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On Geometry of Some Subspaces in Hornich Space

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Abstract

Let $\mathcal{H}$ be the collection of all locally univalent analytic functions f defined on the unit disk $\mathbb{D}$ with the normalization $f(0) = f'(0) - 1 = 0$, and $\mathcal{S} \subseteq \mathcal{H}$ be the class of all univalent functions. For $f, g \in \mathcal{H}$ and $r \in \mathbb{C}$, the Hornich operators are defined as

$$r \odot f(z) := \int_0^z \{f'(\xi)\}^r \mathrm{d}\xi \quad \text{and} \quad f \oplus g(z) := \int_0^z f'(\xi)g'(\xi)\mathrm{d}\xi.$$

We study geometric properties of some subclasses of $\mathcal{S}$ in the sense of the Hornich space $(\mathcal{H}, \odot, \oplus)$. In fact, we prove that the classes of strongly convex functions of order β, Noshiro-Warschawski functions, and strongly Ozaki close-to-convex functions are all convex in $(\mathcal{H}, \odot, \oplus)$, which generalize some known results. Meanwhile, for $M, N \in \mathcal{S}$, let $T[M,N] := \{(r,s) \in \mathbb{C}^2 : r \odot f \oplus s \odot g \in N, \ for \ \forall f, g \in M\}$. We give the precise descriptions of $T[M,N]$ for some $M, N \in \mathcal{S}$.

1 Introduction

In this paper, we let $\mathcal{H}$ be the class of all locally univalent analytic functions f defined on the unit disk $\mathbb{D} = \{z : |z| < 1\}$ normalized by $f(0) = f'(0) - 1 = 0$, and $\mathcal{S} \subseteq \mathcal{H}$ be the class of all univalent functions. In [9], for $f, g \in \mathcal{H}$ and $r \in \mathbb{C}$, Hornich introduced the Hornich operations on $\mathcal{H}$ as follows,

$$r \odot f(z) := \int_0^z \{f'(\xi)\}^r \mathrm{d}\xi \quad \text{and} \quad f \oplus g(z) := \int_0^z f'(\xi)g'(\xi)\mathrm{d}\xi, \tag{1.1}$$

where the branch of $(f')^\alpha = \exp(\alpha \log f')$ is taken so that $(f')^\alpha(0) = 1$. With the operations $\odot$ and $\oplus$, the set $\mathcal{H}$ become a vector space which is now called Hornich space $(\mathcal{H}, \odot, \oplus)$.

If we denote the pre-Schwarzian derivative $T_f = f''/f'$ for $f \in \mathcal{H}$, then it is interesting that, for any $f, g \in \mathcal{H}$ and $r \in \mathbb{C}$,

$$T_{f \oplus g} = T_f + T_g, T_{r \odot f} = rT_f. \tag{1.2}$$

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Which means that pre-Schwarzian derivative is a linear operation on $(\mathcal{H}, \odot, \oplus)$.

As we know, it is of great interesting to describe the geometric shape of $\mathcal{S}$ or its subspaces, such as universal Teichmüller space ([4,34]), under pre-Schwarzian derivative. Since the property (1.2), the shape problem of subclass of $\mathcal{S}$ under pre-Schwarzian derivative is equivalent to the geometry of the corresponding subspace in $(\mathcal{H}, \odot, \oplus)$. Following this idea, many mathematicians are dedicated to research in this area. For example, as early as the 1960s and 1970s, Pfaltzgraff [27] showed that $r \odot \mathcal{S} \subset \mathcal{S}$ if $|r| < \frac{1}{4}$ and Roster [31] proved that $r \odot \mathcal{S}$ is not included in $\mathcal{S}$ when $|r| > 1/3$. The exact value r which makes $r \odot \mathcal{S} \subset \mathcal{S}$ hold is still unknown. For more study on this topic, we refer to [2,3,5,13,14,17,23,25,28].

In this paper, we will continue to study the geometric properties related to Hornich space $(\mathcal{H}, \odot, \oplus)$. The first objective is to extend some known geometric properties to subspaces of $\mathcal{S}$ in $(\mathcal{H}, \odot, \oplus)$. In the following, we will introduce some subspaces of $\mathcal{S}$ and list our results.

