,g)) e^{\frac{1}{\phi-1}(\delta+h(t,g))}. \tag{43}$$

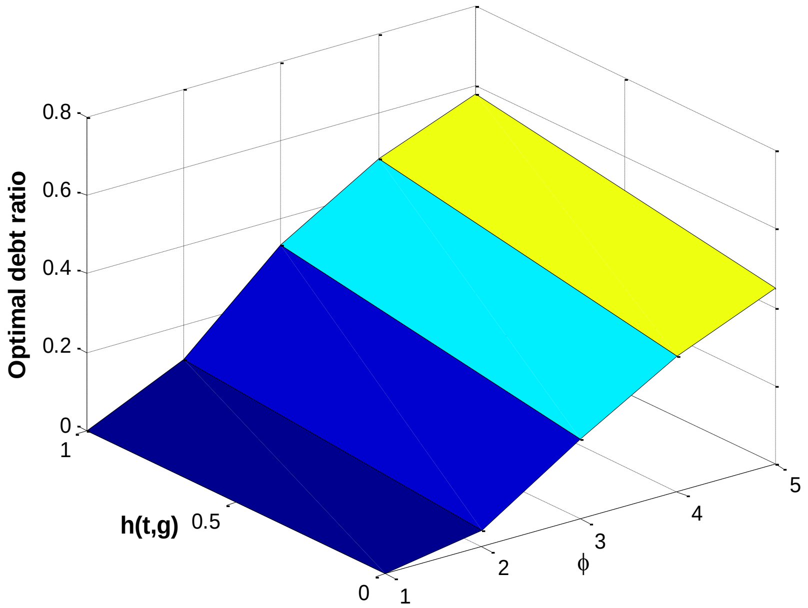

Figure 1: Optimal debt ratio plotted againist ϕ and $h(t,g)$ for $r_k = 0.25$, $r = 0.15$, $\gamma = 0.05$, $\delta = 0.03$ and $g = 0.06$.

Figure 1 shows the change in optimal debt ratio as the values of ϕ varies from 1 to 5 and $h(t,g)$ varies from 0 to 1. Figure 2 shows the change in optimal debt ratio as the values of ϕ varies from 1 to 10 and $h(t,g)$ varies from 0 to 1. Figure 3 shows the change in optimal debt ratio as the values of ϕ varies from 1 to 20 and $h(t,g)$ varies from 0 to 1. Observe that as ϕ becomes so large the optimal debt ratio will be very close to one. Hence, we have that as ϕ increases the optimal debt ratio increases and vice versa.

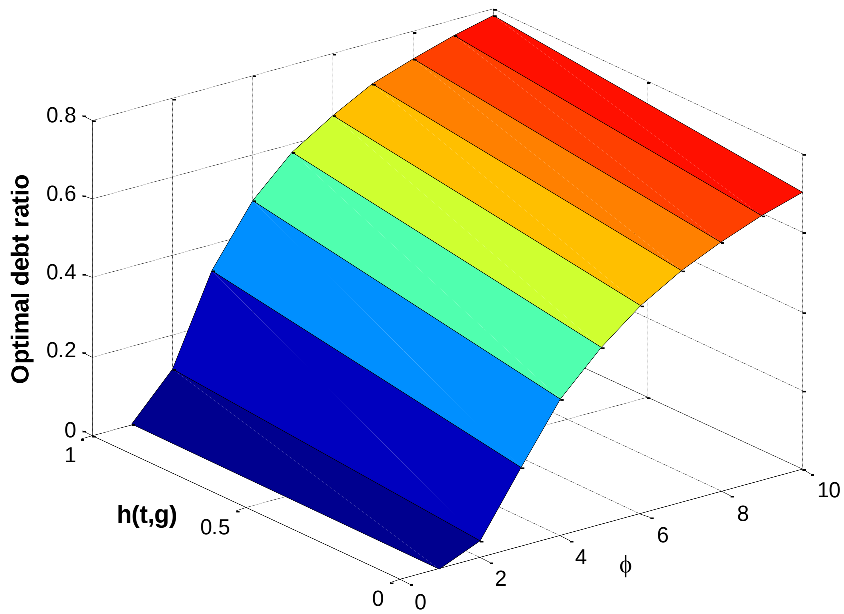

Figure 2: Optimal debt ratio plotted againist ϕ and $h(t,g)$ for $r_k = 0.25$, $r = 0.15$, $\gamma = 0.05$, $\delta = 0.03$ and $g = 0.06$.

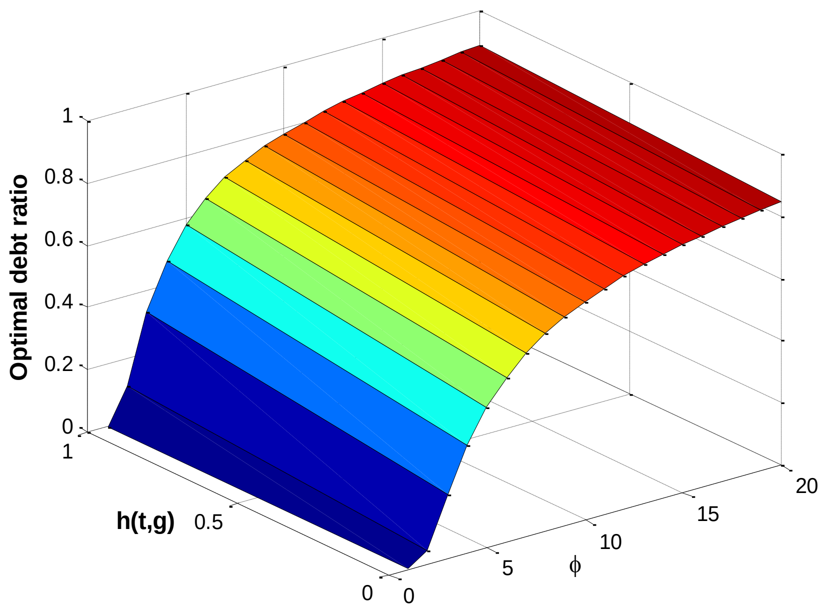

Figure 3: Optimal debt ratio plotted againist ϕ and $h(t,g)$ for $r_k = 0.25$, $r = 0.15$, $\gamma = 0.05$, $\delta = 0.03$ and $g = 0.06$.

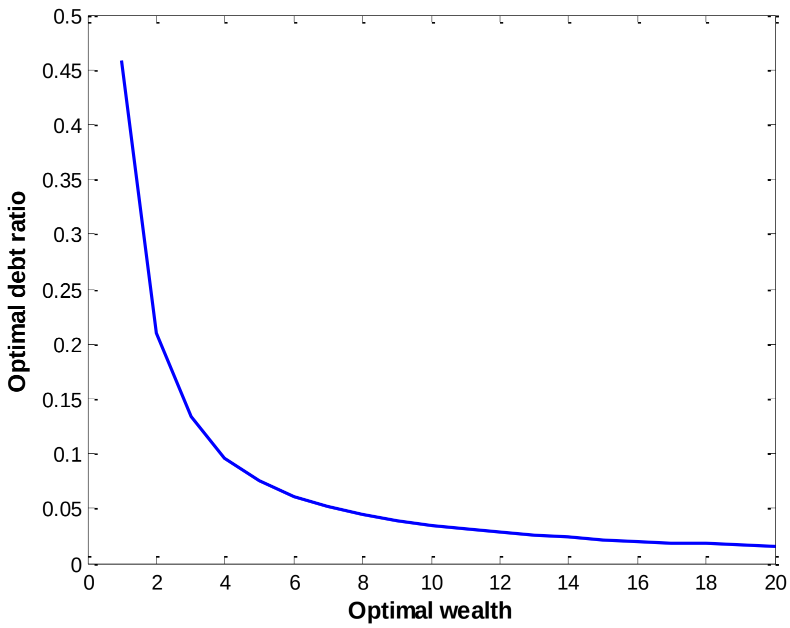

Figure 4: Optimal debt ratio plotted againist optimal wealth for $r_k = 0.25$, $r = 0.15$, $\gamma = 0.05$, $\delta = 0.03$, $\phi = 5$ and $g = 0.06$.

Figure 4 shows the change in optimal debt ratio as the values of optimal wealth changes. Observe that the optimal debt ratio is inversely related to optimal wealth.

3.2 Optimal consumption

In this subsection, we consider the optimal consumption rate at time t.

Definition 4. *The optimal policy for the consumption rate α of the investor is defined as*

$$\alpha^* = arg\max_{\alpha} f_1(\alpha),$$

where $f_1(\alpha) = -\alpha(1+\xi) + \frac{\alpha^{1-\gamma}}{1-\gamma} e^{-h(t,g)}$ is a concave function.

Proposition 4. *The optimal consumption rate α^* of the investor is*

$$\alpha^* = \left((1+\xi)e^{h(t,g)}\right)^{-\frac{1}{\gamma}}. \tag{44}$$

Proof. By the principles of the first order conditions, we have that

$$\frac{\partial f_1}{\partial \alpha} = -(1+\xi) + \alpha^{-\gamma} e^{-h(t,g)} = 0. \tag{45}$$

It then follows that

$$\alpha^* = \left((1+\xi)e^{h(t,g)}\right)^{-\frac{1}{\gamma}}. \tag{46}$$

□

Proposition 4 shows the optimal consumption rate of the investor at time t. It is observe that the optimal consumption rate is a function of the coefficient of risk aversion with respect to investment strategies, tax rate and income growth rate. This result is very interesting as it gives a very fundamental result: when tax rate increases, consumption rate decreases and vice versa, for all other parameters remain the same. This shows that tax rate can be use as a control measure for the consumption rate of goods and services in an investment portfolio.

We also observe that with $h(t,g) > 0$ and ξ fixed, consumption rate will decrease as γ increases and increase as γ decreases. Notice also that as $h(t,g) \to \infty$, for all other parameters remain fixed, $\alpha^* \to 0$. This implies that if $h(t,g)$ is large, optimal consumption rate will reduce drastically. If $h(t,g) \to 0$, optimal consumption rate will tends to $\left(1+\xi\right)^{-\frac{1}{\gamma}}$. This implies that as $h(t,g) \to 0$, optimal consumption rate will depend on tax rate and the coefficient of risk aversion with respect to investment strategies.

