Earthline Journal of Mathematical Sciences
E-ISSN: 2581-8147
Volume 15, Number 1, 2025, Pages 85-103
https://doi.org/10.34198/ejms.15125.085103

Mathematical Models to Determine Optimum Inventory Level in a Supply Chain System

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Abstract

The importance of inventory management for organization's effectiveness and profitability cannot be over-emphasized considering the fact that keeping a large quantity of idle goods increases the holding cost of paying for storage facilities and risk of spoilage and theft. On the other hand, ordering for too little quantities sometimes lead to frequent reordering and thereby increasing charges on processing and receiving the items. In this research, we examine a new approach to inventory management by formulating a dynamic programming model in a linear programming form. Many research inventory models have dealt with either the correlation between the management of inventory and profitability or the impact of inventory management on retailers' and producers' profit. In this research we present a new approach which examines the initial and maximum capacities of a warehouse where inventory products are stocked before selling them to prospective customers. The proposed model is a dynamic programming model in a linear programming form which includes both the quantities supplied and demanded in its formulation. Thus, our proposed dynamic model in linear programming form is more applicable to practical situations than what can be found in other models. The incorporation of supply and demand factors distinguish the model from recent inventory models found in literature which are based only on supply factor and profitability. The proposed model has been applied to a warehouse of cement manufacturing industry in Nigeria. From the results obtained in the numerical example, it is observed that periodic supply and demand quantities to produce optimum inventory cost can be determined through the proposed model's algorithm. We also observed that the optimum values of the dual objective function and that of the primal objective function are equal.

1. Introduction

Inventory management for organizational productivity and profitability has remained a major challenge to both manufacturers and retailers due to the huge costs involved in keeping a large quantity of idle goods. These costs include the holding cost of paying for storage facilities, spoilage and theft. On the other hand, ordering for too little quantities sometimes lead to frequent reordering and thereby increasing charges on processing and receiving the items. Hence the challenge of inventory management centers on how to determine the right quantities to order for and the time to place the order so that the total inventory costs which consist of holding cost and ordering cost can be minimized. There are two extreme phenomena every manufacturer and retailer would want to avoid. The first one is keeping too much quantity of idle goods which increases the holding cost

Received: October 31, 2024; Accepted: November 28, 2024; Published: December 13, 2024

2020 Mathematics Subject Classification: 90C39, 90B06.

Keywords and phrases: inventory, dynamic programming, warehouse, supply, demand.

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of paying for storage facilities and risk of spoilage and theft. Secondly, ordering for too little quantities which implies frequent reordering and increase charges on processing and receiving the products, [1]. Warehouses are where inventory items are kept for future sales. The management of warehouses and distribution centers as components of supply chains is very crucial for the survival of any manufacturing industry as well. Warehouses serve the primary purpose of reducing the total cost of production. Thus, inventory control process ensures retailers have enough products in stock to meet customers demand, minimize holding cost, maximize revenue and minimize the risk of stock outs. Inventory control is an integral part of business activities that can determine company's profits or losses. Inventory items such as raw materials, safety stock, work-in-progress, finished products, and maintenance equipment must be well monitored regularly in order to ensure company's productivity and growth. Other items include work-in-progress or partially completed products and finished products or products ready for customers' consumption. Inventory not properly managed can leads to a lot of inventory problems that reduce productivity and growth. However, customers receive quality services, operations are optimized and retailers remained relevant in a competitive environment if inventory problems are properly managed. Inventory management involves different activities such as replenishing finished stocks, monitoring inventory levels, forecasting demand, allocation and data management. Data management includes information relating to quantities of products, customer locations and qualities of products in stock. Inventory data management helps the retailers to make better decisions about when to order, when to give discount and promote products in order to avoid excess inventory which could be best described and interpreted by mathematical models. The main reason for inventory control is to ensure industries have the right quantity of stock to meet customers demand without incurring excess holding costs or out of stock risk. The idea of inventory control centers on keeping enough inventory products for future use or sale. Demand forecasting is an essential part of inventory management which enables industries to align inventory level with anticipated demand level. Techniques used for predicting inventory level are: (a) Time series analysis which involves analyzing the historical data to study the demand's trends and patterns in order to predict anticipated future demand level. The process includes using smoothing and moving average. (b) Qualitative method which uses opinions and judgments of experts, market research and survey to forecast anticipated future demand level. (c) Quantitative method which involves the use of mathematical models and algorithms to predict anticipated future demand level with the aid of historical data.

2. Literature Review

[2] stated that seasonal fluctuations, demand coverage during supply, and forecasting of future demand are some of the reasons for using warehouses. Hence, there is need to develop mathematical models which determine optimal inventory policies in order to strike a balance among the costs of reordering and holding cost as stated in [3]. Logistics is an aspect of supply chain which plans, executes and controls the efficiency and effectiveness of flow and stocked goods from the point of production to the point of customers' consumption and satisfaction requirements, [4]. [5] defined storage as an essential part of logistics which ensures the supply of goods is not interrupted in the time of scarcity. Management impact of inventory on organization's performance is discussed in [6], while an empirical analysis of the relationship between inventory management and profitability is reported in [7]. A mathematical model of inventory minimization and growth in productivity is developed in [8]. A model describing the behavior of inventory and efficacy of credit control is discussed in [9] and was later expanded in [10] in form of retail services performance turnover. The models on correlation between inventory management and financial performance in both large and small businesses are discussed in [11] and [12]. The management of raw material inventory effects on the profit of a company is reported in [13]

while [14] used a regression analysis model to evaluate the effects of inventory turnover period in connection with a company's gross profit. Some mathematical models on warehousing assessment have been developed by many researchers. For examples, an order batching model was developed in [15], an algorithm driven analytic network of warehouses is reported in [16], while an automatic warehousing model in logistic firms is reported in [17]. Warehouse models which discussed inventory control are found in [18-20]. Similarly, a joint order batching in warehouse management is discussed in [21-23].