We say a function $f \in \mathcal{S}$ is convex if its image $f(\mathbb{D})$ is a convex domain in $\mathbb{C}$, and let $\mathcal{K}$ be the collection of all convex functions. It is well known that $f \in \mathcal{K}$ if and only if

$$\mathrm{Re}\{1 + \frac{zf''(z)}{f'(z)}\} > 0, z \in \mathbb{D}.$$

For $0 < \beta < 1$, we let $\mathcal{SK}(\beta)$ be the strongly convex functions of order β which consisting of all function $f \in \mathcal{S}$ satisfying

$$\left|\arg\{1 + \frac{zf''(z)}{f'(z)}\}\right| < \frac{\pi\beta}{2}, z \in \mathbb{D}.$$

In addition, using subordination, Janowski [11] extended $\mathcal{K}$ to the class $\mathcal{K}(A, B)$ which consists of all $f \in \mathcal{S}$ satisfying

$$1 + \frac{zf''(z)}{f'(z)} \prec \frac{1 + Az}{1 + Bz}, z \in \mathbb{D},$$

where $-1 \leq B < A \leq 1$. As we know, $\mathcal{SK}(\beta)$ and $\mathcal{K}(A, B)$ are generalizations of $\mathcal{K}$, which were studied by many mathematicians [12,21,30,36,37].

In 1974, Kim [14] et al. study the geometric properties of $\mathcal{K}$ and proved that $\mathcal{K}$ is convex in $(\mathcal{H}, \odot, \oplus)$. Since $\mathcal{SK}(\beta)$ and $\mathcal{K}(A, B)$ are generalizations of $\mathcal{K}$, we consider the convexity problem of them in $(\mathcal{H}, \odot, \oplus)$ and prove the following theorem.

Theorem 1.1. *Both the subclasses $\mathcal{SK}(\beta)$ and $\mathcal{K}(A, B)$ are convex in $(\mathcal{H}, \odot, \oplus)$.*

A function $f \in \mathcal{S}$ is called starlike respect to 0 if $f(\mathbb{D})$ is a starlike domain respect to 0 in $\mathbb{C}$, and we let $\mathcal{S}^*$ denote the class of all starlike function. It is well known that $f \in \mathcal{S}^*$ if and only if

$$\mathrm{Re}\{\frac{zf'(z)}{f(z)}\} > 0, z \in \mathbb{D}.$$

In 1973, for $0 < \alpha < 1$, Janowski [11] introduced $\mathcal{S}^*(\alpha)$ the starlike functions of order α which is the collection of all function $f \in \mathcal{S}$ such that

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > \alpha, z \in \mathbb{D}.$$

The growth theorem and other characteristics of $\mathcal{S}^*(\alpha)$ were studied by many mathematicians [35]. By a counterexample, Sugawa [13] have shown that $\mathcal{S}^*$ is not convex in $(\mathcal{H}, \odot, \oplus)$. Later, in 2007, Lamprecht [17] proved that $\mathcal{S}^*$ is starlike respect to identity in $(\mathcal{H}, \odot, \oplus)$. We consider the geometric property of $\mathcal{S}^*(\alpha)$ in $(\mathcal{H}, \odot, \oplus)$, and prove the following theorem.

Theorem 1.2. *For* $\forall \alpha \in [0.156, 1)$*,* $\mathcal{S}^*(\alpha)$ *is not convex in* $(\mathcal{H}, \odot, \oplus)$*.*

