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Variational Iteration Algorithm-I with an Auxiliary Parameter for Solving Fokker-Planck Equation

Hijaz Ahmad

Department of Basic Sciences, University of Engineering and Technology, Peshawar, Pakistan
e-mail: hijaz555@gmail.com

Abstract

In this paper, variational iteration algorithm-I with an auxiliary parameter is implemented to investigate Fokker-Planck equations. To show the accuracy and reliability of the technique comparisons are made between the variational iteration algorithm-I with an auxiliary parameter and classic variational iteration algorithm-I. The comparison shows that variational iteration algorithm-I with an auxiliary parameter is more powerful and suitable method for solving Fokker-Planck equations. Furthermore, the proposed algorithm can successfully be applied to a large class of nonlinear and linear problems.

1. Introduction

The aim of this work is to apply the variational iteration algorithm-I [1] with an auxiliary parameter for the analytical treatment of the Fokker-Planck equation. The method is able to provide analytical results for nonlinear and linear problems, in a direct way very conveniently. One of the main characteristics of this method is that approximate solution of great accuracy can be obtained by only a few iterations. This method has a simple procedure, acceptable results and above all, this method can successfully be applied to a large class of linear and nonlinear problems [2]-[6].

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2. Variational Iteration Algorithm-I

Consider a general differential equation

$$L[u(x)] + N[u(x)] = c(x), \tag{1}$$

the terms $L[u(x)]$ and $N[u(x)]$ represent the linear and nonlinear term respectively, while $c(x)$ is the inhomogeneous source term. Constructing a correction function for Eq. (1) as,

$$u_{k+1}(x) = u_k(x) + \int_0^x \lambda(\eta)[L\{u_k(\eta)\} + N\{\widetilde{u_k(\eta)}\} - c(\eta)]d\eta, \tag{2}$$

where λ is a parameter, which is not known and called the Lagrange multiplier [7].

Taking the variation δ on the one side as well as the other side of Eq. (2) with respect to $u_k(x)$,

$$\delta u_{k+1}(x) = \delta u_k(x) + \delta\int_0^x \lambda(\eta)[L\{u_k(\eta)\} + N\{\widetilde{u_k(\eta)}\} - c(\eta)]d\eta, \tag{3}$$

where $\widetilde{u_k(\eta)}$ is considered as a restricted term which means $\delta\widetilde{u_k(\eta)} = 0$.

Using optimality conditions, the value of Lagrange multiplier $\lambda(\eta)$ can be identified. An exact solution obtains when $k \to \infty$.

$$u(x) = \lim_{k\to\infty} u_k(x). \tag{4}$$

In short, the formula for equation (1) is,

$$\begin{cases} u_0(x) \text{ is an appropriate initial approximation,} \\ u_{k+1}(x) = u_k(x) + \int_0^x \lambda(\eta)[L\{u_k(\eta)\} + N\{u_k(\eta)\} - c(\eta)]d\eta \\ k = 0, 1, 2, 3, ... \end{cases} \tag{5}$$

This technique is called VIA-I, which is a further development of the general Lagrange multiplier technique for solving nonlinear problems by Inokuti et al. [7]. Now this method [8]-[10] has been developed [11-12] to solve a lot of problems arise in various fields of sciences.

3. Insertion of an Auxiliary Parameter in Variational Iteration Algorithm-I

In VIA-I, an auxiliary parameter h can be inserted. The optimal choice of unknown h improves the correctness, precision and effectiveness of the technique. After inserting h, equation (5) will become

$$\begin{cases} u_0(x) \text{ is an appropriate initial approximation,} \\ u_1(x,\, h) = u_0(x) + h\int_0^x \lambda(\eta)[L\{u_0(\eta)\} + N\{u_0(\eta)\} - c(\eta)]d\eta, \\ u_{k+1}(x,\, h) = u_k(x,\, h) + h\int_0^x \lambda(\eta)[L\{u_k(\eta,\, h)\} + N\{u_k(\eta,\, h)\} - c(\eta,\, h)]d\eta, \\ k = 1,\, 2,\, 3,\, \ldots. \end{cases} \tag{6}$$

This technique is known as VIA-I with AP. Actually, this technique is simple, has a lesser size of calculation, not difficult to analyze and have the ability to approximate the solution precisely in solution domain of wide range.