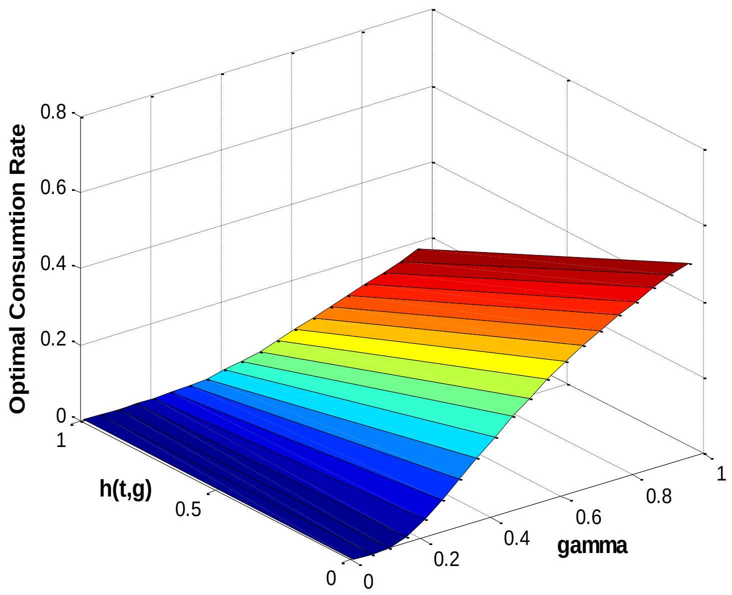

Figure 5: Effects of $h(t,g)$ and γ on optimal consumption rate for $\xi = 0.9$.

Figure 5 shows the optimal consumption rate given that $\xi = 0.9$. We observe that as $h(t,g)$ and γ varies from 0 to 1, the optimal consumption rate increases and vice versa. Figure 6 shows the effect of the coefficient of risk aversion with respect to wealth-after-tax, γ and $h(t,g)$ on the optimal consumption

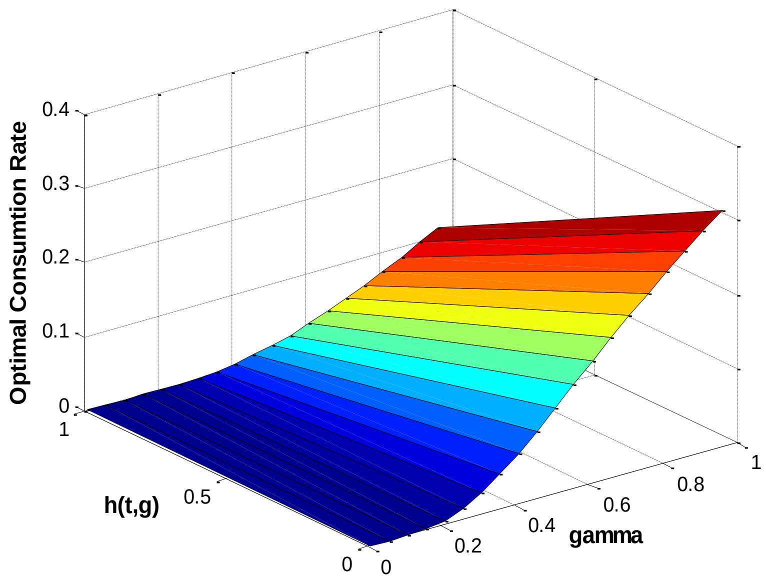

Figure 6: Effects of $h(t,g)$ and γ on optimal consumption rate for $\xi = 2$.

rate of the investor, for the value of $\xi = 2$. We observe that as the investor becomes more risk averse, the fraction of wealth-after-tax spent on consumption increases and vice versa. We found also that for $\xi = 0.9$ and $\xi = 2.0$ respectively, for all other parameters remain fixed, α^* is approximately 18% and 12% for increasing values of $h(t,g)$ and γ. This indicates that with an increasing tax ratio, the optimal consumption rate will decrease and vice versa.

Proposition 5. *The optimal consumption rate α^* changes negatively with taxation provided $\xi > 0$.*

Proof. From (46), we have that

$$\frac{\partial \alpha^*}{\partial \xi} = -\frac{1}{\gamma}\big(1+\xi\big)^{-\frac{1+\gamma}{\gamma}} e^{-\frac{h(t,g)}{\gamma}}. \tag{47}$$

It then follows that $\frac{\partial \alpha^*}{\partial \xi} < 0$, for all $\xi > 0$. □

This indicates that change in the optimal consumption rate with respect to change in the tax rate of the investor, will result to a drastic fall in consumption rate. In other words, consumption rate reacts negatively to an increase in taxation.

3.3 Optimal portfolio strategies

In this section, we presents the portfolio strategies of the investor.

Definition 5. *The optimal portfolio strategy* π^* *of the investor is defined as*

$$\pi^* = arg\max_{\pi} f_2(\pi),$$

where the functions $f_2(\pi) = \pi(\mu - r\kappa) - \frac{1}{2}\pi\pi'\Sigma\Sigma'\gamma + \rho\pi\Sigma\sigma_B'\xi\gamma - \rho_1\pi\Sigma\sigma_g'h_g + \Phi(\pi J\kappa)$ *and* $\Phi(\pi J\kappa) = -\frac{\lambda_S}{(1-\gamma)}\int_{\mathbb{R}}[1-(1+\pi J\kappa)^{1-\gamma}]\nu_S(dz_S)$ *are both concave.*

Proposition 6. *Let* π^* *be the optimal portfolio strategy of the investor, then*

$$\pi'^* = \frac{1}{\gamma}\left(1 - \frac{(\Sigma\Sigma')^{-1}J\kappa}{(J\kappa)'(\Sigma\Sigma')^{-1}J\kappa}J\kappa\right)\Bigg\{(\Sigma\Sigma')^{-1}(\mu - r\kappa) + \rho\xi\gamma(\Sigma\Sigma')^{-1}\Sigma\sigma_B' - h_g\rho_1(\Sigma\Sigma')^{-1}\Sigma\sigma_g'\Bigg\} + \frac{\bar{a}(\Sigma\Sigma')^{-1}J\kappa}{(J\kappa)'(\Sigma\Sigma')^{-1}J\kappa}. \tag{48}$$

Proof. By first order condition for the portfolio process, we have that

$$\frac{\partial f_2(\pi)}{\partial \pi} = (\mu - r\kappa) - \pi'\Sigma\Sigma'\gamma + \rho\Sigma\sigma_B'\xi\gamma - \rho_1\Sigma\sigma_g'h_g + M_S\lambda_S J\kappa\dot{\Phi}(\pi J\kappa). \tag{49}$$

Let $\bar{a} = \pi J\kappa$ be a scalar such that multiplying (49) by $\frac{1}{\gamma}(J\kappa)'(\Sigma\Sigma')^{-1}$, we have that $\bar{a}$ must satisfy

$$\frac{1}{\gamma}(J\kappa)'(\Sigma\Sigma')^{-1}(\mu - r\kappa) - \bar{a} + \rho\xi(J\kappa)'(\Sigma\Sigma')^{-1}\Sigma\sigma_B' - h_g\frac{\rho_1}{\gamma}(J\kappa)'(\Sigma\Sigma')^{-1}\Sigma\sigma_g' + \frac{M_S}{\gamma}\lambda_S(J\kappa)'(\Sigma\Sigma')^{-1}J\kappa\dot{\Phi}(\bar{a}) = 0. \tag{50}$$

From (49), we solve for π'. It follows that

$$\pi'^* = \frac{1}{\gamma}(\Sigma\Sigma')^{-1}(\mu - r\kappa) + \rho\xi(\Sigma\Sigma')^{-1}\Sigma\sigma_B' - h_g\frac{\rho_1}{\gamma}(\Sigma\Sigma')^{-1}\Sigma\sigma_g' + \frac{M_S\lambda_S}{\gamma}(\Sigma\Sigma')^{-1}J\kappa\dot{\Phi}(\bar{a}). \tag{51}$$

We have that

$$\dot{\Phi}(\bar{a}) = \frac{\gamma}{M_S\lambda_S(J\kappa)'(\Sigma\Sigma')^{-1}J\kappa}\Big(-\frac{1}{\gamma}(J\kappa)'(\Sigma\Sigma')^{-1}(\mu - r\kappa) + \bar{a} - \rho\xi(J\kappa)'(\Sigma\Sigma')^{-1}\Sigma\sigma_B' + h_g\frac{\rho_1}{\gamma}(J\kappa)'(\Sigma\Sigma')^{-1}\Sigma\sigma_g'\Big) \tag{52}$$

Substituting (52) into (51), we have

$$\pi'^* = \frac{1}{\gamma}\left(1 - \frac{(\Sigma\Sigma')^{-1}J\kappa}{(J\kappa)'(\Sigma\Sigma')^{-1}J\kappa}J\kappa\right)\Bigg\{(\Sigma\Sigma')^{-1}(\mu - r\kappa) + \rho\xi\gamma(\Sigma\Sigma')^{-1}\Sigma\sigma_B' - h_g\rho_1(\Sigma\Sigma')^{-1}\Sigma\sigma_g'\Bigg\} + \frac{\bar{a}(\Sigma\Sigma')^{-1}J\kappa}{(J\kappa)'(\Sigma\Sigma')^{-1}J\kappa}. \tag{53}$$

□

We observe that it is optimal for the investor to have a portfolio with the following components: a Merton portfolio with jump, a jump risk hedging portfolio, a portfolio component that hedges the risks in the risky assets and a component $\bar{a}$ that determines how much of a risk-free portfolio value the investor holds only in the riskless asset.

The optimal portfolio of the investor can be wriiten as

$$\begin{aligned}\pi^* &= \underbrace{\frac{1}{\gamma}\left(1-\frac{(\Sigma\Sigma')^{-1}J\kappa}{(J\kappa)'(\Sigma\Sigma')^{-1}J\kappa}J\kappa\right)(\Sigma\Sigma')^{-1}(\mu-r\kappa)}_{\zeta_1}\\ &+\underbrace{\left(1-\frac{(\Sigma\Sigma')^{-1}J\kappa}{(J\kappa)'(\Sigma\Sigma')^{-1}J\kappa}J\kappa\right)\rho\xi(\Sigma\Sigma')^{-1}\Sigma\sigma_B'}_{\zeta_2}\\ &-\underbrace{\frac{1}{\gamma}\left(1-\frac{(\Sigma\Sigma')^{-1}J\kappa}{(J\kappa)'(\Sigma\Sigma')^{-1}J\kappa}J\kappa\right)h_g\rho_1(\Sigma\Sigma')^{-1}\Sigma\sigma_g'}_{\zeta_3}+\underbrace{\frac{\bar{a}(\Sigma\Sigma')^{-1}J\kappa}{(J\kappa)'(\Sigma\Sigma')^{-1}J\kappa}}_{\zeta_4}.\end{aligned} \tag{54}$$

From (54), we observe that it is optimal for the investor to invest in an investment portfolio that has the following components:

1. a speculative portfolio ζ_1 proportional to the market price of risk with respect to the risky assets and the inverse of relative risk aversion coefficient, $\frac{1}{\gamma}$,
2. a tax rate hedging portfolio strategy ζ_2 proportional to the diffusion term of tax payment and tax rate through the coefficient of correlation between taxes and risky assets,
3. an income growth rate hedging portfolio strategy ζ_3 proportional to the volatility of the income growth process, the coefficient of correlation between risky assets and income growth rate through the cross derivative of functional h with respect to the income growth rate g and the inverse of relative risk aversion coefficient with respect to investment process $\frac{1}{\gamma}$,
4. the component ζ_4 of the risky assets shows how much of this portfolio the investor is willing to take and is controlled by the quantity $\bar{a}$, which is a risk-free fund that holds only the riskless asset.