A function $f \in \mathcal{S}$ is a close-to-convex function if and only if there exists a function $g \in \mathcal{K}$ such that

$$\operatorname{Re}\{\frac{f'(z)}{g'(z)}\} > 0.$$

Let $\mathcal{C}$ be the class of all close-to-convex functions. Allu et al. [19] generalized the concept of close-to-convex functions to the class of strongly Ozaki close-to-convex functions $\mathcal{F}(\lambda, \alpha)$, by introducing two parameters λ and α, where $\alpha \in (0, 1]$ and $\lambda \in [\frac{1}{2}, 1]$. A function $f \in \mathcal{S}$ is a strongly Ozaki close-to-convex function if

$$\left|\arg\left[\frac{2\lambda - 1}{2\lambda + 1} + \frac{2}{2\lambda + 1}\left(1 + \frac{zf''(z)}{f'(z)}\right)\right]\right| < \frac{\alpha\pi}{2}, \ z \in \mathbb{D}. \tag{1.3}$$

Later, $\mathcal{F}(\lambda, \alpha)$ was studied by Sevtap et al. [8, 35].

Since Kim et al. [14] gave the convexity of $\mathcal{C}$ in $(\mathcal{H}, \odot, \oplus)$, we consider the geometric property of $\mathcal{F}(\lambda, \alpha)$ and derive the following theorem.

Theorem 1.3. *For* $\alpha \in (0, 1]$ *and* $\lambda \in [\frac{1}{2}, 1]$*,* $\mathcal{F}(\lambda, \alpha)$ *is convex in* $(\mathcal{H}, \odot, \oplus)$*.*

A function $f \in \mathcal{H}$ is called a Noshiro-Warschawski function if f satisfies

$$\operatorname{Re} f'(z) > 0, \forall z \in \mathbb{D},$$

and we denoted by $\mathcal{R}$ the class of Noshiro-Warschawski functions. Hotta et al. [10] proved that class $\mathcal{R}$ is starlike in $(\mathcal{H}, \odot, \oplus)$. In this paper, We show stronger geometric properties of $\mathcal{R}$ as follows.

Theorem 1.4. *Let* $\mathcal{R}$ *be the class of Noshiro-Warschawski functions, then* $\mathcal{R}$ *is convex in* $(\mathcal{H}, \odot, \oplus)$*.*

To state the second objective of this paper, let us introduce some notations. For $f,\ g \in \mathcal{H}$ and $r, s \in \mathbb{C}$, we define

$$I_r(z) := r \odot f = \int_0^z \left(f'(\xi)\right)^r \, \mathrm{d}\xi \quad \textit{and}$$

$$I_{r,s}(z) := (r \odot f) \oplus (s \odot g)(z) = \int_0^z \left(f'(\xi)\right)^r \left(g'(\xi)\right)^s \, \mathrm{d}\xi. \tag{1.4}$$

For M and N be two non-empty subclasses of $\mathcal{H}$, we denote

$$T[M,N] := \left\{(r,s) \in \mathbb{C}^2 : \forall f,\ g \in M, I_{r,s} \in N\right\}. \tag{1.5}$$

By the definition of $T[M,N]$, it is not difficult to find that, for any $a, b \in \mathbb{C}$,

$$T[a \odot M, b \odot N] = \frac{b}{a} T[M,N], \tag{1.6}$$

where $a \odot M = \{f = a \odot g : g \in M\}$ and $\frac{b}{a}T[M,N] = \{(\frac{b}{a}r, \frac{b}{a}s), (r,s) \in T[M,N]\}$.

For $M, N \subset \mathcal{S}$, it is interesting to describe the shape of $T[M,N]$. For example, Kim [15] et al. proved that $T[\mathcal{K},\mathcal{C}] = \left\{(r,s) \in \mathbb{C}^2 : -\frac{1}{2} \le r, s \le \frac{3}{2}, -\frac{1}{2} \le r+s \le \frac{3}{2} \right\}$, and $T[\mathcal{C},\mathcal{C}] = \left\{(r,s) \in \mathbb{C}^2 : -\frac{1}{3} \le r - 3s \le 1, -\frac{1}{3} \le r + 3s \le 1 \right\}$. For more results related to this topic, we refer to [1, 2, 6, 10, 13, 16, 20, 25, 26, 33]. In this paper, we will continue to describe $T[M,N]$ precisely for some subspace $M, N \in \mathcal{S}$.