4. The Fokker-Planck Equation [2]

In this section, the general form of Fokker-Planck equation which is also called forward Kolmogorov equation is

$$\frac{\partial u}{\partial t} = \left[\frac{\partial}{\partial t}A(x) + \frac{\partial^2}{\partial x^2}B(x)\right]u(x,\, t), \tag{7}$$

with conditions:

$$u(x,\, 0) = f(x), \quad x \in \mathcal{R}.$$

It is the equation for the motion of concentration field $u(x,\, t)$. The backward Kolmogorov equation can be written in the following form

$$\frac{\partial u}{\partial t} = -\left[\frac{\partial}{\partial t}A(x,\, t) + \frac{\partial^2}{\partial x^2}B(x,\, t)\right]u(x,\, t). \tag{8}$$

Let the initial conditions,

$$u(x,\, 0) = f(x), \quad x \in \mathcal{R}$$

and

$$A(x,\, t) = -(x + 1),$$

$$B(x, t) = x^2 e^t.$$

Then equation (8) becomes

$$\frac{\partial u}{\partial t} = \left[\frac{\partial^2}{\partial x^2} x^2 e^t - \frac{\partial}{\partial t}(x+1)\right] u(x, t). \tag{9}$$

First, we solve this example by VIA-I.

Constructing the correction function for equation (9) as,

$$u_{k+1}(x, t) = u_k(x, t) + \int_0^t \lambda(\eta)\left\{\frac{\partial u_k(x, \eta)}{\partial \eta} + \frac{\partial u_k(x+\tilde{1})}{\partial \eta} u_k(x, \eta) - \frac{\partial^2(\tilde{x}^2 e^{\eta})}{\partial x^2} u_k(x, \eta)\right\} d\eta. \tag{10}$$

Taking the variation δ on the one side as well as the other side with respect to $u_k(x, t)$

$$\delta u_{k+1}(x, t) = \delta u_k(x, t) + \delta\int_0^t \lambda(\eta)\left\{\frac{\partial u_k(x, \eta)}{\partial \eta} + \frac{\partial u_k(x+\tilde{1})}{\partial \eta} u_k(x, \eta) - \frac{\partial^2(\tilde{x}^2 e^{\eta})}{\partial x^2} u_k(x, \eta)\right\} d\eta.$$

Ignoring the restricted terms

$$\delta u_{k+1}(x, t) = \delta u_k(x, t) + \delta\int_0^t \lambda(\eta)\left\{\frac{\partial u_k(x, \eta)}{\partial \eta}\right\} d\eta$$

$$= \delta u_k(x, t) + \lambda(\eta)\delta u_k(x, t) - \int_0^t \lambda'(\eta)\delta u_k(x, \eta) d\eta$$

$$= (1 + \lambda(\eta))\delta u_k(x, t) - \int_0^t \lambda'(\eta)\delta u_k(x, \eta) d\eta.$$

The stationary conditions are:

$$\lambda'(\eta) = 0,$$

$$1 + \lambda(\eta) = 0,$$

we get the value of $\lambda(\eta)$ which is $\lambda(\eta) = -1$.

Using this value of $\lambda(\eta)$ in equation (10) results in the below iterative scheme:

$$u_{k+1}(x, t) = u_k(x, t) - \int_0^t \left\{ \frac{\partial u_k(x, \eta)}{\partial \eta} + \frac{\partial u_k(x+1)}{\partial \eta} u_k(x, \eta) - \frac{\partial^2 (x^2 e^{\eta})}{\partial x^2} u_k(x, \eta) \right\} d\eta. \tag{11}$$

starting with

$$u_0(x, t) = x + 1,$$

other approximations by using the scheme (11),

$$u_1(x, t) = (t+1)(x+1),$$

$$u_2(x, t) = \frac{(x+1)((t^2 + 2t + 2))}{2},$$

$$u_3(x, t) = \frac{(x+1)((t^3 + 3t^2 + 6t + 6))}{6},$$

$$\vdots$$

we stop the procedure at $u_{10}(x, t)$. The absolute error of $u_{10}(x, t)$ in the solution domain $(x, t) \in [0, 5] \times [0, 1]$ can be seen in Figure 1.