It is observed that if $\frac{(\Sigma\Sigma')^{-1}J\kappa}{(J\kappa)'(\Sigma\Sigma')^{-1}J\kappa}J\kappa < 1$, $\xi > 0$ and $\rho(\Sigma\Sigma')^{-1}\Sigma\sigma_B' < 0\kappa$, tax rate will pose a serious risk on the portfolio by reducing the optimal portfolio value of the investor.

3.4 Optimal debt and optimal tax payment of the investor

Here, we consider the optimal debt and the optimal tax payment of the investor at time t. First, we consider the optimal debt of the investor at time t. It is defined as the product of the optimal debt ratio and the optimal wealth-after-tax process of the investor at time t. That is, $\varphi^*(t) = \frac{K^*(t)}{\bar{X}^*(t)}$. Hence, we have from (25) and (40) that

$$K^*(t) = \bar{X}^*(t)\varphi^*(t) = \bar{X}^*(t)\big[(r_k(t) - r(t) - g(t))e^{h(t,g)}\big]^{\frac{1}{\phi-1}}, \tag{55}$$

where we obtain the optimal wealth of the investor at time t to be

$$\begin{aligned}\bar{X}^*(t) = \bar{x}_0 \exp\{&\textstyle\int_0^T [\big(\xi(t)(r(t) + g(t) - \mu_B(t) - \alpha^*(t)) + \pi^*(t)(\mu(t) - r(t)\kappa) \\ &+\varphi^*(t)(r(t) + g(t) - r_k(t)) + (r(t) + g(t) - \alpha^*(t))\big)]dt \\ &+ \textstyle\int_0^T [\pi^*(t)\Sigma(t)dW_S(t) - \xi(t)\sigma_B(t)dW_B(t)] \\ &+ \textstyle\int_0^T [\pi^*(t)J(t)dN_S(t) - \xi(t)J_B(t)dN_B(t)]\}.\end{aligned} \tag{56}$$

(55) represents that optimal debt accrued to the investor at time t. It is observe that if ϕ tends to infinity, the entire wealth-after-tax will be equal to the optimal debt of the investor at time t. It simply implies that the investor only traded for the creditor over time. Hence, the investor should ensure that ϕ is small as possible, but must be greater than one.

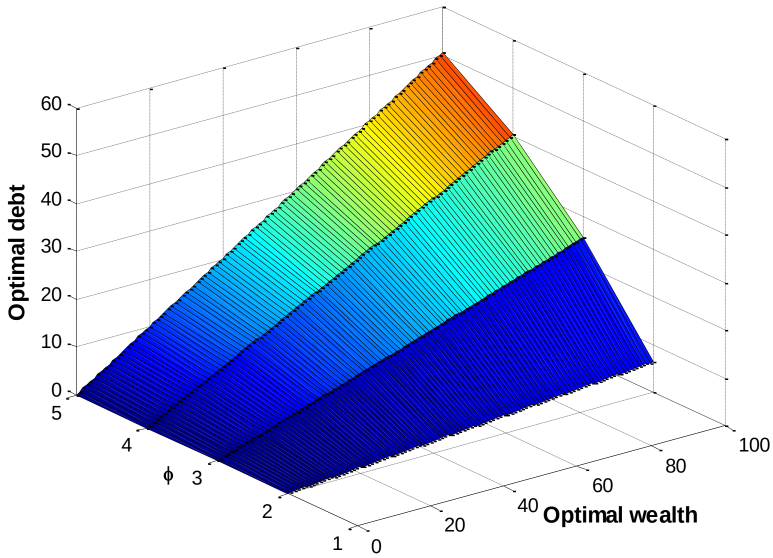

Figure 7: Optimal Debt of the Investor for $r_k = 0.25$, $r = 0.15$, $g = 0.06$ and $h(t, g) = 0.5$

From figure 7, we found that, for the optimal wealth varying from 0 to 100 and ϕ varying from 1 to 5, the optimal debt of the investor varies from 0 to about 55. From figure 8, we observe that as the optimal wealth varies from 0 to 100 and ϕ varies from 1 to 20, the optimal debt of the investor varies from 0 to 86.6669. Also, as the optimal wealth varies from 0 to 100 and ϕ varies from 1 to 10000, the optimal debt of the investor varies from 0 to 99.9728. Further, as the optimal wealth varies from 0 to 100 and ϕ varies from 1 to 100000, the optimal debt of the investor varies from 0 to 99.9973. This implies that as ϕ tends to infinity, the optimal debt of the investor will tends to the optimal wealth-after-tax of the investor.

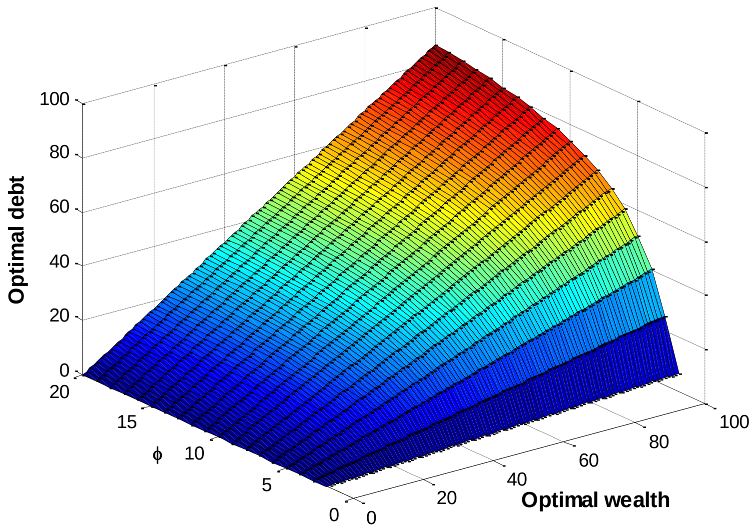

Figure 8: Optimal Debt of the Investor for $r_k = 0.25$, $r = 0.15$, $g = 0.06$ and $h(t, g) = 0.5$.

Next, we consider the optimal tax payment of the investor at time t. It the product of the tax rate and the optimal wealth-after-tax of the investor at time t. That is, $B^*(t) = \xi(t)\bar{X}^*(t)$. It then follows from (46) that

$$B^*(t) = (\alpha^{*-\gamma}e^{-h(t,g)} - 1)\bar{X}^*(t). \tag{57}$$

Figure 9 shows the tax payment of the investor for three different values of optimal consumption rate over time. It is observe that as the consumption rate increases, the tax payment by the investor decreases. Hence, the higher the consumption rate, the lower the tax payment of the investor over time.

3.5 Explicit form for the parameter $\bar{a}$

In this subsection, we obtain the explicit expression for the parameter $\bar{a}$. We assume that the Levy measure $\nu_S(dz_S)$ is allowed to generate uniform jumps following Ait-Sahalia et al (2009). We now have the following proposition.

Proposition 7. *Given that the investor has power utlity with CRRA coefficient $\gamma = 2$ and the Levy measure $\nu_S(dz_S)$ is allowed to generate assymmetric uniform jumps, then*

$$\bar{a} = \frac{1}{3}\left[-2A_1 - \frac{2^{\frac{1}{3}}\left[A_1^2 + 3(1 - B_1(\lambda_{S-} + \lambda_{S+}))\right]}{\Psi^{\frac{1}{3}}} - (2\Psi)^{\frac{1}{3}}\right] \tag{58}$$

where $A_1 = -\frac{1}{2}(J\kappa)'(\Sigma\Sigma')^{-1}(\mu - r\kappa) + \rho(J\kappa)'(\Sigma\Sigma')^{-1}\Sigma\sigma_B'\xi - \frac{\rho_1}{2}(J\kappa)'(\Sigma\Sigma')^{-1}\Sigma\sigma_g'h_g$ *and* $B_1 = \frac{1}{2}(J\kappa)'(\Sigma\Sigma')^{-1}J\kappa$.

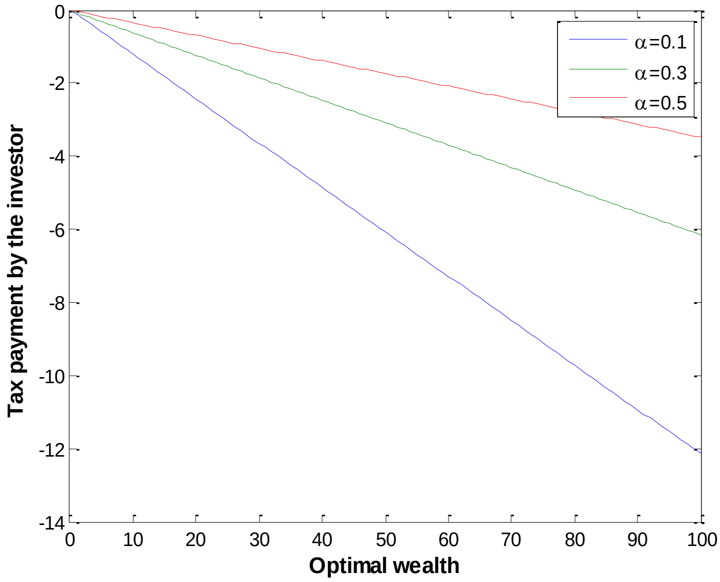

Figure 9: Effects of optimal wealth and optimal consumption rate on tax payment for $\gamma = 0.05$ and $h(t, g) = 0.0005$.