$\mathcal{F}(\lambda) := \mathcal{F}(\lambda, 1)$ is called the connection of Ozaki close-to-convex functions, and $\mathcal{K}(\alpha) := \mathcal{K}(1-2\alpha, -1)$ with $0 \le \alpha < 1$ is known as the class of α-convex functions. By definition, a function $f \in \mathcal{S}$ is a Ozaki close-to-convex function if and only if

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \frac{1}{2} - \lambda.$$

$\mathcal{F}(\lambda)$ and $\mathcal{K}(\alpha)$ were studied by Ponnusamy et al. [29, 37]. For $\mathcal{F}(\lambda)$, $\mathcal{K}(\alpha)$ and $\mathcal{R}$, we give the description of $T[M,N]$ and get the following three theorems.

Theorem 1.5. *For* $\forall \lambda \in [1/2, 1]$,

1. $T[\mathcal{F}(\lambda), \mathcal{K}] = \left\{(r,s) \in \mathbb{C}^2 : r, s \ge 0,\ 0 \le r+s \le \frac{2}{1+2\lambda}\right\}$;
2. $T[\mathcal{F}(\lambda), \mathcal{C}] = \left\{(r,s) \in \mathbb{C}^2 : -\frac{1}{1+2\lambda} \le r, s \le \frac{3}{1+2\lambda}, -\frac{1}{1+2\lambda} \le r+s \le \frac{3}{1+2\lambda}\right\}$.

Theorem 1.6. $T[\mathcal{K}(\alpha), \mathcal{K}(\alpha)] = \{(r,s) \in \mathbb{R}^2 : r, s \ge 0, 0 \le r+s \le 1\}$.

Theorem 1.7. $T[\mathcal{R}, \mathcal{R}] = \{(r,s) \in \mathbb{R}^2 : -1 \le r, s \le 1, -1 \le r+s \le 1\}$.

2 Convexity of some subclasses in the Hornich space

We will give the proofs of Theorems 1.1-1.4 in this section. As a preparation for proving our main theorem, we need to introduce some results as lemmas.

By definition, it is not difficult to derive the following result about $I_{r,s}$ in equation (1.4).

Lemma 2.1. *For* $\forall f,\ g \in \mathcal{S}$ *and* $\forall r, s \in \mathbb{C}$,

$$1 + \frac{zI''_{r,s}}{I'_{r,s}} = r(1 + \frac{zf''(z)}{f'(z)}) + s(1 + \frac{zg''(z)}{g'(z)}) + (1 - r - s). \tag{2.1}$$

Proof of Lemma 2.1. For $\forall f,\ g \in \mathcal{H}$ and $s, r \in \mathbb{C}$, we have

$$\begin{aligned} 1 + \frac{zI''_{r,s}}{I'_{r,s}} &= 1 + \frac{rzf''(z)}{f'(z)} + \frac{szg''(z)}{g'(z)} \\ &= r(1 + \frac{zf''(z)}{f'(z)}) + s(1 + \frac{zg''(z)}{g'(z)}) + (1 - r - s). \end{aligned}$$

□

By Lemma 2.1, we give the proof of Theorem 1.1.

Proof of Theorem 1.1. We divide the proof of Theorem 1.1 into two parts.

Part (i): $\mathcal{SK}(\beta)$ is convex

For $\forall \beta \in (0, 1)$, $\forall f,\ g \in \mathcal{SK}(\beta)$ and $\forall r \in [0, 1]$, it follows from the equation (2.1) that

$$\left| \arg\{1 + \frac{zI''_{r,1-r}}{I'_{r,1-r}}\} \right| = \left| \arg\{r(1 + \frac{zf''(z)}{f'(z)}) + (1 - r)(1 + \frac{zg''(z)}{g'(z)})\} \right| < \frac{\pi\beta}{2}.$$

This shows that $I_{r,1-r}(z) \in \mathcal{SK}(\beta)$, i.e. the class $\mathcal{SK}(\beta)$ is convex.

Part (ii): $\mathcal{K}(A, B)$ is convex in $(\mathcal{H}, \odot, \oplus)$.