Warehouse management includes organization and maintenance of all activities that takes place in a warehouse in order to enhance smooth and efficient flow of supply chains. Warehouse operations are prone to errors and challenges no matter how the process seems to be automated and error free. [24] reported that almost \$385 billion is spent annually on warehouses worldwide. Poor management of warehouses can hinder the performance of the entire operation of a warehouse. If the problems associated with a warehouse are identified early enough, solution process could be initiated to avert damages and losses. According to [25], logistics problem arises in industries' warehouse if the inventory control is not effective enough to minimize the lead time, while [26] remarked that effective production activities can be guaranteed if there is an accurate logistics control process in the manufacturing industry. Nigerian industries are already experiencing inventory problems which border on overstock, under-stock, out of stock, and delay in raw material supply, [1]. [27] remarked that more attention should be given to effective management of inventory since inventory forms a critical part of business operations. As earlier stated in [1], keeping a large quantity of idle goods will increase the holding cost of paying for storage facilities, risk of spoilage and theft. On the other hand, ordering for too little quantities will lead to frequent reordering and increase reordering cost as a result of charges on processing and receiving the items. Holding cost, Shortage cost and Salvage cost are the three types of financial costs associated with inventory problems. Holding cost which is also known as storage cost, shortage cost or unsatisfied demand cost is the cost incurred when the demand quantity exceeds supply quantity, while salvage cost is the cost incurred as a result of deteriorating values of goods or discount as stated in [1] and [3]. In this work, we wish to extend the inventory model in [1] in which optimum replenishment policies were derived without numerical illustration. Unlike other existing models, a new approach to determine what quantity to order for at each period is adopted with the aim of maintaining an optimum inventory level which will maximize the revenue accruable to the company. The existing models used only supply factor to the exclusion of demand factor in their mathematical formulations. In a practical situation both supply and demand occur regularly in an ideal supply chain. Hence our proposed dynamic programming model incorporates both supply and demand costs to represent practical situations.

In this article, we proposed dynamic programming model in a linear programming form which considers both supply and demand quantities. The need for consideration of these two quantities in inventory problem becomes imperative because organizations' existence depends on both demand and supply of their products. Hence, the dynamic programming model in linear programming form represents a real situation more than other inventory models. The inclusion of supply and demand factors makes this model an extension of the model in [1] and other recent inventory models such as the ones developed in [21], [22], and [24] discussed earlier in this paper.

3. Problem Definition

General problems commonly associated with warehouse operations are as follow:

Human error: The negative effects of human error in operating a warehouse cannot be over-emphasized. Human errors occur in warehouses' operation in form of poor space management, parcel exceeding a fixed storage limit, products not being able to fit into designated space, wrong labeling etc. Frequent cleaning at times causes wear and tear of standard labels which make the labels unreadable. This can lead to delay in purchase of such products and reduce warehouse efficiency. Labeling problem also includes missed compliance code, scratched print-heads, peeling edges and color of products.

Accidental redundancy: This is a problem created as a result of too many people picking too many products from several sections of the warehouse to satisfy a particular order. This increases manpower cost and time due to misplacement of products.

Messy warehouse layout or poor space management: As reported in [27], only 68% of average warehouse capacity is being utilized by manufacturers and retailers due to poor space management in many warehouses. This problem can be solved by maximizing floor space/vertical space usage at the same time reserving adequate space for staff to walk through. Automation and equipment can also be deployed to reduce manpower cost, enhance easy access to products in the warehouse and classify inventory products in order to ensure their safety.

Bad inventory management: An inventory can be said to be badly managed if: (a) A product that's supposed to be found in a particular section of the warehouse is placed somewhere else. (b) If we refuse to order for products, hoping that we have enough to supply, only to realize that we don't. (c) We place an order for products with the hope that we don't have enough to supply, only to realize that we do. (d) If we have to do away with stocks already received, and have problem of where to keep or supply them.

The above problems can be overcome if we keep accurate and up to date record of our inventory consistently. A study by [28] revealed that 43% of many small businesses have no track record of their inventory. Another survey conducted by [29] reported that 34% of businesses did not get their order in time due to depleted stock. Checking inventory manually leads to miscalculation due to human errors. Hence, many researchers advocate for the use of mathematical models, automated software. A typical example is the one that first collects inventory data through a fixed device and sends the information to a software solution which classifies and track the inventory.

Some Inventory Problem the proposed model is set to address includes the following:

(i) Inaccurate demand forecasting: When demand forecast is inaccurate, it becomes impossible to determine the exact levels of inventory to keep.

(ii) Overstock: This is when a retailer orders for quantities more than customers demand. This leads to wasted resources, tied up capital, excess inventory and expire products.

(iii) Under-stock: This occurs when retailer fails to order for enough inventory products hence there will be shortage of products, lost sales, customer dissatisfaction. Under-stock has the most damaging effects when it occurs during a peak demand and holiday seasons.

(iv) Poor visibility into inventory: This occurs when the retailer does not have knowledge concerning what he has in stock, hence he cannot fulfill orders on time. This leads to loss of sales and poor revenue generation.

4. The Mathematical Model

Model Assumptions

The following assumptions are made in course of formulating the proposed model:

(a) Supply and demand of a particular brand of products are considered.

(b) Periodic supply cost (k'_j) and demand cost (k_j) are known.

(c) The initial quantity of the products required for the warehouse to start off and maximum number of the products it can accommodate are known.

(d) Excess supply and excess demand are considered.

Notations

d_j = Quantity of the product demanded in period j.

s_j = Quantity of the product supplied in period j.

k_j = Average accrued revenue to the company from each product sold in period j.

k'_j = Average cost per product supplied in period j.

q = Initial quantity of the products required for the warehouse to start off.

Q = Maximum number of the products the warehouse can accommodate.

Model Description

Suppose $s_j(t+\delta)$ quantity of the products are supplied at $(t+\delta)$ time in period j into the warehouse and $d_j(t+\delta)$ quantity of the products are demanded or sold in $(t+\delta)$ time of period j where δ is the time difference between supply and demand at time $(t+\delta)$. Let $d_j(t+\delta)$ and $k_j(t+\delta)$ be the number of quantities demanded and the average profit generated by the company from every product sold in period j. Let $k'_j(t+\delta)$ be the average cost per product supplied at time $(t+\delta)$ of period j. As $\delta \to 0$, the notations above become $d_j(t), s_j(t), k_j(t)$ and $k'_j(t) d_j, s_j, k_j$ and k'_j. Hence, if the variables $q, Q, k_j(t)$ and $k'_j(t)$ of an inventory problem are given, the job of the production manager is to determine the optimal quantities d_j and s_j in order to maximize the accruable net revenue of the company. Since the proposed inventory model is a dynamic type that involves many periods, we divide the time horizon into small time intervals. Hence, $d_j(t)$, $s_j(t)$, $k_j(t)$ and $k'_j(t)$ are considered not to vary during the time intervals but in between the intervals, they are assumed to be discontinuous.