Proof. Given that the Levy measure generates assymmetric positive and negative jumps and satisfies the power law, we have that

$$m\nu_S(dz_S) = \begin{cases} \lambda_{S+}\frac{dz}{z}, \ if \ z \in (0, 1] \\ -\lambda_{S-}\frac{dz}{z}, \ if \ z \in [-1, 0), \end{cases} \tag{59}$$

such that $\lambda_{S+} > 0$ is the intensity of positive jumps and $\lambda_{S-} > 0$ is the intensity of negative jumps. Given that $\gamma = 2$, we have that $\dot{\Phi}(\bar{a}) = -\lambda_{S+}\log(1 + \bar{a}) - \lambda_{S-}\log(1 - \bar{a})$. This implies that (50) becomes a cubic equation in $\bar{a}$ such that

$$f'(\bar{a}) = \bar{a} + A_1 + B_1[-\lambda_{S+}\log(1 + \bar{a})^{-1} - \lambda_{S-}\log(1 - \bar{a})^{-1}] = 0 \tag{60}$$

with solvency constraint $|\bar{a}| < 1$. We obtain the $\bar{a}$ as

$$\bar{a} = \frac{1}{3}\left[-2A_1 - \frac{2^{\frac{1}{3}}\left[A_1^2 + 3(1 - B_1(\lambda_{S-} + \lambda_{S+}))\right]}{\Psi^{\frac{1}{3}}} - (2\Psi)^{\frac{1}{3}}\right], \tag{61}$$

where

$$\Psi = B_2 + \sqrt{(-4(A_1^2 - 3(-1 + B_1(\lambda_{S-} + \lambda_{S+})))^3 + B_2)},$$

$$B_2 = (2A_1^3 + 27B_1(\lambda_{S-} - \lambda_{S+}) - 9A_1(2 + B_1(\lambda_{S-} - \lambda_{S+})).$$

□

Suppose $-\frac{1}{2}(\mu - r\kappa) + \rho\Sigma\sigma_B'\xi - \frac{\rho_1}{2}\Sigma\sigma_g' h_g = \frac{1}{2}(\lambda_{S-} + \lambda_{S+})J\kappa$. This implies that the investor is free or has no exposuure to jump risk. This is because jumps in one investment portfolio is used to hedge jump risk in another portfolio.

4 Empirical Analysis of our Models

In this section, we provide numerical results for our models. For the purpose of relating the results generated to practical issues in finance and the economics, we collected stock price data between Feburary 2015 to June 2020 from the Nigerian Stock Exchange. The data collected were the stock prices of six firms and it include: Guaranty Trust Bank (GTB), Access Bank (AB), PZ cussons (PZ), Oando, Zenith Bank and United Bank for Africa (UBA) . We analyse the collected data using SPSS package in order to obtain the necessary statistical informations (i.e., mean, standard deviation (SD) and jump-sizes of the various stocks) required for the implementation of our models. Table 1 show the means, standard deviations and jump-sizes of the stock prices collected. The resulting optimization problems were solved using MatLab R2007b software, and the results are presented in Table 2 to Table 5.

Table 1: Mean, SD and jump-size of stock prices

Stocks	Mean	Standard deviation	Jump-sizes
Guarantee Trust Bank	0.2766	0.6904	0.1425
Access Bank	0.0788	0.6405	0.0581
PZ cussons	0.1688	0.7236	0.1192
Oando	0.0823	0.6115	0.2498
Zenith Bank	0.1724	0.6721	0.2051
United Bank for Africa	0.0727	0.5815	0.1122

Table 2 shows the optimal portfolio returns of the investor in stocks and risk-free asset for varying values of γ and r, and $\xi = 0.07$. We found that as the coefficient of risk aversion γ and nominal interest rate increasing, portfolio returns in stocks, decreasing and the risk-free asset is increasing. This indicates that as the investor becomes more risk averse and nominal interest rate increases, more of the investments should be moved from risky stocks to risk-free asset. Also, we observe that with an increase in nominal interest rate, the portfolio returns in stocks decrease and risk-free asset increase. This suggest that in periods of increasing interest rate, the investor should be encouraged to invest more in risk-free aseet.

Table 3 shows the optimal portfolio returns of the investor in stocks and risk-free asset, for varying values of γ and ξ, and $r = 0.05$. We observe that portfolio returns in stocks decrease with an increasing tax rate form 0.03 to 0.10 for $\gamma = 0.1$ and start to increase as tax rate increases from 0.12. This indicates that an increase in the tax rate form 0.03 to 0.10, for $\gamma = 0.1$ will reduce the portfolio value in risky assets and increases the portfolio value in riskless asset. But, when the value of γ increases from 0.3, the portfolio returns in stocks decrease with an increasing tax ratio and increase in riskless asset of the investor's portfolio. It then implies that taxation on an investment portfolio will effect the portfolio returns in the risky assets negatively and riskless asset positively. Hence, in the presents of high tax in an investment portfolio, more of the investment should be shifted to the riskless asset over time.

Table 4 shows the portfolio return values for both risky and riskless assets for changes in γ and $\bar{a}$ (a control parameter that determines how much of this portfolio value the investor is willing to take). We observed the following:

1. as the value of $\bar{a}$ increases, the portfolio return in risky assets increases and vice versa.
2. as the value of $\bar{a}$ increases, the portfolio return in risk-free asset decreases and vice versa.

Table 5 shows the portfolio returns for both risky and riskless assets, for changes in h_g and γ. We observed the following:

1. as the value of h_g increases, the portfolio return in risky assets increases and vice versa.
2. as the value of h_g increases, the portfolio return in risk-free asset decrease and vice versa.

Table 2: Portfolio returns in stocks for varying r and γ, and $\xi = 0.07$.

γ	r	GTB	AB	PZ	Oando	Zenith	UBA	π_0^*
0.1	0.01	136.7862	14.5300	91.8414	36.8810	89.2369	38.9355	-407.2110
	0.02	129.3769	12.7815	86.5817	34.0825	84.3012	36.6210	-382.7448
	0.05	107.1489	7.5359	70.8028	25.6869	69.4940	29.6776	-309.3461
	0.10	70.1022	-1.2067	44.5046	11.6944	44.8152	18.1052	-187.0150
	0.15	33.0555	-9.9493	18.2064	-2.2981	20.1365	6.5328	-64.6839
	0.30	-78.0845	-36.1771	-60.6883	-44.2756	-53.8997	-28.1845	302.3095
	0.50	-222.2711	-71.1475	-165.8811	-100.2456	-152.6146	-74.4741	791.6341
0.3	0.01	45.6604	4.8679	30.6779	12.3200	29.7890	12.9937	-135.3087
	0.02	43.1907	4.2851	28.9244	11.3871	28.1438	12.2222	-127.1533
	0.05	35.7813	2.5366	23.6648	8.5866	23.2080	12.2080	-120.6871
	0.10	23.4324	-0.3776	14.8987	3.9245	14.9818	6.0503	-61.9100
	0.15	11.0835	-3.2918	6.1327	-0.7397	6.7555	2.1928	-21.1330
	0.30	-25.9631	-12.0345	-20.1656	-14.7322	-17.9232	-9.3796	101.1981
	0.50	-75.3587	-23.6913	-55.2298	-33.3889	-50.8282	-24.8095	264.3063
0.5	0.01	27.4034	2.9168	18.4204	7.3929	17.8758	7.8044	-80.8137
	0.02	25.9216	2.5671	17.3685	6.8332	16.8887	7.3415	-75.9205
	0.05	21.4760	1.5180	14.2127	5.1541	13.9272	5.9528	-61.2407
	0.10	14.0666	-0.2306	8.9531	2.3556	8.9915	3.6383	-36.7745
	0.15	6.6573	-1.9791	3.6934	-0.4429	4.0557	1.3238	-12.3083
	0.30	-15.5707	-7.2247	-12.0855	-8.8384	-10.7515	-5.6196	61.0904
	0.50	-45.2080	-14.2187	-33.1241	-2.0324	-30.4945	-14.8775	158.9553
1.0	0.01	13.7306	1.4651	9.2428	3.7069	8.9557	3.9130	-40.0140
	0.02	12.9897	1.2903	8.7168	3.4270	8.4621	3.6815	-37.5674
	0.05	10.7669	0.7657	7.1389	2.5875	6.9814	2.9872	-302275
	0.10	7.0622	-0.1086	4.5091	1.1882	4.5135	1.8299	-17.9944
	0.15	3.3575	-0.9828	1.8793	-0.2110	2.0456	0.6727	-5.7613
	0.30	-7.7565	-3.6056	-6.0102	-4.4088	-5.3580	-2.7990	30.9380
	0.50	-22.5751	-7.1026	-16.5295	-10.0058	-15.2295	-7.4280	79.8705
3.0	0.01	4.6207	0.4996	3.1294	1.2517	3.0125	1.3197	-12.8337
	0.02	4.3737	0.4413	2.9541	1.1584	2.8480	1.2426	-12.0182
	0.05	3.6328	0.2664	2.4282	0.8786	2.3544	1.0111	-9.5715
	0.10	2.3979	-0.0250	1.5516	0.4122	1.5318	0.6254	-5.4938
	0.15	1.1630	-0.3164	0.6749	-0.0542	0.7092	0.2396	-1.4161
	0.30	-2.5417	-1.1907	-1.9549	-1.4535	-1.7587	-0.9176	10.8170
	0.50	-7.4182	-2.3563	-5.4613	-3.3192	-5.0492	-2.4606	27.1278
5.0	0.01	2.7923	0.3038	1.9007	0.7581	1.8196	0.7999	-7.3743
	0.02	2.6441	0.2688	1.7955	0.7021	1.7208	0.7536	-6.8550
	0.05	2.1996	0.1639	1.4799	0.5342	1.4247	0.6147	-5.4170
	0.10	1.4586	-0.0109	0.9540	0.2543	0.9311	0.3833	-2.9704
	0.15	0.7177	-0.1858	0.4280	-0.0255	0.4376	0.1518	-0.5238
	0.30	-1.5051	-0.7104	-1.1499	-0.8651	-1.0432	-0.5425	6.8161
	0.50	-4.4688	-1.4098	-3.2538	-1.9845	-3.0175	-1.4683	16.6026

Table 3: Portfolio returns in stocks for varying ξ and γ, and $r = 0.05$.