For $\forall A, B$ satisfying $-1 \leq A < B \leq 1$, $\forall r \in [0, 1]$ and $\forall f,\ g \in \mathcal{SK}(A, B)$, by the equation (2.1) we have

$$1 + \frac{zI'_{r,1-r}(z)}{I''_{r,1-r}(z)} = r\{1 + \frac{zf''(z)}{f'(z)}\} + (1 - r)\{1 + \frac{zg''(z)}{g'(z)}\} \prec \frac{1 + Az}{1 + Bz},$$

from which $\mathcal{K}(A, B)$ is convex.

□

To prove Theorem 1.2, we need to introduce some results on the Hankel determinant. As a generalization of coefficient estimates, the Hankel determinant was introduced by Noonan and Thomas [22] in 1976 and was studied by Noor et al. [7, 18, 24]. Here we only consider the second Hankel determinant $H_2(2)$ of $f \in \mathcal{S}$, defined by $H_2(2) := |a_2a_4 - a_3^2|$ for $f(z) = z + \sum_{n=2}^{\infty} a_nz^n$. In 2022, by $H_2(2)$, Thomas et al. [32] provided a necessary condition for $f \in \mathcal{S}^*(\alpha)$ as follows.

Lemma A. *Let* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ *and* $\alpha \geq 0$. *If* $f \in \mathcal{S}^*(\alpha)$, *then* $H_2(2) \leq \frac{1}{3}(1-\alpha)^2|(3-2\alpha)(2\alpha-1)|$.

We prove Theorem 1.2 by a counterexample.

Proof of Theorem 1.2. For $\forall \alpha \in [0, 1)$, we define two functions f_1 and f_2 by $f_1(z) = \gamma z/(1-\gamma z)^{2-2\alpha}$ and $f_2(z) = \bar{\gamma} z/(1-\bar{\gamma} z)^{2-2\alpha}$, where $\gamma = e^{i\pi/4}$. Notice that f_1 and f_2 both are rotations of the function $z/(1-z)^{2-2\alpha}$. Therefore, $f_1, f_2 \in \mathcal{S}^*(\alpha)$. Let G be the midpoint of f_1 and f_2, we have

$$\begin{aligned} G(z) &= 0.5 \odot f_1(z) \oplus 0.5 \odot f_2(z) \\ &= \int_0^z [f_1'(\zeta)]^{0.5} [f_2'(\zeta)]^{0.5} d\zeta \\ &= \int_0^z \sqrt{\frac{1+(1-2\alpha)\gamma\zeta}{(1-\gamma\zeta)^{3-2\alpha}}} \sqrt{\frac{1+(1-2\alpha)\overline{\gamma}\zeta}{(1-\overline{\gamma}\zeta)^{3-2\alpha}}} d\zeta \\ &= \int_0^z \sqrt{\frac{1+\sqrt{2}\,(1-2\alpha)\,\zeta + (1-2\alpha)^2\,\zeta^2}{\left(1-\sqrt{2}\zeta+\zeta^2\right)^{3-2\alpha}}} d\zeta. \end{aligned}$$

With the help of Matlab, we calculate the values of the second Hankel determinant $H_2(2)$ of $G(z)$ and $\frac{1}{3}(1-\alpha)^2|(3-2\alpha)(2\alpha-1)|$, denoted by H_G and $H_{\mathcal{S}^*(\alpha)}$ respectively, and give the following graph.