In this research, a dynamic programming technique involving backward algorithm is formulated in a linear programming form and used to evaluate periodic supply and demand quantity of trailer bags of cement to a warehouse. The procedure is a backward recursive type of dynamic programming whose computations start from the last stage and ends in the first stage. The proposed model has been applied to a case of a warehouse of cement manufacturing industry in Nigeria. Data used are collected from a cement warehouse which has initial and full capacity to accommodate 163 and 394 bags of cement respectively. Based on the collected data, we are to determine the optimal monthly demand and supply number of bags of cement that will maximize total gross profit accruable to the company in the next 12 months when the warehouse strength is planned to accommodate 394 trailer bags of cement. This implies that the initial quantity of the products (q) required for the warehouse to start off and the maximum quantity (Q) of the products it can accommodate are known.

Model Formulation

The problem of the inventory is to maximize the total cost of the quantity demanded less the total cost of the product sold in period j. i.e. $\sum_{j=1}^{n}\left(k_j d_j - k'_j s_j\right)$.

While the objective function can be stated as:

$$Max \quad z = \sum_{j=1}^{n}\left(k_j d_j - k'_j s_j\right). \tag{1}$$

The inventory model has two types of warehouse constraints.

(i) Excess supply constraints: This states that the total excess inventory supplied in the first i periods must not be above the available space $(Q - q)$ in the warehouse, i.e.

$$\sum_{j=1}^{i}\left(s_j - d_j\right) = -\sum_{j=1}^{i} d_j + \sum_{j=1}^{i} s_j \leq Q - q, \quad i = 1(1)n \tag{2}$$

where $(s_j - d_j) > 0$ is the sum total of inventory by which the warehouse is overstock in j period. The left-hand side of (2) is the net surplus inventory supplied in the first periods of i.

(ii) Excess demand constraints: This occurs when the quantity demanded exceeds the quantity supplied to the warehouse. The demand constraints denote the level to which the warehouse is under-stocked in the first period $(i-1)$ plus the demand quantity in period i and this must not be less than q the initial supply available in the warehouse. If it does, it will imply that the warehouse has only human resources which cannot be since existence of any warehouse in practical situation is based on the combination of stocked products and the seller. This can be expressed mathematically as:

$$\sum_{j=1}^{i-1}\left(d_j - s_j\right) + d_i = \sum_{j=1}^{i} d_j - \sum_{j=1}^{i-1} s_j \leq q, \quad i = 1(1)n \tag{3}$$

where $(d_j - s_j) > 0$ is the sum total of inventory by which the warehouse is under-stock in j period. This can also be referred to as excess demand. In other words, it is the level by which demand quantity exceed supply quantity. The left-hand side of (3) is the net surplus in the inventory minus demand quantity in the first periods $(i-1)$ plus the demand in period i.

(iii) Non-negativity constraints: This implies that the supply and demand variables must be non-negative. That is

$$d_j, s_j \geq 0, \quad j = 1(1)n. \tag{4}$$

Equation (1) denotes the total inventory cost from all the periods. Equations (1)-(4) represent a dynamic programming problem which could be expressed as follows:

Primal Linear Programming Problem

$$\left.\begin{aligned}
&Max\ z=\sum_{j=1}^{n}\left(k_j d_j - k'_j s_j\right)\\
&s.t.\\
&\qquad -\sum_{j=1}^{i} d_j + \sum_{j=1}^{i} s_j \le Q-q,\ \ i=1(1)n\\
&and\quad \sum_{j=1}^{i} d_j - \sum_{j=1}^{i-1} s_j \le q,\ \ i=1(1)n\\
&\qquad d_j, s_j \ge 0,\quad j=1(1)n
\end{aligned}\right\}. \tag{5}$$

Equation (5) is the dynamic programming inventory model consisting of both supply and demand factors. The dynamic programming model in system (5) has two linear constraints and two non-negativity constraints with two types of variables. Equation (5) can be expanded to produce the system in (6).

$$\left.\begin{array}{llll}
Max & z = k_1 d_1 + k_2 d_2 + \cdots + k_n d_n - k'_1 s_1 - k'_2 s_2 - \cdots - k'_n s_n & & \\
s.t. & & & \\
 & -d_1 & +s_1 & \le Q-q\\
 & -d_1 - d_2 & +s_1 + s_2 & \le Q-q\\
 & -d_1 - d_2 - d_3 & +s_1 + s_2 + s_3 & \le Q-q\\
\cdots\cdots & \cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots & \cdots\cdots\\
 & \multicolumn{3}{l}{-d_1 - d_2 - d_3 - \cdots - d_n + s_1 + s_2 + s_3 + \cdots + s_n \le Q-q}\\
 & d_1 & & \le q\\
 & d_1 + d_2 & -s_1 & \le q\\
 & d_1 + d_2 + d_3 & -s_1 - s_2 & \le q\\
\cdots\cdots & \cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots & \cdots\cdots\\
 & \multicolumn{3}{l}{d_1 + d_2 + d_3 + \cdots + d_n - s_1 - s_2 - s_3 - \cdots - s_{n-1} \le q}\\
 & x_j, y_j \ge 0,\quad j=1(1)n & &
\end{array}\right\}. \tag{6}$$

The matrix skeleton of the system (6) is shown in Figure 1

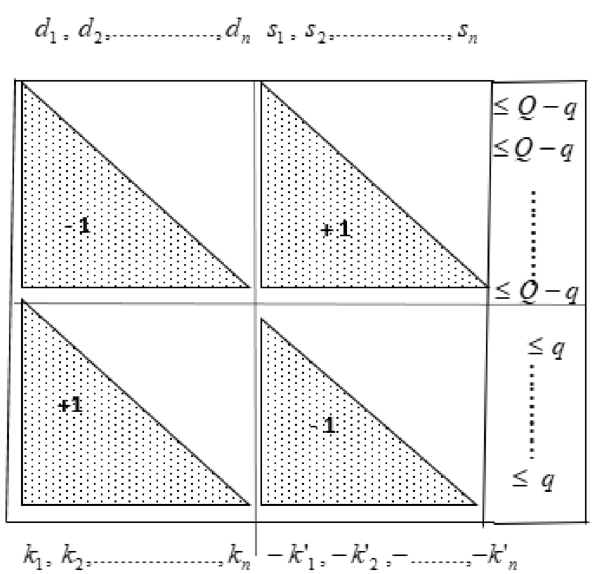

Figure 1: Matrix skeleton of the primal LP model coefficient arrays.