γ	ξ	GTB	AB	PZ	Oando	Zenith	UBA	π_0^*
0.1	0.03	107.1496	7.5369	70.8039	25.6871	69.4946	29.6772	-309.6494
	0.04	107.1494	7.5367	70.8036	25.6871	69.4945	29.6773	-309.6486
	0.06	107.1491	7.5362	70.8031	25.6870	69.4941	29.6775	-309.6469
	0.10	107.1955	7.5649	70.8361	25.7095	69.5291	29.6774	-309.5126
	0.12	107.2187	7.5793	70.8526	25.7208	69.5466	29.6774	-309.5954
	0.20	107.2172	7.5773	70.8504	25.7205	69.5453	29.6782	-309.5889
	0.35	107.2143	7.5736	70.8463	25.7200	69.5428	29.6796	-309.5767
0.3	0.03	35.7821	2.5375	23.6659	8.5888	23.2087	9.9074	-102.6903
	0.04	35.7819	2.5373	23.6656	8.5887	23.2085	9.9075	-102.6895
	0.06	35.7815	2.5368	23.6651	8.5887	23.2082	9.9077	-102.6879
	0.10	35.7808	2.5358	23.6640	8.5885	23.2075	9.9080	-102.6846
	0.12	35.7804	2.5353	23.6634	8.5885	23.2072	9.9082	-102.6830
	0.20	35.7789	2.5334	23.6613	8.5882	23.2058	9.9090	-102.6765
	0.35	35.7761	2.5297	23.6572	8.5876	23.2033	9.9104	-102.6643
0.5	0.03	21.4909	1.5278	14.2240	5.1610	13.9386	5.9523	-61.2947
	0.04	21.4907	1.5276	14.2238	5.1610	13.9384	5.9524	-61.2938
	0.06	21.4903	1.5271	14.2232	5.1609	13.9381	5.9526	-61.2922
	0.10	21.4896	1.5261	14.2221	5.1608	13.9374	5.9529	-61.2890
	0.12	21.4892	1.5256	14.2216	5.1607	13.9371	5.9531	-61.2873
	0.20	21.4877	1.5237	14.2194	5.1604	13.9357	5.9539	-61.2808
	0.35	21.4848	1.5200	14.2153	5.1599	13.9332	5.9553	-61.2686
1.0	0.03	10.7747	0.7711	7.1452	2.5910	6.9874	2.9867	-30.2561
	0.04	10.7745	0.7709	7.1449	2.5910	6.9872	2.9868	-30.2553
	0.06	10.7741	0.7704	7.1443	2.5909	6.9869	2.9870	-30.2537
	0.10	10.7734	0.7694	7.1432	2.5908	6.9862	2.9874	-30.2504
	0.12	10.7730	0.7689	7.1427	2.5907	6.9859	2.9876	-30.2488
	0.20	10.7715	0.7670	7.1405	2.5904	6.9845	2.9884	-30.2423
	0.35	10.7687	0.7633	7.1364	2.5898	6.9820	2.9898	-30.2301
3.0	0.03	3.6335	0.2674	2.4293	0.8787	2.3551	1.0108	-9.5748
	0.04	3.6333	0.2672	2.4290	0.8787	2.3549	1.0109	-9.5740
	0.06	3.6330	0.2667	2.4284	0.8786	2.3546	1.0110	-9.5724
	0.10	3.6322	0.2657	2.4273	0.8785	2.3539	1.0114	-9.5691
	0.12	3.6318	0.2652	2.4268	0.8784	2.3536	1.0116	-9.5675
	0.20	3.6303	0.2633	2.4246	0.8781	2.3523	1.0124	-9.5610
	0.35	3.6275	0.2596	2.4205	0.8776	2.3497	1.0138	-9.5487
5.0	0.03	2.2017	0.1658	1.4820	0.5350	1.4264	0.6143	-5.4253
	0.04	2.2016	0.1655	1.4818	0.5350	1.4263	0.6144	-5.4245
	0.06	2.2012	0.1650	1.4812	0.5349	1.4259	0.6146	-5.4229
	0.10	2.2004	0.1641	1.4801	0.5348	1.4253	0.6150	-5.4196
	0.12	2.2000	0.1636	1.4796	0.5347	1.4249	0.6152	-5.4180
	0.20	2.1985	0.1616	1.4774	0.5344	1.4236	0.6159	-5.4115
	0.35	2.1957	0.1580	1.4733	0.5388	1.4211	0.6174	-5.3993

Table 4: Portfolio returns in stocks for varying $\bar{a}$ and γ, $r = 0.05$.

γ	$\bar{a}$	GTB	AB	PZ	Oando	Zenith	UBA	π_0^*
0.1	0.5	107.3082	7.6032	70.9546	25.7527	69.6025	29.7083	-309.9296
	1.0	107.4559	7.6411	71.1223	25.8055	69.6943	29.7603	-310.4798
	2.0	107.7512	7.7170	71.4578	25.9112	69.8779	29.8653	-311.5802
	5.0	108.6373	7.9445	72.4641	26.2281	70.4286	30.1788	-314.8814
	10.0	110.1140	8.3236	74.1413	26.7563	71.3464	30.7016	-320.3833
	15.0	111.5907	8.7028	75.8185	27.2845	72.2643	31.2244	-325.8852
	20.0	113.0674	9.0820	77.4957	27.8128	73.1821	31.7492	-331.3871
0.3	0.5	35.8676	2.5585	23.7621	8.6193	23.2612	9.9381	-103.0062
	1.0	36.0146	2.5965	23.9298	8.6721	23.3530	9.9904	-103.5564
	2.0	36.3100	2.6723	24.2652	8.7777	23.5366	10.0949	-104.6568
	5.0	37.1960	2.8998	25.2716	9.0947	24.0873	10.4086	-107.9579
	10.0	38.6762	3.2790	26.9488	9.6229	25.0051	10.9314	-113.4599
	15.0	40.1495	3.6581	28.6260	10.1511	25.9230	11.4541	-118.9618
	20.0	41.6262	4.0373	30.3032	10.6793	26.8408	11.9769	-124.4637
0.5	0.5	21.5787	1.5496	14.3236	5.1926	13.9330	5.9840	-61.6215
	1.0	21.7264	1.5875	14.4913	5.2454	14.0848	6.0363	-62.1717
	2.0	22.0217	1.6634	14.8267	5.3511	14.2683	6.1409	-63.2721
	5.0	22.9078	1.8909	15.8331	5.6680	14.8190	6.4545	-66.2721
	10.0	24.3845	2.2700	17.5103	6.1962	15.7369	6.9773	-72.0752
	15.0	25.8612	2.6492	19.1875	6.7244	16.6547	7.5001	-77.5771
	20.0	27.3379	3.0284	20.8647	6.8501	17.5726	8.0229	-83.0790
1.0	0.5	10.8625	0.7929	7.2447	2.6226	7.0418	3.0185	-30.5830
	1.0	11.0102	0.8308	7.4124	2.6754	7.1336	3.0708	-31.1332
	2.0	11.3056	0.9067	7.7479	2.7810	7.3172	3.1753	-32.2336
	5.0	12.1916	1.1342	8.7542	3.0980	7.8679	3.4890	-35.5347
	10.0	13.6683	1.5133	10.4314	3.6262	8.7857	4.0118	-41.0367
	15.0	15.1450	1.8925	12.1086	4.1544	9.7035	4.5346	-46.5386
	20.0	16.6217	2.2716	13.7558	4.6826	10.6214	5.0573	-52.0405
3.0	0.5	3.7184	0.2884	2.5254	0.9092	2.4077	1.0415	-9.8907
	1.0	3.8661	0.3264	2.6932	0.9621	2.4995	1.0937	-10.4409
	2.0	4.1614	0.4022	3.0286	1.0677	2.6830	1.1937	-11.5412
	5.0	5.0475	0.6297	4.0349	1.3846	3.2337	1.5120	-14.8424
	10.0	6.5242	1.0089	5.7121	1.9128	4.1516	2.0347	-20.3443
	15.0	8.0009	1.3880	7.3893	2.4411	5.0694	2.5575	-25.8462
	20.0	9.4776	1.7672	9.0665	2.9693	5.9872	3.0803	-31.3482
5.0	0.5	2.2896	0.1875	1.5816	0.5666	1.4808	0.6461	-5.7522
	1.0	2.4373	0.2255	1.7493	0.6194	1.5726	0.6983	-6.3024
	2.0	2.7326	0.3013	2.0847	0.7250	1.7562	0.8029	-7.4028
	5.0	3.6186	0.5288	3.0911	1.0420	2.3069	1.1166	-10.7039
	10.0	5.0954	0.9080	4.7683	1.5702	3.2247	1.6393	-16.2059
	15.0	6.5721	1.2871	6.4455	2.0984	4.1426	2.1621	-21.7078
	20.0	8.0488	1.6663	8.1227	2.6266	5.0604	2.6849	-27.2097

Table 5: Portfolio returns in stocks for varying h_g and γ, and $r = 0.05$.