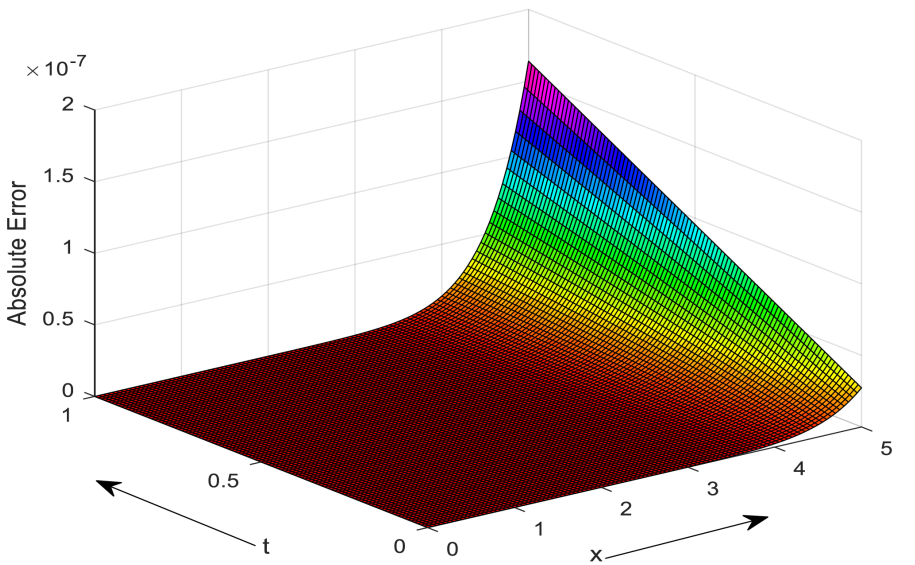

Figure 1. Absolute error betwixt the exact and approximate solutions by VIA-I.

Now we want to solve this problem by VIA-I with AP.

Using VIA-I with AP, the recurrence relation for equation (9) is

$$u_{k+1}(x, t, h) = u_k(x, t, h) - h\int_0^t \left\{ \frac{\partial u_k(x, \eta, h)}{\partial \eta} + \frac{\partial u_k(x+1)}{\partial \eta} u_k(x, \eta, h) - \frac{\partial^2 (x^2 e^{\eta})}{\partial x^2} u_k(x, \eta, h) \right\} d\eta. \quad (12)$$

Starting with

$$u_0(x, t) = x + 1.$$

Other approximations can be get by using the recurrence relation (12),

$$u_1(x, t, h) = (ht + 1)(x + 1),$$

$$u_2(x, t, h) = (ht + 1)(x + 1) + \frac{ht(x+1)((ht - 2h + 2))}{2},$$

$$u_3(x, t, h) = \frac{(x+1)(h^3t^3 - 6h^3t^2 + 6h^3t + 9h^2t^2 - 18h^2t + 18ht + 6)}{6},$$

$$\vdots$$

we stop the procedure at $u_{10}(x, t, h)$.

The following residual function is defined

$$r_{10}(x, t, h) = \frac{\partial u_{10}(x, t, h)}{\partial \eta} + \frac{\partial u_{10}(x+1)}{\partial \eta} u_{10}(x, t, h) - \frac{\partial^2 (x^2 e^{\eta})}{\partial x^2} u_{10}(x, t, h). \quad (13)$$

The square of residual function for 10th-order approximation with respect to h for $(x, t) \in [0, 5] \times [0, 1]$ is

$$\frac{1}{(11)^2} \sum_{i=0}^{10} \sum_{j=0}^{10} \left(r_{10}\left(\frac{5i}{10}, \frac{j}{10}, h \right) \right)^2. \quad (14)$$

The minimum value of above square residual function occurs at $h = 1.02408718538499$. Using this value of h in $u_{10}(x, t, h)$ in the solution domain $(x, t) \in [0, 5] \times [0, 1]$, error betwixt the exact and approximate solutions can be seen in Figure 2.