The matrix array is made of triangular blocks which represents dynamic situations. Their coefficients are either $+1$ or -1 depending on the block. We observe that the triangular block in the lower right-hand corner is fewer by one row and one column than the other three. Hence, the DP model for inventory problem based on supply and demand quantities is sparse and consequently has the advantage getting solution fast when computer is used.

If we represent the first n dual variables for the first n constraints in system (6) with $x_1, x_2, \cdots, x_n$ and $y_1, y_2, \cdots, y_n$ for the last n dual variables for dual DP model of the inventory problem, then we shall have:

Dual DP Problem

$$Minimize\ \ w = (Q-q)\sum_{i=1}^{n} x_i + q\sum_{i=1}^{n} x_i \tag{7}$$

$s.t.$

$$-\sum_{i=k}^{n} x_i + \sum_{i=k}^{n} y_i \geq k_l,\ \ l = 1(1)n \tag{8}$$

$$\sum_{i=l}^{n} x_i - \sum_{i=l+1}^{n} y_i \geq -k'_l,\ \ l = 1(1)n \tag{9}$$

$$x_i, y_i \geq 0, \qquad i = 1(1)n. \tag{10}$$

If $l = n$, then, the second summation in equation (9) will not exist. Equations (7)-(10) corresponding to the matrix skeleton of the dual DP problem is shown in Figure 2.

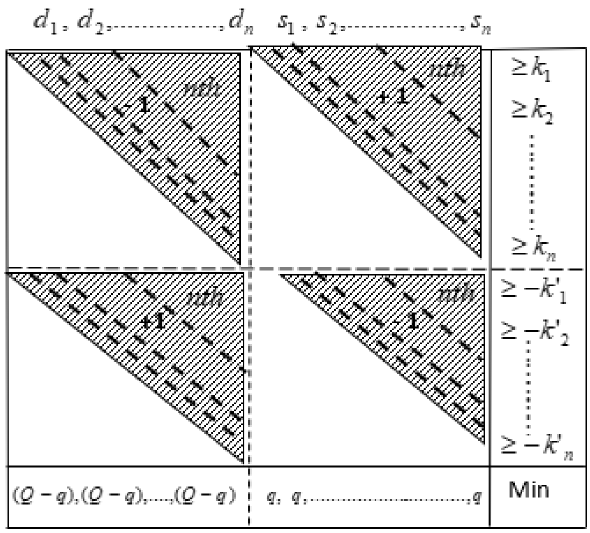

Figure 2: Matrix skeleton of the dual DP model coefficient arrays.

The next step is to define new variables X_l and Y_l such that:

$$X_l = \sum_{i=k}^{n} x_i,\ \ \ l = 1(1)n \tag{11}$$

$$Y_l = \sum_{i=l}^{n} y_i, \quad l = 1(1)n. \tag{12}$$

By duality theorem of dynamic programming, if x_i and y_i are nonnegative, then X_l and Y_l must be nonnegative, but X_l and Y_l are nonnegative does not necessarily imply that $x_i \geq 0$ and $y_i \geq 0, \forall i$. In order to ensure the non-negativity of x_i and y_i, we augment the dual LP problem, expressed in terms of X_l and Y_l by the constraints:

$$X_l \geq X_{l+1}, \quad l = 1(1)n - 1 \tag{13}$$

$$Y_l \geq Y_{l+1}, \quad l = 1(1)n - 1 \tag{14}$$

where the constraints in equations (13) and (14) can only exist if l is not equal to zero since $X_{n+1} = Y_{n+1} = 0$. Hence, we have extra constraints $2(n-1)$ imposed on equations (13) and (14) in order to ensure that the original variables turn out to be nonnegative in the optimal solution.

If we replace $\sum_{i=1}^{n} x_i$ and $\sum_{i=1}^{n} y_i$ in the dual DP problem in equations (7)-(10) and incorporate the constraints in (13) and (14), we obtain the dual stated in system (15).

$$\left.\begin{array}{l} Min \;\; w = (Q-q)X_1 + qY_1 \\ s.t. \\ \qquad X_l \geq Y_{l+1} - k'_l, \;\; l = 1(1)n \\ \qquad X_l \geq X_{k+1}, \;\; l = 1(1)n-1 \\ \qquad X_l \geq 0, \;\; l = 1(1)n \\ \qquad Y_l \geq X_l + k_l, \;\; l = 1(1)n \\ \qquad Y_l \geq Y_{l+1}, \;\; l = 1(1)n-1 \\ \qquad Y_k \geq 0, \;\; l = 1(1)n \end{array}\right\}. \tag{15}$$

The system (15) is the dual dynamic programming problem which begins with period 1 while X_1 and Y_1 are the least values in their solution set. There are *n* dual DP sub-problems in system (15) with each involving the pair $(X_l, Y_l), l = 1(1)n$. The sub-problem consisting of X_l and Y_l is expressed as:

$$\left.\begin{array}{l} Min \;\; w = (Q-q)X_1 + qY_1 \\ s.t. \\ \qquad X_1 \geq X_2 - k'_1 \\ \qquad X_1 \geq X_2 \\ \qquad X_1 \geq 0 \\ \qquad Y_1 \geq X_1 + k_1 \\ \qquad Y_1 \geq Y_2 \\ \qquad Y_1 \geq 0 \end{array}\right\}. \tag{16}$$

The dual DP sub-problem in (16) represents a situation when each of the triangles in Figure 2 is complete. The constraints in system (16) make it difficult to solve the sub-problem in (16) except we first solve X_2 and Y_2. This means that the procedure is a backward recursive type of dynamic programming whose computational procedure starts from the last stage and ends in the first stage. We begin the computation from the last (nth) pair (X_n, Y_n) with sub-problem given as:

$$\left.\begin{array}{ll} Min & w=(Q-q)X_n+qY_n \\ s.t. & \\ & X_n \geq -k'_n \\ & X_n \geq 0 \\ & Y_n \geq X_n+k_n \\ & X_n \geq 0 \end{array}\right\}. \tag{17}$$

The sub-problem in (17) represents the nth triangles in Figure 2. (X_1, Y_1) can be evaluated by using the backward recursive approach. The backward recursive method is used to evaluate the n optimal pairs $(X_n, Y_n), (X_{n-1}, Y_{n-1}), \cdots, (X_1, Y_1)$ as suboptimal solutions to the n sub-problems. The procedure is as follows:

We begin the computation from the last sub-problem in system (17) which is the nth sub-problem. The constraints are:

$\left.\begin{array}{l} X_n \geq -k'_n \\ X_n \geq 0 \end{array}\right\} \Rightarrow$ solution set is $X_n \geq 0$, minimizing w, means we must select the least value of X_n in the solution set.

$$X_n = max(-k'_n, 0) \tag{18}$$

$$\left.\begin{array}{l} Y_n \geq X_n+k_n \\ Y_n \geq 0 \end{array}\right\} \text{ solution set is } Y_n \geq (X_n+k_n) \text{ and } Y_n = max(X_n+k_n, 0). \tag{19}$$

The general case for all the sub-problems is written as follows:

$$\left.\begin{array}{ll} Min & w=(Q-q)X_l+qY_l \\ s.t. & \\ & X_l \geq Y_{l+1}-k'_l \\ & X_l \geq X_{l+1} \\ & X_l \geq 0 \\ & Y_l \geq X_l+k_l \\ & Y_l \geq Y_{l+1} \\ & Y_l \geq 0 \end{array}\right\}. \tag{20}$$

The optimal values of X_l and Y_l are obtained as follows:

$\left.\begin{array}{l} X_l \geq Y_l+k_l \\ X_l \geq X_{l+1} \\ X_l \geq 0 \end{array}\right\}$ solution set is $X_l \geq max(Y_{l+1}-k'_l, X_{l+1}, 0)$

i.e. $X_l = max(Y_{l+1}-k'_l, X_{l+1}, 0)\,, l=(n-1),(n-2),\cdots,3,2,1$

$\left.\begin{array}{l} Y_l \geq X_l+k_l \\ Y_l \geq Y_{l+1} \\ Y_l \geq 0 \end{array}\right\}$ solution set is $Y_l \geq max(X_{l+1}-k_l, Y_{l+1}, 0)$

i.e. $Y_l = max(X_l+k_l, Y_{l+1}, 0)\,, l=(n-1),(n-2),\cdots,3,2,1$

$$\left.\begin{array}{l} X_l = max(Y_{l+1}-k'_l, X_{l+1}, 0)\,, \\ Y_l = max(X_l+k_l, Y_{l+1}, 0) \end{array}\right\} \; l=(n-1),(n-2),\cdots,3,2,1. \tag{21}$$

In each sub-problem, X_l are obtained before the Y_l. The last of these are which are partial sums of k_l and k'_l. X_1 and Y_1 are then put into equation (7) to produce the minimum value of the objective function of the dual dynamic programming problem.

Evaluation of the Original Primal Variables

The objective function value can be obtained by using equation (7) $(\text{i.e.}\,(Q-q)X_1+qY_1)$. After solving the original dual LP problem in equations (7)-(10) to obtain the dual variables x_i and y_i, by the simplex method then, the optimal solution of the primal (in terms of d_j and s_j) would have been obtained using the optimal simplex multipliers of the dual surplus variables. This is impossible due to the modification of the procedure to a DP approach for the dual LP problem. But since X_l and Y_l are partial sums of k_l and k'_l in equations (18), (19), and (21), the objective function in (7) i.e. $[(Q-q)X_1+qY_1]$ can be simplified to the form:

$$[(Q-q)X_1+qY_1] = -\alpha_1 c_1' - \alpha_2 c_2' - \cdots - \alpha_n c_n' + \beta_1 c_1 + \beta_2\beta_2 + \cdots + \beta_n c_n. \tag{22}$$

Equation (22) can further be equated to equation (1), the original primal objective function. That is

$$\sum_{j=1}^{n}\left(k_j d_j - k'_j s_j\right) = -\alpha_1 k'_1 - \alpha_2 k'_2 - \cdots - \alpha_n k'_n + \beta_1 k_1 + \beta_2 k_2 + \cdots + \beta_n k_n. \tag{23}$$

Since α_j and β_j are nonnegative, from equation (23), we have

$$\alpha_j = y_j \quad \text{and} \quad \beta_j = x_j.$$

The dynamic programming technique for the inventory problem discussed in this section can be stated as follows:

Dynamic Programming Algorithm for Manpower Planning

Step 1: Formulate the linear programming model of the inventory problem with known warehouse storage capacity at both initial and last stage subject to excess supply and excess demand quantity constraints.

Step 2: Formulate the dual linear programming problem of the primal dynamic programming problem of step 1.

Step 3: Transform the dual dynamic programming problem in step 2 by representing the dynamic programming variables say X_l and Y_l as sums of the dual variables say x_i and y_i respectively, i.e.

$$X_l = \textstyle\sum_{i=l}^{n} x_i \quad \text{and} \quad Y_l = \textstyle\sum_{i=k}^{n} y_i.$$

Step 4: Start computation by evaluating the nth stage, X_n and Y_n using the constraints of the DP model as:

$$X_n = max(-k'_n, 0) \quad \text{and} \quad Y_n = max(X_n + k_n, 0).$$

Step 5: For the lth stage$(l=(n-1),(n-2),\ldots,3,2,1)$ evaluate X_{l-1} and Y_{l-1} as:

$$\left.\begin{aligned} X_{l-1} &= max(Y_l - k'_{l-1}, X_l, 0) \\ Y_{l-1} &= max(X_{l-1} + k_{l-1}, Y_l, 0) \end{aligned}\right\}, \quad l=(n-1),(n-2),\ldots,3,2,1.$$

Step 6: Beginning from stage $l=n$, and continuing to $l=2$ in step 5, X_1 and Y_1 are calculated as partial sums of k_j and k'_j for evaluation of the dual objective function value i.e. $min\, z = (Q-q)X_1 + qY_1$.

Step 7: By using the duality theorem, equate the dual objective expression to the primal objective expression $(\text{i.e. } Max\ z = \sum_{j=1}^{n} c_j x_j - c_j' y_j)$ in order to determine the periodic quantity demanded (d_j) and the periodic quantity supplied (s_j) that will maximize total accruable revenue to the company. This is stated as:

$$-\alpha_1 ck'_1 - \alpha_2 k'_2 - \cdots - \alpha_n k'_n + \beta_1 k_1 + \beta_2 k_2 + \cdots + \beta_n k_n = \sum_{j=1}^{n} \left(k_j d_j - k'_j s_i\right).$$

Step 8: Calculate the maximum total revenue accruable to the company and stop.

Imposition of Extra Constraints on the Inventory Problem

The two additional constraints that can be imposed on the inventory problem in equation (6) are (i) that the number of products supplied be nonnegative periodically, i.e. $(s_j - d_j) \geq 0,\ j = 1(1)n$. (ii) Initially quantity supplied (q) equal to Q the full capacity of the warehouse, $(\text{i.e.}\, q = Q)$.