γ	h_g	GTB	AB	PZ	Oando	Zenith	UBA	π_0^*
0.1	0.01	107.6168	7.6324	71.3606	25.8465	69.7762	29.8663	-311.0988
	0.02	107.6239	7.6368	71.3657	25.8499	69.7816	29.8663	-311.1241
	0.05	107.6451	7.6502	71.3810	25.8601	69.7976	29.8661	-311.2001
	0.10	107.6805	7.6724	71.4066	25.8771	69.8244	29.8658	-311.3268
	0.15	107.7159	7.6947	71.4322	25.8941	69.8511	29.8655	-311.4535
	0.20	107.7512	7.7170	71.4578	25.9112	69.8779	29.8652	-311.5802
	0.30	107.8220	7.7615	71.5090	25.9452	69.9314	29.8645	-311.8336
0.3	0.01	36.2643	2.6430	24.2316	8.7560	23.5019	10.0957	-104.4925
	0.02	36.2667	2.6444	24.2333	8.7571	23.5037	10.0957	-104.5009
	0.05	36.2737	2.6489	24.2384	8.7605	23.5091	10.0957	-104.5263
	0.10	36.2855	2.6563	24.2469	8.7662	23.5158	10.0956	-104.5685
	0.15	36.2973	2.6637	24.2554	8.7719	23.5269	10.0955	-1046107
	0.20	36.3091	2.6712	24.2640	8.7776	23.5358	10.0954	-104.6530
	0.30	36.3327	2.6860	24.2810	8.7889	23.5536	10.0952	-104.7374
0.5	0.01	21.9949	1.6464	14.8073	5.3381	14.2480	6.1411	-63.1758
	0.02	21.9963	1.6473	14.8083	5.3388	14.2491	6.1411	-63.1809
	0.05	21.0005	1.6500	14.8114	5.3408	14.2523	6.1410	-63.1961
	0.10	22.0076	1.6545	14.8165	5.3442	14.2576	6.1410	-63.2214
	0.15	22.0147	1.6589	14.8216	5.3477	14.2630	6.1409	-63.2468
	0.20	22.0217	1.6634	14.8267	5.3511	14.2682	6.1409	-63.2721
	0.30	22.0359	1.6723	14.8370	5.3579	14.2790	6.1407	-63.3228
1.0	0.01	11.2921	0.8982	7.7381	2.7746	7.3070	3.1754	-32.1854
	0.02	11.2928	0.8986	7.7386	2.7749	7.3075	3.1754	-32.1880
	0.05	11.2949	0.9000	7.7402	2.7759	7.3091	3.1754	-32.1956
	0.10	11.2985	0.9022	7.7427	2.7776	7.3118	3.1754	-32.2082
	0.15	11.3020	0.9044	7.453	2.7793	7.3145	3.1754	-32.2209
	0.20	11.3056	0.9067	7.7479	2.7810	7.3172	3.1753	-32.2336
	0.30	11.3126	0.9111	7.7530	2.7844	7.3225	3.1753	-32.2589
3.0	0.01	4.1561	0.3982	3.0241	1.0654	2.6788	1.1988	-11.5214
	0.02	4.1563	0.3984	3.0243	1.0655	2.6790	1.1988	-11.5222
	0.05	4.1570	0.3988	3.0248	1.0658	2.6796	1.1988	-11.5248
	0.10	4.1582	0.3996	3.0256	1.0664	2.6805	1.1988	-11.5290
	0.15	4.1594	0.4003	3.0265	1.0670	2.6813	1.1988	-11.5332
	0.20	4.1606	0.4011	3.0273	1.0675	2.6822	1.1987	-11.5374
	0.30	4.1629	0.4025	3.0290	1.0687	2.6840	1.1987	-11.5459
5.0	0.01	2.7290	0.2985	2.0815	0.7236	1.7534	0.8034	-7.3893
	0.02	2.7292	0.2986	2.0816	0.7236	1.7535	0.8034	-7.3898
	0.05	2.7296	0.2988	2.0819	0.7238	1.7538	0.8034	-7.3914
	0.10	2.7303	0.2993	2.0824	0.7242	1.7543	0.8034	-7.3939
	0.15	2.7310	0.2997	2.0830	0.7245	1.7549	0.8033	-7.3964
	0.20	2.7317	0.3002	2.0835	0.7249	1.7554	0.8033	-7.3990
	0.30	2.7331	0.3010	2.0845	0.7255	1.7565	0.8033	-7.4040

5 Verification Theorem

In this section, we provide a verification theorem for our problem. First, we give the following lemmas that will be useful in establishing the verification theorem.

Lemma 1. $E|B(t)| \leq 4E|B_0|e^{4(\Theta_{\mu_B}+\Theta_{JB}\lambda_B)t}$.

Proof. We have from (19), that

$$B(t) = B_0 + \int_0^t B(s)\mu_B(s)ds + \int_0^t B(s)\sigma_B(s)dW_B(s) + \int_0^t B(s)J_B(s)dN_B(s). \tag{62}$$

Applying the Cauchy-Schwarz inequality and taking mathematical expectation of both sides of (62), we have

$$E|B(t)| \leq 4E|B_0| + 4E\left|\int_0^t \mu_B(s)B(s)ds\right| + 4E\left|\int_0^t B(t)\sigma_B(s)dW_B(s)\right| + 4E\left|\int_0^t B(s)J_B(s)dN_g(s)\right|. \tag{63}$$

But,

$$E\left|\int_0^t B(t)\sigma_B(s)dW_B(s)\right| = 0$$

and

$$E\left|\int_0^t B(s)J_B dN_g(s)\right| \leq \lambda_B \int_0^t E|B(s)J_B(s)\kappa|ds.$$

It follows that

$$E|B(t)| \leq 4E|B_0| + 4\int_0^t E|\mu_B(s)B(s)|ds + 4\lambda_B \int_0^t E|B(s)J_B(s)\kappa|ds. \tag{64}$$

It also follows that

$$E|B(t)| \leq 4E|B_0| + 4\int_0^t E|\mu_B(s)||B(s)|ds + 4\lambda_B \int_0^t E|B(s)||J_B(s)\kappa|ds. \tag{65}$$

Hence, using (20), we have

$$E|B(t)| \leq 4E|B_0| + (4\Theta_{\mu_B} + 4\Theta_{JB}\lambda_B)\int_0^t E|B(s)|ds. \tag{66}$$

Differentiating both sides of (66) with respect to t, we have that

$$\frac{dE|B(t)|}{dt} \leq E|B(t)|(4\Theta_{\mu_B} + 4\Theta_{JB}\lambda_B), \quad E|B(0)| = 4E|B_0|. \tag{67}$$

Solving (67), we have

$$E|B(t)| \leq 4E|B_0|e^{4(\Theta_{\mu_B}+\Theta_{JB}\lambda_B)t}. \tag{68}$$

□

Lemma 2. *Let $\omega \in \mathbb{R}$ and suppose $\tilde{\eta}(\omega)$ is uniformly bounded, then there exist a positive constant κ_g, such that $\forall t \in [0,T]$, $|\tilde{\eta}(\omega)| \leq L$ and (11) holds, then (11) has a unique strong solution. It then follows*

$$E|g(t)| \leq 4E|g_0| + \frac{2L + \lambda_g \Theta_g}{2|\tilde{\eta}(\omega)|}(e^{8|\tilde{\eta}(\omega)|t} - 1). \tag{69}$$

Proof. From equation (11), we have that

$$g(t) = g_0 + \int_0^t \Lambda(g(s))ds + \int_0^t g(s)\tilde{\eta}(\omega)ds + \int_0^t \sigma_g(s)dW(s) + \int_0^t J_g dN_g(s). \tag{70}$$

By applying the Cauchy-Schwarz inequality and taking mathematical expectation of both sides of (70), we have

$$E|g(t)| \leq 4E|g_0| + 8E\bigg|\int_0^t (\Lambda(g(s))ds + g(s)\tilde{\eta}(\omega))ds\bigg| + 4E\bigg|\int_0^t \sigma_g dW_g(s)\bigg| + 4E\bigg|\int_0^t J_g dN_g(s)\bigg|. \tag{71}$$

But,

$$E\bigg|\int_0^t \sigma_g dW_g(s)\bigg| = 0,$$

since its a Martingale and

$$E\bigg|\int_0^t J_g dN_g(s)\bigg| \leq E\bigg|\lambda_g \int_0^t J_g \kappa ds\bigg| \leq \lambda_g \int_0^t E|J_g\kappa|ds = \lambda_g \Theta_g t.$$

It follows that

$$E|g(t)| \leq 4E|g_0| + 8E\int_0^t \bigg|\Lambda(g(s)) + g(s)\tilde{\eta}(\omega)\bigg|ds + 4\lambda_g\Theta_g t. \tag{72}$$

Whence,

$$E|g(t)| \leq 4E|g_0| + 8Lt + 8\bigg|\tilde{\eta}(\omega)\bigg|E\int_0^t \bigg|g(s)\bigg|ds + 4\lambda_g\Theta_g t. \tag{73}$$

Differentiating (73) with respect to t, we have the following ordinary differential inequality (ODI):

$$\frac{dE|g(t)|}{dt} - 8|\tilde{\eta}(\omega)|E|g(t)| \leq 4(2L + \lambda_g\Theta_g), \;\; E|g(0)| = 4E|g_0|. \tag{74}$$

Solving (74), we have

$$E|g(t)| \leq 4E|g_0| + \frac{2L + \lambda_g \Theta_g}{2|\tilde{\eta}(\omega)|}(e^{8|\tilde{\eta}(\omega)|t} - 1).$$

□

Theorem 1. *Let $\tilde{y}(g)$ be the classical solution to (31) such that $U(\bar{x}, g) = \frac{\bar{x}^{1-\gamma}}{1-\gamma}e^{h(t,g)}$, $V(\alpha, \bar{x}) = \frac{(\alpha\bar{x})^{1-\gamma}}{1-\gamma}$, $V^k(\varphi, \bar{x}) = \frac{(\varphi\bar{x})^{\phi}}{\phi}$ and $h(g(T)) = 1$. For admissible control strategies $u = (\varphi, \alpha, \pi) \in \mathcal{A}$,*

$$U(\bar{X}, g) \geq F(t, \bar{x}, g) = E_{\bar{x},g}\int_0^\infty e^{-\delta t}\bigg(V(\alpha(t)\bar{x}(t)) + V^k(\varphi(t)\bar{x}(t))\bigg)dt. \tag{75}$$

Let $u^* = (\varphi^*, \alpha^*, \pi^*) \in \mathcal{A}$ satisfy the following optimal control policies:

$$\varphi^* = \left[(r_k - r - g)e^{h(t,g)}\right]^{\frac{1}{\phi-1}}, \quad \alpha^* = \left((1+\xi)e^{h(t,g)}\right)^{-\frac{1}{\gamma}},$$

$$\pi'^* = \frac{1}{\gamma}\left(1 - \frac{(\Sigma\Sigma')^{-1}J\kappa}{(J\kappa)'(\Sigma\Sigma')^{-1}J\kappa}J\kappa\right)\left\{(\Sigma\Sigma')^{-1}(\mu - r\kappa) + \rho\xi\gamma(\Sigma\Sigma')^{-1}\Sigma\sigma'_B - h_g\rho_1(\Sigma\Sigma')^{-1}\Sigma\sigma'_g\right\}$$
$$+\frac{\bar{a}(\Sigma\Sigma')^{-1}J\kappa}{(J\kappa)'(\Sigma\Sigma')^{-1}J\kappa}.$$

Then $u^* \in \mathcal{A}$ and $U(\bar{X}, g) \geq F(t, \bar{x}, g) = E_{\bar{x},g}\int_0^\infty e^{-\delta t}\left(V(\alpha(t)\bar{x}(t)) + V^k(\varphi(t))\right)dt$.

Thus, $U(\bar{x}^*, g)$ is optimal for $F(t, \bar{x}, g)$, such that $U(\bar{x}, g) = U(\bar{x}^*, g)$ and $u = u^*$.

Proof. For any admissible control $u \in \mathcal{A}$, we apply the Itô lemma to the function $f(t, U) = e^{-\delta t}U(\bar{X}(t), g(t))$.