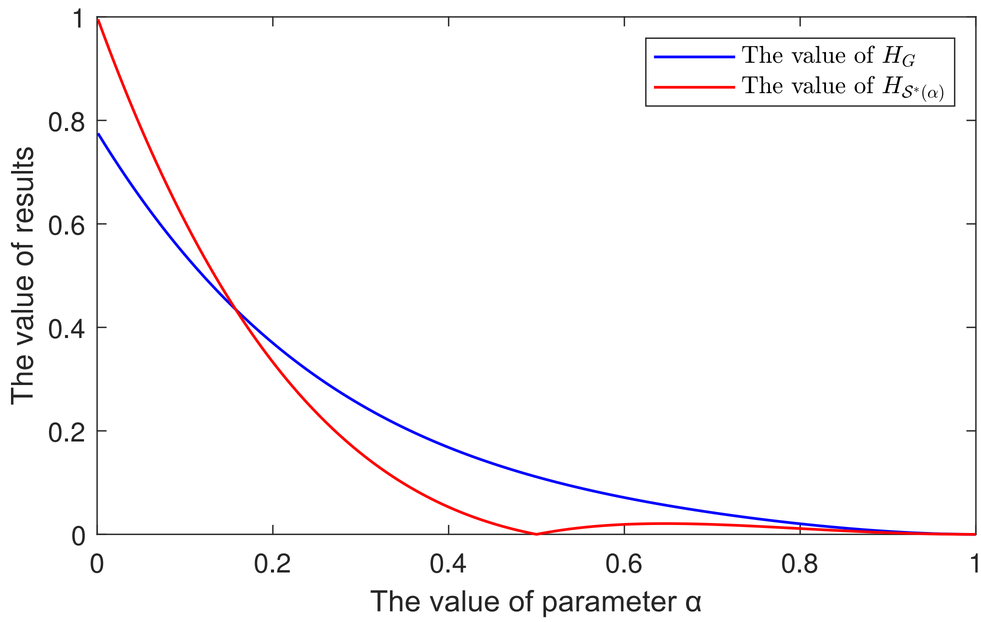

Figure 1: Comparison of H_G and $H_{\mathcal{S}^*(\alpha)}$.

As shown in Figure 1, by Lemma A, we can see that for $\forall\alpha \in [0.156, 1)$, $H_G \geq \frac{1}{3}(1-\alpha)^2|(3-2\alpha)(2\alpha-1)|$, which implies $\mathcal{S}^*(\alpha)$ is not convex in $(\mathcal{H}, \odot, \oplus)$. □

Next, we turn our attention from the proof of Theorem 1.2 to the proof of Theorem 1.3, where we will discuss the convexity of $\mathcal{F}(\lambda, \alpha)$.

Proof of Theorem 1.3. For $\alpha \in (0, 1]$, $\lambda \in [\frac{1}{2}, 1]$ and $\forall r \in [0, 1]$, let f and g be two functions belonged to the class $\mathcal{F}(\lambda, \alpha)$, we have

$$\begin{aligned}
&\left|\arg\left[\frac{2\lambda-1}{2\lambda+1} + \frac{2}{2\lambda+1}\left(1 + \frac{zI''_{r,1-r}(z)}{I'_{r,1-r}(z)}\right)\right]\right| \\
=&\left|\arg\left[\frac{2\lambda-1}{2\lambda+1} + \frac{2}{2\lambda+1}\left(r(1 + \frac{zf''(z)}{f'(z)}) + (1-r)(1 + \frac{zg''(z)}{g'(z)})\right)\right]\right| \\
=&\left|\arg[r(\frac{2\lambda-1}{2\lambda+1} + \frac{2}{2\lambda+1}(1 + \frac{zf''(z)}{f'(z)})) + (1-r)(\frac{2\lambda-1}{2\lambda+1} + \frac{2}{2\lambda+1}(1 + \frac{zg''(z)}{g'(z)}))]\right| \\
<&\frac{\alpha\pi}{2},
\end{aligned}$$

which implies that $\mathcal{F}(\lambda, \alpha)$ is convex in $(\mathcal{H}, \odot, \oplus)$. □

Since the proof of Theorem 1.4 is parallel to those of Theorems 1.1 and 1.3, we only give the outline of the proof here.

Proof of Theorem 1.4. For $\forall f, g \in \mathcal{R}$ and $\forall r \in [0, 1]$, we have

$$\mathrm{Re}I'_{r,1-r}(z) = r\mathrm{Re}f' + (1-r)\mathrm{Re}g' > 0,$$

which implies $\mathcal{R}$ is convex in $(\mathcal{H}, \odot, \oplus)$. □

3 Precise description of $T[M, N]$

In this section, by some lemmas, we will give the proofs of Theorems 1.5-1.7.

To begin our proofs, we first characterize the structure of the class $\mathcal{F}(\lambda)$, which can be expressed in the following form.