The system (6) of the DP problem can be augmented with the two constraints as follows:

$$\left.\begin{array}{ll}
Max\ \ z = c_1 s_1 + c_2 s_2 + \cdots + c_n s_n - c_1^{'} d_1 - c_2^{'} d_2 - \cdots - c_n^{'} d_n & (a) \\
s.t. & \\
\quad -d_1 + s_1 \leq 0 & (b_1) \\
\quad -d_1 - d_2 + s_1 + s_2 \leq 0 & (b_2) \\
\quad -x_1 - x_2 - x_3 + s_1 + s_2 + s_3 \leq 0 & (b_3) \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\
\quad -d_1 - d_2 - d_3 - \cdots - d_n + s_1 + s_2 + s_3 + \cdots + s_n \leq 0 & (b_n) \\
\quad d_1 \leq Q & (c_1) \\
\quad d_1 + d_2 - s_1 \leq Q & (c_2) \\
\quad d_1 + d_2 + d_3 - s_1 - s_2 \leq Q & (c_3) \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\
\quad d_1 + d_2 + d_3 + \cdots + d_n - s_1 - s_2 - s_3 - \cdots - s_{n-1} \leq Q & (c_n) \\
\quad -d_1 + s_1 \geq 0 & (d_1) \\
\quad -d_2 + s_2 \geq 0 & (d_2) \\
\quad -d_3 + s_3 \geq 0 & (d_3) \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\
\quad -d_n + s_n \geq 0 & (d_n) \\
\quad d_1, d_2, \cdots, x_d, s_1, s_2, \cdots, s_n \geq 0 & (e)
\end{array}\right\} \quad (24)$$

Equation (a) in system (24) is the objective function, which is the measure of effectiveness of the inventory system while equations (b_1)-(c_n) are the linear constraints. Equation (e) is the non-negativity constraints. Particularly system (d_1)-(d_n) form the nonnegative excess supply in n periods. The inventory problem described in equations (a) to (e) is similar to the primal dynamic programming in system (6) because they have the same type of objective functions and non-negativity constraints. The warehouse began operation with full capacity inventory (Q) in the second dynamic programming problem. When n extra nonnegative excess supply constraints are added to the second dynamic programming problem, it then has a total of three n-linear constraints instead of two *n-linear constraint* in the former including two different variables. A theorem based on the dynamic programming problem (a)-(e) in system (24) can be stated and proved as follows:

Theorem 1. *Suppose that the warehouse started operation with initial inventory at full capacity* $(i.e.\, q = Q)$ *and that the periodic surplus supply quantity* $(s_j - d_j) \geq 0, j = 1(1)n$ *is nonnegative, then*

(i) $d_j = s_j$;

(ii) $0 \leq d_j \leq Q$ *and* $0 \leq s_j \leq Q$;

(iii) *The dynamic programming problem in equation* (a)-(e) *can be simplified to have only n-variables*, $x_j, (j - 1(1)n)$ *which is the number of quantities demanded in period* j.

Proof. The theorem can be proved as follows:

From equations ($24b_1$) and ($24d_1$), we have

$$-d_1 + s_1 = 0. \tag{25}$$

Putting equation (25) into equation ($24b_2$) and comparing it with equation ($24d_2$), we have

$$-d_2 + s_2 = 0 \tag{26}$$

$$\text{i.e. } d_2 = s_2.$$

Similarly,

$$-d_n + s_n = 0 \tag{27}$$

$$\text{i.e. } d_n = s_n.$$

From equation ($24c_1$)

$$d_1 \le Q. \tag{28}$$

By putting equation (24) into equation ($24c_2$), we have

$$d_2 \le Q. \tag{29}$$

Similarly, putting (25) and (26) into ($24c_3$), we have

$$d_3 \le Q. \tag{30}$$

Similarly, by putting equations (25)-(27) into equation ($24c_n$), we have

$$d_n \le Q. \tag{31}$$

Since $d_j = s_j$ for $j = 1,2,\dots,n$, the objective function can now be stated only in terms of the d_j variables and the linear programming problem in (24a)-(24e) is simplified to:

$$\left.\begin{array}{ll} Max & z = (k_1 - k'_1)d_1 + (k_2 - k'_2)d_2 + \cdots + (k_n - k'_n)d_n \\ s.t. & \\ & d_1 \le Q \\ & d_2 \le Q \\ & \vdots \quad \vdots \quad \vdots \\ & d_n \le Q \\ & d_1, d_2, d_3, \dots, d_n \ge 0 \end{array}\right\}. \tag{32}$$

This completes the required proof.

Numerical Illustration of the Model

In the next section, we present a numerical illustration of the proposed inventory DP model by using data obtained from one of the major Dangote cement distribution centre located in Effurun-Warri metropolitan city in Delta, Nigeria.

Table 1: Annual Demand and Supply Cost from January - December, 2023.

Year	JAN (1)	FEB (2)	MAR (3)	APR (4)	MAY (5)	JUN (6)	JUL (7)	AUG (8)	SEP (9)	OCT (10)	NOV (11)	DEC (12)
k_j	33287	32046	35771	35919	36638	37553	38438	39127	33066	32282	38085	40125
k'_j	30149	32282	33666	34306	37546	34306	37895	36158	32982	30468	37689	36646

In 2023, a Dangote cement depot in Effurun-Warri, Delta State, Nigeria had initial and full capacity to accommodate 162 and 393 trailer bags of cement respectively. Based on the data in Table 1, determine the optimal monthly demand and supply number of trailer bags of cement that will maximize total accruable revenue to the company in the next 12 months when the warehouse strength is planned to accommodate 393 trailer bags of cement. The initial quantity of the products (q) required for the warehouse to start off and maximum quantity (Q) of the products it can accommodate are known.

Solution.

The dual of the dynamic programming model to solve is stated in system (15).

$$\left.\begin{array}{l} Min\ \ w = (Q-q)X_1 + qY_1 \\ s.t. \\ \qquad X_l \geq Y_{l+1} - k'_l, \qquad l = 1(1)n \\ \qquad X_l \geq X_{k+1}, \qquad l = 1(1)n-1 \\ \qquad X_l \geq 0, \qquad l = 1(1)n \\ \qquad Y_l \geq X_l + k_l, \qquad l = 1(1)n \\ \qquad Y_l \geq Y_{l+1}, \ \ l = 1(1)n-1 \\ \qquad Y_k \geq 0, \ \ l = 1(1)n \end{array}\right\} \tag{33}$$

where $X_l = \sum_{i=k}^{n} x_i, Y_l = \sum_{i=k}^{n} y_i, l = 1(1)n\ k_j, k'_j$ are periods j demand and supply costs respectively, and q, Q are the initial quantity of the products required for the warehouse to start off and the maximum quantity of the products it can accommodate respectively. We have 12 periods in this example, $n = 12$ and we proceed to evaluate the X_l and $Y_l (l = 12,11,10,\cdots,2,1)$ using system (21).