It follows that

$$d[e^{-\delta t}U(\bar{X}(t), g(t))] = e^{-\delta t}dU(\bar{X}(t), g(t)) - \delta e^{-\delta t}U(\bar{X}(t), g(t))dt. \tag{76}$$

Applying Itô lemma to $U(\bar{X}(t), g(t))$, we have that

$$\begin{aligned} dU(\bar{X}(t), g(t)) &= U_{\bar{x}}d\bar{X}(t) + U_g dg(t) + \frac{1}{2}U_{\bar{x}\bar{x}}(d\bar{X}(t))^2 + \frac{1}{2}U_{gg}(dg(t))^2 \\ &\quad + U_{\bar{x}g}d\bar{X}(t)dg(t) \\ &= \mathcal{L}^u\tilde{U}(\bar{X}(t), g(t))dt + (\pi\Sigma dW_S(t) - \xi\sigma_B dW_B(s))\bar{x}U_{\bar{x}} \\ &\quad + \sigma_g dW_g U_g + (\pi J dN_S(t) - \xi J_B dN_B(t))\bar{x}U_{\bar{x}} + J_g dN_g(t)U_g \\ &= \mathcal{L}^u\tilde{U}(\bar{X}(t), g(t))dt + N_1 + N_2, \end{aligned} \tag{77}$$

where

$$\begin{aligned} \mathcal{L}^u\tilde{U}(\bar{X}(t), g(t)) &= \bar{x}(\xi(r + g - \alpha - \mu_B) + \pi(\mu - r\kappa) + \varphi(r + g - r_k) \\ &\quad + (r + g - \alpha))U_{\bar{x}} + (\Lambda(g) + g\tilde{\eta}(\omega))U_g + \frac{1}{2}\bar{x}^2\pi\Sigma(\pi\Sigma)'U_{\bar{x}\bar{x}} + \frac{1}{2}\bar{x}^2\xi^2\sigma_B\sigma'_B U_{\bar{x}\bar{x}} \\ &\quad - \rho\bar{x}\pi\xi\sigma_B\sigma'_g U_{\bar{x}g} + \frac{1}{2}\sigma_g\sigma'_g U_{gg} + \rho_1\bar{x}\pi\Sigma\sigma'_g U_{\bar{x}g}, \end{aligned}$$

$$\begin{aligned} N_1 &= (\pi\Sigma dW_S(t) - \xi\sigma_B dW_B(s))\bar{x}U_{\bar{x}} + \sigma_g dW_g U_g, \\ N_2 &= (\pi J dN_S(t) - \xi J_B dN_B(t))\bar{x}U_{\bar{x}} + J_g dN_g(t)U_g. \end{aligned}$$

It follows that

$$U(\bar{X}(t), g(t)) - U(\bar{X}_0, g_0) = \int_0^t \mathcal{L}^u\tilde{U}(\bar{X}(s), g(s))ds + \bar{N}_1 + \bar{N}_2, \tag{78}$$

where

$\bar{N}_1 = \int_0^t U_{\bar{x}}\bar{x}(s)\big(\pi\Sigma dW_S(s) - \xi\sigma_B dW_B(s)\big) + U_g\sigma_g dW_g(s),$

$\bar{N}_2 = \int_0^t (\pi J dN_S(s) - \xi J_B dN_B(s))\bar{x}U_{\bar{x}} + J_g dN_g(s)U_g.$

We note that $\bar{N}_1$ is a Martingale. Also,

$$E\bar{N}_2 = \lambda_S \int_0^t U_{\bar{x}}\bar{x}(s)\pi J dN_S(s) - \lambda_B \int_0^t U_{\bar{x}}\bar{x}(s)\xi J_B dN_B(s) + \lambda_g \int_0^t U_g J_g ds < \infty.$$

This implies that the summation of the jumps are finite and thus taken as a constant.

Using (31), for any control $u \in \mathcal{A}$ we have that

$$\mathcal{L}^u\tilde{U}(\bar{X}(t), g(t)) - \delta U(\bar{X}(t), g(t)) \leq -V(\bar{x}(t)\alpha(t)) - V^k(\varphi(t)\bar{x}(t)). \tag{79}$$

From (76) , we have that

$$d\big[e^{-\delta t}U(\bar{X}(t), g(t))\big] = e^{-\delta t}\Big[(\mathcal{L}^u\tilde{\mathrm{U}}(\bar{X}(t), g(t)\big) - \delta U(\bar{X}(t), g(t))dt + d\bar{\mathrm{N}}_2\Big]. \tag{80}$$

Integrating both sides of (80) over $[0, T]$ and taking the expectation, we have

$$\begin{aligned} E\int_0^T d\Big[e^{-\delta t}U(\bar{X}(t), g(t))\Big] &\leq -E\int_0^T e^{-\delta t}V(\alpha(t)\bar{x}(t))dt \\ - E\int_0^T e^{-\delta t}V^k(\varphi(t)\bar{x}(t))dt &+ E\int_0^T e^{-\delta t}d\bar{\mathrm{N}}_2(t)dt. \end{aligned} \tag{81}$$

This implies that

$$\begin{aligned} &Ee^{-\delta T}U(\bar{X}(T), g(T)) - \bar{U}(\bar{x}_0, g_0) \leq -E\textstyle\int_0^T e^{-\delta t}V(\alpha(t)\bar{x}(t))dt \\ &-E\textstyle\int_0^T e^{-\delta t}V^k(\varphi(t)\bar{x}(t))dt + E\int_0^T e^{-\delta t}d\bar{N}_2(t)dt. \end{aligned} \tag{82}$$

Integrating the last component of (82), we have that

$$E\textstyle\int_0^T e^{-\delta t}d\bar{N}_2(t)dt = e^{-\delta T}\Big[\bar{N}_2(T) + \frac{\delta(1-2\delta)}{1-\delta}\int_0^T \bar{N}_2(t)dt\Big]. \tag{83}$$

It then follows that

$$\begin{aligned} U(\bar{x}_0, g_0) \geq\; & E\int_0^T e^{-\delta t}V(\alpha(t)\bar{x}(t))dt + E\int_0^T e^{-\delta t}V^k(\varphi(t)\bar{x}(t))dt \\ & + Ee^{-\delta T}U(\bar{X}(T), g(T)) - e^{-\delta T}\Big[\bar{N}_2(T) + \frac{\delta(1-2\delta)}{1-\delta}\int_0^T \bar{N}_2(t)dt\Big] \\ & \geq E\int_0^T e^{-\delta t}V(\alpha(t)\bar{x}(t))dt + E\int_0^T e^{-\delta t}V^k(\varphi(t)\bar{x}(t))dt + E\frac{e^{-\delta T + h(g(T))}\bar{X}(T)^{1-\gamma}}{1-\gamma} \\ & - e^{-\delta T}\Big[\bar{N}_2(T) + \frac{\delta(1-2\delta)}{1-\delta}\int_0^T \bar{N}_2(t)dt\Big]. \end{aligned} \tag{84}$$

Clearly, $\bar{N}_2(T)$ is a constant and $\int_0^T \bar{N}_2(t)dt < \infty$. This implies that

$$\limsup_{T\to\infty} e^{-\delta T} E\left[\bar{N}_2(T) + \frac{\delta(1-2\delta)}{1-\delta}\int_0^T \bar{N}_2(t)dt\right] = 0.$$

To show that (75) holds, we need to establish that

$$\limsup_{T\to\infty} Ee^{-\delta T} U(\bar{X}(t), g(t)) \geq 0.$$

It is indeed sufficient to show that

$$\limsup_{T\to\infty} E\frac{e^{-\delta T + h(g(T))}\bar{X}(T)^{1-\gamma}}{1-\gamma} \geq 0. \tag{85}$$

But, $\bar{X}(t)$ is given by

$$\bar{X}(t) = \bar{x}_0 e^{\chi(t)}, \tag{86}$$

where

$$\begin{aligned}\chi(t) = \int_0^t \{&\big[\xi(s)(r(s)+g(s)-\mu_B(s)-\alpha(s)) + \pi(s)(\mu(s)-r(s)\kappa) + \varphi(s)(r(s)+g(s)-r_k(s))\\ &+ (r(s)+g(s)-\alpha(s)) - \frac{1}{2}(\pi(s)\Sigma(s))(\pi(s)\Sigma(s))' + \frac{1}{2}\xi^2(s)\sigma_B(s)\sigma_B(s)\big]ds + \pi(s)\Sigma(s)dW(s)\\ &- \xi(s)\sigma_B(s)dW_B(s) + \pi(s)JdN(s) - \xi(s)J_B dN_B(s)\}.\end{aligned} \tag{87}$$

Thus,

$$E\frac{e^{-\delta T + h(g(T))}\bar{X}(T)^{1-\gamma}}{1-\gamma} = E\frac{e^{-\delta T + h(g(T))}\bar{x}_0^{1-\gamma}e^{(1-\gamma)\chi(T)}}{1-\gamma} \tag{88}$$

We set $\bar{\mu}(t) = (r(t)+g(t)-\alpha(t)-\mu_B(t)) + \pi(t)(\mu(t)-r(t)\kappa) + \varphi(t)(r(t+g(t)-r_k(t)) + (r(t+g(t)-\alpha(t)) - \frac{1}{2}(\pi(t)\Sigma(t))(\pi(t)\Sigma(t))' + \frac{1}{2}\xi^2(t)\sigma_B(t)\sigma_B'(t)$, $\bar{v} = \pi(t)J$, $\bar{v}_B = \xi(t)J_B$.

Hence, we have

$$\begin{aligned}\chi(t) = &\int_0^t \bar{\mu}(s)ds + \int_0^t \pi(s)\sigma(s)dW(s) - \int_0^t \xi(s)\sigma_B(s)dW_B(s)\\ &+ \int_0^t \log(1+\bar{v}(s))dN(s) - \int_0^t \log(1+\bar{v}_B(s))dN_B(s).\end{aligned} \tag{89}$$

Assume that the rate coefficients $\bar{\mu}(s)$, $\varphi(t)$, $\alpha(s)$, $\xi(s)$ and $g(s)$ are bounded, while $\bar{v}(s) > -1$, $\bar{v}_B(s) > -1$, $0 < \phi < 1$ and $\gamma \neq 1$. Taking the expectation of $\chi(t)$, we have

$$E\big[\chi(t)\big] = E\int_0^t \bar{\mu}(s)ds + E\int_0^t \log(1+\bar{v}(s))dN(s) - E\int_0^t \log(1+\bar{v}_B(s))dN_B(s). \tag{90}$$

We note that $Ee^{\int_0^t \log(1+\bar{v}(s))dN(s)} = e^{\int_0^t \lambda\bar{v}(s)ds}$ and $Ee^{\int_0^t \log(1+\bar{v}_B(s))dN(s)} = e^{\int_0^t \lambda\bar{v}_B(s)ds}$.