Lemma 3.1. *For $\lambda \in [1/2, 1]$, then $\mathcal{F}(\lambda) = \frac{1+2\lambda}{2} \odot \mathcal{K}$.*

Proof of Lemma 3.1. For $\forall\lambda \in [1/2, 1]$ and $\forall f \in \mathcal{F}(\lambda)$, let $g(z) = \frac{2}{1+2\lambda} \odot f(z)$. It is not difficult to derive $g(z) \in \mathcal{K}$, which implies $\mathcal{F}(\lambda) \subseteq \frac{1+2\lambda}{2} \odot \mathcal{K}$. On the other hand, letting $f(z) = \frac{1+2\lambda}{2} \odot g(z)$ for each function $g(z) \in \mathcal{K}$, we have $f(z) \in \mathcal{F}(\lambda)$, which means $\mathcal{F}(\lambda) \supseteq \frac{1+2\lambda}{2} \odot \mathcal{K}$. □

By Lemma 3.1, we give the proof of Theorem 1.5 as follow.

Proof of Theorem 1.5. We divide this proof into two parts.

Part (i): $T[\mathcal{F}(\lambda), \mathcal{K}] = \left\{(r, s) \in \mathbb{C}^2 : r, s \geq 0,\ 0 \leq r + s \leq \frac{2}{1+2\lambda}\right\}$.

It was shown in [15] that

$$T[\mathcal{K}, \mathcal{K}] = \{(r, s) \in \mathbb{C}^2 : r, s \geq 0, 0 \leq r + s \leq 1\},$$
$$T[\mathcal{K}, \mathcal{C}] = \{(r, s) \in \mathbb{C}^2 : -\frac{1}{2} \leq r, s \leq \frac{3}{2}, -\frac{1}{2} \leq r + s \leq \frac{3}{2}\}. \tag{3.1}$$

By equation (1.6) and Lemma 3.1, we have

$$\begin{aligned} T[\mathcal{F}(\lambda), \mathcal{K}] &= \frac{2}{1+2\lambda} T[\mathcal{K}, \mathcal{K}] \\ &= \frac{2}{1+2\lambda}\left\{(r, s) \in \mathbb{C}^2 : r, s \geq 0, 0 \leq r + s \leq 1\right\} \\ &= \left\{(r, s) \in \mathbb{C}^2 : r, s \geq 0, 0 \leq r + s \leq \frac{2}{1+2\lambda}\right\}, \end{aligned}$$

which completes the proof of the part (i) in Theorem 1.5.

Part (ii): $T[\mathcal{F}(\lambda), \mathcal{C}] = \left\{(r, s) \in \mathbb{C}^2 : -\frac{1}{1+2\lambda} \leq r, s \leq \frac{3}{1+2\lambda}, -\frac{1}{1+2\lambda} \leq r + s \leq \frac{3}{1+2\lambda}\right\}$.

By the equation (3.1), similar to the part (ii), we have

$$\begin{aligned} T[\mathcal{F}(\lambda), \mathcal{C}] &= \frac{2}{1+2\lambda} T[\mathcal{K}, \mathcal{C}] \\ &= \frac{2}{1+2\lambda}\left\{(r, s) \in \mathbb{C}^2 : -\frac{1}{2} \leq r, s \leq \frac{3}{2}, -\frac{1}{2} \leq r + s \leq \frac{3}{2}\right\} \\ &= \left\{(r, s) \in \mathbb{C}^2 : -\frac{1}{1+2\lambda} \leq r, s \leq \frac{3}{1+2\lambda}, -\frac{1}{1+2\lambda} \leq r + s \leq \frac{3}{1+2\lambda}\right\}. \end{aligned}$$

which completes the proof of the Part (ii) in Theorem 1.5. □

Next, we give the proof of Theorem 1.6.

Proof of Theorem 1.6. The proof is divided into two steps.