$X_{12} = max(-k'_{12}, 0) = 0$

$X_{12} = max(X_{12} + k_{12}, 0) = X_{12} + k_{12} = k_{12}$

$X_{11} = max(Y_{12} - k_{11}, X_{12}, 0) = Y_{12} - k'_{11} = k_{12} - k'_{11}$

$Y_{11} = max(X_{11} + k_{11}, Y_{12}, 0) = X_{11} + k_{11} = k_{12} - k'_{11} + k_{11}$

$X_{10} = max(Y_{11} - k'_{10}, X_{11}, 0) = Y_{11} - k'_{10} = k_{12} - k'_{11} + k_{11} - k'_{10}$

$Y_{10} = max(X_{10} + k_{10}, Y_{11}, 0) = X_{10} + k_{10} = k_{12} - k'_{11} + k_{11} - k'_{10} + k_{10}$

$X_9 = max(Y_{10} - k'_9, X_{10}, 0) = X_{10} = k_{12} - k'_{11} + k_{11} - k'_{10}$

$Y_9 = max(X_9 + k_9, Y_{10}, 0) = X_9 + k_9 = k_{12} - k'_{11} + k_{11} - k'_{10} + k_9$

$X_8 = max(Y_9 - k'_8, X_9, 0) = X_9 = k_{12} - k'_{11} + k_{11} - k'_{10}$

$Y_8 = max(X_8 + k_8, Y_9, 0) = X_8 + k_8 = k_{12} - k'_{11} + k_{11} - k'_{10} + k_8$

$X_7 = max(Y_8 - k'_7, X_8, 0) = Y_8 - k'_7 = k_{12} - k'_{11} + k_{11} - k'_{10} + k_8 - k'_7$

$Y_7 = max(X_7 + k_7, Y_8, 0) = X_7 + k_7 = k_{12} - k'_{11} + k_{11} - k'_{10} + k_8 - k'_7 + k_7$

$X_6 = max(Y_7 - k'_6, X_7, 0) = Y_7 - k'_6 = k_{12} - k'_{11} + k_{11} - k'_{10} + k_8 - k'_7 + k_7 - k'_6$

$Y_6 = max(X_6 + k_6, Y_7, 0) = X_6 + k_6 = k_{12} - k'_{11} + k_{11} - k'_{10} + k_8 - k'_7 + k_7 - k'_6 + k_6$

$X_5 = max(Y_6 - k'_5, X_6, 0) = Y_6 - k'_5 = k_{12} - k'_{11} + k_{11} - k'_{10} + k_8 - k'_7 + k_7 - k'_6 + k_6 - k'_5$

$Y_5 = max(X_5 + k_5, Y_6, 0) = Y_6 = k_{12} - k'_{11} + k_{11} - k'_{10} + k_8 - k'_7 + k_7 - k'_6 + k_6$

$X_4 = max(\rightleftarrows Y_5 - k'_4, Y_5, 0) = Y_5 - k'_4 = k_{12} - k'_{11} + k_{11} - k'_{10} + k_8 - k'_7 + k_7 - k'_6 + k_6 - k'_4$

$Y_4 = max(X_4 + k_4, Y_5, 0) = X_4 + k_4 = k_{12} - k'_{11} + k_{11} - k'_{10} + k_8 - k'_7 + k_7 - k'_6 + k_6 - k'_4 + k_4$

$$\begin{aligned} X_3 &= max(Y_4 - k'_3, X_4, 0) = Y_4 - k'_3 \\ &= k_{12} - k'_{11} + k_{11} - k'_{10} + k_8 - k'_7 + k_7 - k'_6 + k_6 - k'_4 + k_4 - k'_3 \end{aligned}$$

$$\begin{aligned} Y_3 &= max(X_3 + k_3, Y_4, 0) = X_3 + k_3 \\ &= k_{12} - k'_{11} + k_{11} - k'_{10} + k_8 - k'_7 + k_7 - k'_6 + k_6 - k'_4 + k_4 - k'_3 + k_3 \end{aligned}$$

$$\begin{aligned} X_2 &= max(Y_3 - k'_2, X_3, 0) = Y_3 - k'_2 \\ &= k_{12} - k'_{11} + k_{11} - k'_{10} + k_8 - k'_7 + k_7 - k'_6 + k_6 - k'_4 + k_4 - k'_3 + k_3 - k'_2 Y_2 \\ &= max(X_2 + k_2, Y_3, 0) = Y_3 \\ &= k_{12} - k'_{11} + k_{11} - k'_{10} + k_8 - k'_7 + k_7 - k'_6 + k_6 - k'_4 + k_4 - k'_3 + k_3 \end{aligned}$$

$$\begin{aligned} X_1 &= max(Y_2 - k'_1, X_2, 0) = Y_2 - k'_1 \\ &= k_{12} - k'_{11} + k_{11} - k'_{10} + k_8 - k'_7 + k_7 - k'_6 + k_6 - k'_4 + k_4 - k'_3 + k_3 - k'_1 \end{aligned}$$

$$\begin{aligned} Y_1 &= max(X_1 + k_1, Y_2, 0) = X_1 + k_1 \\ &= k_{12} - k'_{11} + k_{11} - k'_{10} + k_8 - k'_7 + k_7 - k'_6 + k_6 - k'_4 + k_4 - k'_3 + k_3 - k'_1 + k_1 X_1 \\ &= (k_3 + k_4 + k_6 + k_7 + k_8 + k_{11} + k_{12}) - (k'_1 + k'_3 + k'_4 + k'_6 + k'_7 + k'_{10} + k'_{11}) \\ &= 265018 - 238479 = \mathbf{26,539} \end{aligned}$$

$Y_1 = (k_1 + k_3 + k_4 + k_6 + k_7 + k_8 + k_{11} + k_{12}) - (k'_1 + k'_3 + k'_4 + k'_6 + k'_7 + k'_{10} + k'_{11})$

$= 298305 - 238479 = \mathbf{59,826}.$

The dual objective function value is given as:

$$(Q - q)X_1 + qY_1 = 231(26539) + 162(59826) = 6130509 + 9691812 = \text{₦}\mathbf{15,822,321}.$$

Thus, **₦15,822,321** is the supply/demand policy **cost** of the inventory problem obtained from the dynamic programming algorithm of the model.