It follows

$$E\frac{e^{-\delta T}\bar{X}(T)^{1-\gamma}}{1-\gamma} = E\frac{e^{-\delta T}\bar{x}_0^{1-\gamma}}{1-\gamma}e^{(1-\gamma)\left[\int_0^T \bar{\mu}(t)dt+\lambda_S\int_0^T \bar{v}(t)dt-\lambda_B\int_0^T \bar{v}_B(t)dt\right]}. \tag{91}$$

We now show that

$$\limsup_{T\to\infty}\left[e^{-\delta T}\frac{E(\bar{X}(T))^{1-\gamma}}{1-\gamma}\right] \geq 0. \tag{92}$$

Since $\int_0^T \bar{\mu}(t)dt < \infty$, $\int_0^T \bar{v}(t)dt < \infty$, $\int_0^T \bar{v}_B(t)dt < \infty$, $\int_0^T \alpha(t)\bar{x}(t)dt < \infty$, $\lambda_S > 0$, $\lambda_B > 0$, $\phi > 0$ and $\gamma \neq 1$.

It implies that

$$\begin{aligned}\limsup_{T\to\infty}\left[e^{-\delta T}\frac{E(\bar{X}(T))^{1-\gamma}}{1-\gamma}\right] = \limsup_{T\to\infty}E\frac{e^{-\delta T}x_0^{1-\gamma}}{1-\gamma}\times\\ e^{(1-\gamma)\left[\int_0^T\left(\bar{\mu}(t)+\lambda_S\bar{v}(t)+\lambda_B\bar{v}_B(t)\right)dt\right]}.\end{aligned} \tag{93}$$

However,

$$\begin{aligned}&\limsup_{T\to\infty}E\frac{e^{-\delta T}x_0^{1-\gamma}}{1-\gamma}e^{(1-\gamma)\left[\int_0^T\left(\bar{\mu}(t)+\lambda_S\bar{v}(t)+\lambda_B\bar{v}_B(t)\right)dt\right]}\\ &\geq \lim_{T\to\infty}E\frac{e^{-\delta T}x_0^{1-\gamma}}{1-\gamma}e^{(1-\gamma)\left[\int_0^T\left(\bar{\mu}(t)+\lambda_S\bar{v}(t)+\lambda_B\bar{v}_B(t)\right)dt\right]} = 0.\end{aligned} \tag{94}$$

It follows that

$$\limsup_{T\to\infty}Ee^{-\delta T}\left[\frac{(\bar{X}(T))^{1-\gamma}}{1-\gamma}\right] \geq 0. \tag{95}$$

Therefore, (75) is satisfied.

We now consider the optimal debt ratio, optimal portfolio strategy and consumption rate such that $u^* = (\varphi^*, \pi^*, \alpha^*)$. If $u^* \in \mathcal{A}$, where $(\bar{X}^*(t), g^*(t))$ are the corresponding trajectories of the optimal policies of u^*. We have that

$$\mathcal{L}^u\tilde{U}(\bar{X}(t), g(t)) - \delta U(\bar{X}(t), g(t)) = -V(\alpha(t)\bar{x}(t)) - V^k(\varphi(t)\bar{x}(t)). \tag{96}$$

This implies that

$$\begin{aligned}U(\bar{x}_0, g_0) = E\int_0^T e^{-\delta t}V(\alpha^*(t)\bar{x}^*(t))dt + E\int_0^T e^{-\delta t}V(\varphi^*(t)\bar{x}^*(t))dt\\ + Ee^{-\delta T}U(\bar{X}^*(T), g(T)) - e^{-\delta T}E\left[\bar{N}_2(T) + \frac{\delta(1-2\delta)}{1-\delta}\int_0^T \bar{N}_2(t)dt\right].\end{aligned} \tag{97}$$

Note that we have defined earlier that $\bar{X}^*(t) = \bar{x}^*(t)$, so they can be use interchangeably. We now show that

$$U(\bar{x}_0, g_0) \leq E\int_0^T e^{-\delta t} V(\alpha^*(t)\bar{x}^*(t))dt + E\int_0^T e^{-\delta t} V^k(\varphi^*(t)\bar{x}^*(t))dt. \tag{98}$$

But observe that

$$\liminf_{T\to\infty} e^{-\delta T} E\Big[\bar{N}_2(T) + \frac{\delta(1-2\delta)}{1-\delta}\int_0^T \bar{N}_2(t)dt\Big] = 0.$$

It follows therefore that it is sufficient to show that

$$\liminf_{T\to\infty} Ee^{-\delta T} U(\bar{X}^*(T), g(T)) \leq 0.$$

But, the optimal terminal wealth is given by

$$\begin{aligned}\bar{X}^*(T) = \bar{x}_0 \exp\Big[\int_0^T \{\big[&\xi(t)(r(t) + g(t) - \mu_B(t) - \alpha(t)) + \pi(t)(\mu(t) - r(t)\kappa) + (r(t + g(t) - \alpha(t))\\ &+ \varphi(t)(r(t + g(t) - r_k(t)) - \frac{1}{2}(\pi(t)\Sigma(t))(\pi(t)\Sigma(t))' + \frac{1}{2}\xi^2(t)\sigma_B(t)\sigma_B(t)\big]dt\\ &+ \pi(t)\Sigma(t)dW(t) - \xi(t)\sigma_B(t)dW_B(t) + \pi(t)JdN(t) - \xi(t)J_B dN_B(t)\}dt\Big].\end{aligned} \tag{99}$$

Multiplying (99) by $e^{-\delta T}$ and taking expectation, we have

$$\begin{aligned}Ee^{-\delta T}\bar{X}^*(T) = e^{-\delta T}\bar{x}_0 \exp\Big[\int_0^T &E\{\xi(t)(r(t) + g(t) - \mu_B(t) - \alpha(t)) + \pi(t)(\mu(t) - r(t)\kappa)\\ &+ \varphi(t)(r(t) + g(t) - r_k(t)) + (r(t) + g(t) - \alpha(t)) - \frac{1}{2}(\pi(t)\Sigma(t))(\pi(t)\Sigma(t))'\\ &+ \frac{1}{2}\xi(t)\sigma_B(t)\sigma_B(t) + \pi(t)\Sigma(t)dW(t) - \xi(t)\sigma_B(t)dW_B(t) + \lambda_S\pi(t)J - \xi(t)\lambda_B J_B\}dt\Big].\end{aligned} \tag{100}$$

It follows that the inequality holds

$$Ee^{-\delta T}|\bar{X}^*(T)| \leq e^{-\delta T}\bar{x}_0\chi_1(T) \leq e^{-\delta T}\bar{x}_0\chi_2(T), \tag{101}$$

where

$$\begin{aligned}\chi_1(T) =& e^{-\delta T}\exp\big(\Big|\int_0^T E\{\xi(t)(r(t) + g(t) - \mu_B(t) - \alpha^*(t)) + \pi^*(t)(\mu(t) - r(t)\kappa\\ &+ \varphi^*(t)(r(t) + g(t) - r_k(t)) + (r(t) + g(t) - \alpha(t)) - \frac{1}{2}(\pi^*(t)\Sigma(t))(\pi^*(t)\Sigma(t))'\\ &+ \frac{1}{2}\xi(t)\sigma_B(t)\sigma_B(t) + \pi^*(t)\Sigma(t) - \xi(t)\sigma_B(t) + \lambda_S\pi^*(t)J - \xi(t)\lambda_B J_B\}dt\Big|\big),\end{aligned} \tag{102}$$

$$\begin{aligned}\chi_2(T) =& e^{-\delta T}\exp\big(\int_0^T E\{|\xi(t)||r(t)| + |g(t)| - |\mu_B(t)| - |\alpha^*(t)|) + |\pi^*(t)(\mu(t) - r(t)\kappa)|\\ &+ |\varphi^*(t)|(|r(t)| + |g(t)| - |r_k(t)|) + |r(t)| + |g(t)| - \alpha(t))| - |\frac{1}{2}(\pi^*(t)\Sigma(t))(\pi^*(t)\Sigma(t))'\\ &+ \frac{1}{2}|\xi(t)|\sigma_B(t)\sigma_B(t) + \pi^*(t)\Sigma(t) - |\xi(t)|\sigma_B(t) + \lambda_S\pi^*(t)J - |\xi(t)|\lambda_B J_B|\}dt\big).\end{aligned} \tag{103}$$

Using (2), (7), (20), Lemma 1 and Lemma 2, and taking the limit and infmum of both sides of (101), we have

$$\lim\inf_{T\to\infty} Ee^{-\delta T}|\bar{X}^*(T)| \le 0. \tag{104}$$

Thus,

$$\lim\inf_{T\to\infty} Ee^{-\delta T}\bar{U}(\bar{X}^*(T), g(T)) \le 0.$$

Therefore, (98) is satisfied and we have that

$$U(\bar{x}_0, g_0) \le E\int_0^T e^{-\delta t}V(\alpha^*(t)\bar{x}^*(t))dt + E\int_0^T e^{-\delta t}V^k(\varphi^*(t)\bar{x}^*(t))dt.$$

Hence, the desire result. □

6 Conclusion

In this section, we give the concluding remarks to this paper. This paper presented a theoretical and an empirical study of the optimal debt ratio, investment management strategy and consumption plan of an investor in a jump diffusion framework under four background risks: investment, income growth rate, taxation and jump risks. The stocks, income growth rate and taxation dynamics were allowed to follow a jump-diffusion process. We investigated the optimal debt ratio, consumption rate and optimal investment strategy for an investor under the power utility function. The optimal investment strategies of the investor include four conponents: a speculative portfolio, a tax risks hedging portfolio strategy, an income growth rate risks hedging portfolio strategy and a risk-free fund that holds only the riskless asset. We found that before loan is taken or given, interest rate on loan to be taken or given, the nominal interest rate, income growth rate, optimal wealth, coefficient of risk averse with respect to investment and coefficient of risk aversion with respect to debt of the investor must be considered. The income growth rate of the investor was found to decrease the debt profile of the investor as it increases. We found in this paper that as the coefficient of risk aversion with respect to debt ratio tends to one, the amount of debt of the investor will be unbearable. It was also found that the higher an investor risk aversion towards debt, the smaller the optimal debt ratio of the investor. It was further found that when tax rate increases, consumption rate decreases and vice versa. Also obtained in this paper is the empirical results using MATLAB R2007b software.

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