Step (i): We prove $T[\mathcal{K}(\alpha),\mathcal{K}(\alpha)] \supseteq \{(r,s) : r,s \geq 0, 0 \leq r+s \leq 1\}$ firstly. Given $\alpha \in [0,1)$, for $f,\ g \in \mathcal{K}(\alpha)$, it follows from the equation (2.1) that

$$1+\frac{zI''_{r,s}(z)}{I'_{r,s}(z)} = (1-r-s)+r\{1+\frac{zf''(z)}{f'(z)}\}+s\{1+\frac{zg''(z)}{g'(z)}\} > \alpha$$

when $r,s \geq 0,\ 0 \leq r+s \leq 1$.

Step (ii): We prove $T[\mathcal{K}(\alpha),\mathcal{K}(\alpha)] \subseteq \{(r,s) : r,s \geq 0, 0 \leq r+s \leq 1\}$ by showing that for each $(r_0,s_0) \notin \{(r,s) : r,s \geq 0, 0 \leq r+s \leq 1\}$, $(r_0,s_0) \notin T[\mathcal{K}(\alpha),\mathcal{K}(\alpha)]$.

If $r_0 < 0$, let $f = (2\alpha-1)(1+z)^{2\alpha-1}$ and $g(z) = z$, then

$$1+\frac{zI''_{r_0,s_0}(z)}{I'_{r_0,s_0}(z)} = 1+r_0\frac{zf''(z)}{f'(z)} = \frac{1+[1+r_0(2\alpha-2)]z}{1+z}. \tag{3.2}$$

When $z \to -1$, the real part of equation (3.2) is less than α. Similarly, if $s_0 < 0$, just choose $g = (2\alpha-1)(1+z)^{2\alpha-1}$ and $f(z) = z$.

If $r_0+s_0 > 1$, let $f = g = (2\alpha-1)(1+z)^{2\alpha-1}$, we have

$$1+\frac{zI''_{r_0,s_0}(z)}{I'_{r_0,s_0}(z)} = \frac{1+[1+(r_0+s_0)(2\alpha-2)]z}{1+z}. \tag{3.3}$$

When $z \to 1$, the real part of the equation (3.3) is less than α. These complete the proof of the Step (ii). □

Finally, we introduce the proof of Theorem 1.7.

Proof of Theorem 1.7. We also divide the proof into two steps.

Step (i): We prove $T[\mathcal{R},\mathcal{R}] \supseteq \{(r,s) \in \mathbb{R}^2 : r,s \geq 0, 0 \leq r+s \leq 1\}$.

For $\forall r,s \in \{(r,s) \in \mathbb{R}^2 : r,s \geq 0, 0 \leq r+s \leq 1\}$ and $\forall f,\ g \in \mathcal{R}$, we have

$$\arg I'_{r,s}(z) = r\arg\{f'(z)\}+s\arg\{g'(z)\} \in (-\frac{\pi}{2},\frac{\pi}{2}).$$

This show that $I'_{r,s}(z) \in \mathcal{R}$.

Step (ii): We prove $T[\mathcal{R},\mathcal{R}] \subseteq \{(r,s) \in \mathbb{R}^2 : r,s \geq 0, 0 \leq r+s \leq 1\}$ by showing that for each $(r_0,s_0) \notin \{(r,s) \in \mathbb{R}^2 : r,s \geq 0, 0 \leq r+s \leq 1\}$, $(r_0,s_0) \notin T[\mathcal{K}(\alpha),\mathcal{K}(\alpha)]$.

For $\forall r_0 \notin [-1,1]$, let $f'(z) = \{(1+z)(1-z)\}^{2/r_0}$ and $g'(z) = 1$, $I'_{r_0,s_0}(z)$ maps the unit disk $\mathbb{D}$ onto the complex plane minus the negative real axis, which implies $I'_{r_0,s_0} \notin \mathcal{R}$. Because of the symmetry of r_0 and s_0, the case of $s_0 \notin [-1,1]$ is obvious. If r_0 and s_0 satisfy $r_0,s_0 \in [-1,1]$, $r_0+s_0 \notin [-1,1]$, just let $f'(z) = g'(z) = \{(1+z)(1-z)\}^{2/(r_0+s_0)}$. These complete the proof of the Step (ii). □

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