In order to obtain the solution to the primal dynamic programming problem, we expand the LHS (34) and substitute as follows:

$$(Q - q)X_1 + qY_1 = \sum_{j=1}^{12}(k_j d_j - k'_j s_j) \tag{34}$$

L.H.S $= 231(k_3 + k_4 + k_6 + k_7 + k_8 + k_{11} + k_{12}) - 231(k'_1 + k'_3 + k'_4 + k'_6 + k'_7 + k'_{10} + k'_{11})$

$+162(k_1 + k_3 + k_4 + k_6 + k_7 + k_8 + k_{11} + k_{12}) - 162(k'_1 + k'_3 + k'_4 + k'_6 + k'_7 + k'_{10} + k'_{11})$

$$= 162k_1 + 0k_2 + 393k_3 + 393k_4 + 0k_5 + 393k_6 + 393k_7 + 393k_8 + 0k_9 + 0k_{10} + 393k_{11} + 393k_{12} - 393k'_1 - 0k'_2 - 393k'_3 - 393k'_4 - 0k'_5 - 393k'_6 - 393k'_7 - 0k'_8 - 0k'_9 - 393k'_{10} - 393k'_{11} - 0k'_{12}$$

R.H.S $= d_1k_1 + d_2k_2 + d_3k_3 + d_4k_4 + d_5k_5 + d_6k_6 + d_7k_7 + d_8k_8 + d_9k_9 + d_{10}k_{10} + d_{11}k_{11} + d_{12}k_{12} - s_1k'_1 - s_2k'_2 - s_3k'_3 - s_4k'_4 - s_5k'_5 - s_6k'_6 - s_7k'_7 - s_8k'_8 - s_9k'_9 - s_{10}k'_{10} - s_{11}k'_{11} - s_{12}k'_{12}$

$\Rightarrow d_1 = 162, d_2 = 0, d_3 = 393, d_4 = 393, d_5 = 0, d_6 = 393, d_7 = 393, d_8 = 393, d_9 = 0, d_{10} = 0, d_{11} = 393, d_{12} = 393,$

$s_1(\text{i.e. } d_{13}) = 393, s_2(\text{i.e. } d_{14}) = 0, s_3(\text{i.e. } d_{15}) = 393, s_4(\text{i.e. } d_{16}) = 393, s_5(\text{i.e. } d_{17}) = 0,$

$s_6(\text{i.e. } d_{18}) = 393, s_7(\text{i.e. } d_{19}) = 393, s_8(\text{i.e. } d_{20}) = 0, s_9(\text{i.e. } d_{21}) = 0, s_{10}(\text{i.e. } d_{22}) = 393,$

$s_{11}(\text{i.e. } d_{23}) = 393, s_{12}(\text{i.e. } d_{24}) = 0.$

The value of the primal objective function is:

$$162(33287) + 393(35771 + 35919 + 37553 + 38438 + 39127 + 38085 + 40125)$$

$$-393(30149 + 33666 + 34306 + 34306 + 37895 + 30468 + 37689)$$

$$= ₦15{,}822{,}321.$$

Thus, the value of the primal objective function is ₦15,822,321.

$$= 5392332 + 393(265018) - 393(238479) = 15822321.$$

The dual objective function value (i.e. $(Q-.q)X_1 + hY_1$) and that of the primal (i.e. $\sum_{j=1}^{12}(k_jd_j - k'_js_j)$) are equal in this solution. This results tally with the Duality Theorem for symmetric duals.

5. Result Analysis

The results obtained from the solution to the inventory problem presented in Section 4.0 show that the demand for the month of January was 162 bags of cement (i.e. $d_1 = 162$,) while supply was 393 bags. In the months of February, May, September and October there should be no supply since $d_2 = 0, d_5 = 0, d_9 = 0$ and $d_{10} = 0$ while in the months of March, April, June, July, August, November and December the demand quantity is 393 bags respectively. That is $d_3 = 393$, $d_4 = 393$, $d_6 = 393$, $d_7 = 393$, $d_8 = 393$, $d_{11} = 393$ and $d_{12} = 393$. We also have the following values regarding the supply quantities from the solution: $s_1(\text{i.e. } d_{13}) = 393$, $s_2(\text{i.e. } d_{14}) = 0, s_3(\text{i.e. } d_{15}) = 393, s_4(\text{i.e. } d_{16}) = 393, s_5(\text{i.e. } d_{17}) = 0, s_6(\text{i.e. } d_{18}) = 393, s_7(\text{i.e. } d_{19}) = 393$.

This implies that in the months of February, May, August, September and December there should be no supply if the inventory cost is to be minimized. While every other month should have up to 393 bags of cement supplied.

The dual objective function value (i.e. $(Q-.q)X_1 + hY_1$) and that of the primal (i.e. $\sum_{j=1}^{12}(k_jd_j - k'_js_j)$) have the same value. That is ₦15,822,321, which is the optimum cost of the inventory. This results tally with the Duality Theorem for symmetric duals.

6. Conclusion

In this article we have examined two extreme phenomena every manufacturer and retailer would want to avoid: keeping a large quantity of idle goods which increases the holding cost of paying for storage facilities and risk of spoilage and theft. On the other hand, ordering for too little quantities which imply frequent reordering which increases charges on processing and receiving the items. Hence, the effects of inventory management on organization's effectiveness and profitability cannot be over-emphasized. In this research, we examine a new approach to inventory management by formulating a dynamic programming model in a linear programming form. While many of the existing inventory models have focused on the correlation between inventory management and profitability, our interest is to examine the initial and maximum capacities of a warehouse where inventory products are stocked before they are sold. The aim of the study is to determine optimum inventory cost using supply and demand factors in formulating dynamic programming in a linear programming form. The work has some similar concepts with batching routing problem in warehouse management discussed in [21], but more of an extension of earlier work of [1] in which optimum replenishment policies were derived without numerical illustration. The proposed model has been applied to a cement warehouse of a manufacturing industry in Nigeria. From the results obtained in the numerical example, it is observed that periodic supply and demand quantities to produce optimum inventory cost can be determined through the proposed model's algorithm. We also observed that the optimum values of the dual objective function and that of the primal objective function are the same.

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