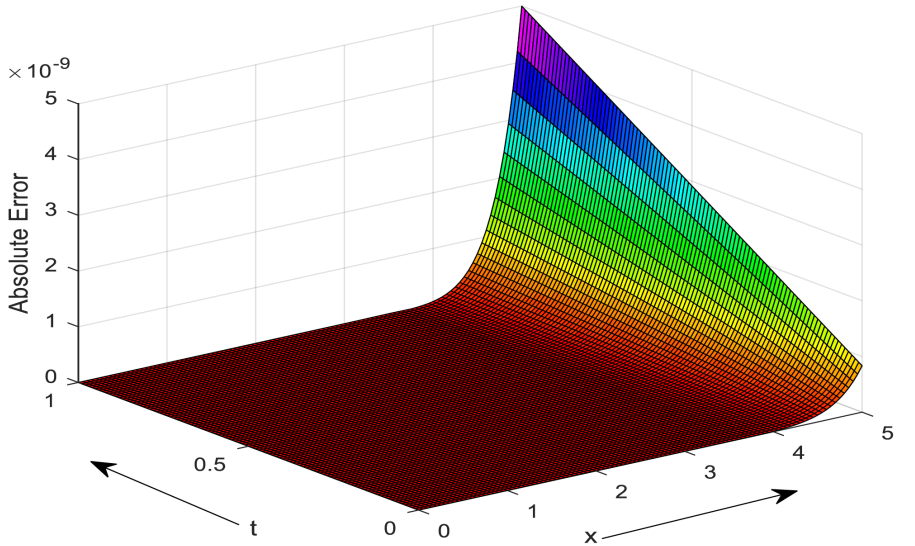

Figure 2. Absolute error betwixt the approximate and exact solutions by VIA-I with AP.

Comparing Figure 1 and Figure 2, it is clear that VIA-I with AP gives better results as compared to VIA-I. Numerical comparison betwixt the exact and approximate solutions of both methods is given in the table below.

Table 1. Comparison of absolute errors for 6th order approximation by VIA-I and VIA-I with AP.

x	***t***	**Absolute Error in VIA-I with AP**	**Absolute Error in VIA-I**
0.5	0.1	3.553×10^{-16}	2.220×10^{-16}
1.0	0.2	2.665×10^{-15}	1.332×10^{-15}
1.5	0.3	2.398×10^{-14}	1.137×10^{-13}
2.0	0.4	7.994×10^{-15}	3.261×10^{-12}
2.5	0.5	6.750×10^{-14}	4.467×10^{-11}
3.0	0.6	4.174×10^{-13}	3.826×10^{-10}
3.5	0.7	3.608×10^{-12}	2.376×10^{-09}
4.0	0.8	3.947×10^{-11}	1.152×10^{-08}
4.5	0.9	6.701×10^{-10}	4.672×10^{-08}
5.0	1.0	4.997×10^{-09}	1.639×10^{-07}

The above table shows that VIA-I with AP is better for a large domain of t as compared to VIA-I.

5. Conclusions

In this paper, variational iteration algorithm-I with an auxiliary parameter has been used in a way that accomplished the desired aim for solving Fokker-Planck equation. This work has made sure that the variational iteration algorithm-I with an auxiliary parameter offers noteworthy advantages in terms of its easy applicability, its computational success, and its adequacy to solve a wide class of differential equations. Graphical and numerical results reveal that this modification of variational iteration algorithm-I is suitable for all linear and nonlinear problems arise in physical sciences and engineering, superior to the variational iteration algorithm-I.

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