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Resolvent Dynamical Systems Technique for Mixed General Variational Inequalities

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Abstract

In this paper, we introduce new second order dynamical system approach for solving a class of mixed general variational inequalities. Using the forward finite difference schemes, we suggest some multi-step iterative methods for solving the mixed variational inequalities. Convergence analysis is investigated under certain mild conditions. Some special cases are discussed as applications of the results. It is an interesting problem to compare these methods with other technique for solving mixed variational inequalities and related optimizations.

1 Introduction

Variational inequality theory, which was introduced by Stampacchia [64] and Lions et al. [27] in early sixteen, provides us with a simple, natural, unified, novel and general framework to study an extensive range of unilateral, obstacle, free, moving and equilibrium problems arising in fluid flow through porous media, elasticity, circuit analysis, transportation, oceanography, operations research, finance, economics, and optimization. It is worth mentioning that the variational inequalities can be viewed as a significant and novel generalization of the variational principles. It is known that the minimum of a differentiable convex functions on the convex sets can be characterized by the variational inequality. It is amazing that variational inequalities have influenced various areas of pure and applied sciences and are still continue to influence the recent research, see [6, 7, 9–12, 17–19, 21–24, 27–29, 31–51, 54–59, 59–67, 70].

In recent years, variational inequalities have been extended and generalized in various directions by using novel and innovative ideas and techniques, both for their own sake and for their applications. An

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important and useful generalization is called the mixed variational inequality or the variational inequality of the second kind. For the applications, formulations and numerical methods, see [7,9–11,18,19,23,24,27, 31,36,39,40,42,43,47,48,50,58,59,66,67] and the references therein. Due to the presence of the nonlinear term in the mixed variational inequality, projection method and its variant forms cannot be used to suggest numerical methods for solving the mixed variational inequalities. These facts motivated us to use the technique of the resolvent operator, the origin of which can be traced back to Brezis [11]. In this technique, the given operator is decomposed into the sum of two maximal monotone operators, whose resolvent are easier to evaluate than the resolvent of the original operator. Such a method is known as operator splitting method. This can lead to every efficient methods, since one can treat each part of the original operator independently. The operator splitting methods and related techniques have been analyzed and studied by many researchers including Glowinski and Le Tallec [19], and Tseng [67]. In the context of the mixed variational inequalities, Noor [28, 41] has used the projection operator technique to suggest some splitting type methods applying the approach of updating the solution. These three-step methods are also known as Noor's iterations. It is noted that these forward-backward splitting algorithms are similar to those of Glowinski et al. [19], which they suggested by using the Lagrangian technique. It is known that three-step schemes are versatile and efficient. These three-step schemes are a natural generalization of the splitting methods for solving partial differential equations. For applications of the splitting techniques to partial differential equations, see Ames [1]. A useful feature of the forward-backward splitting method is that the resolvent step involves the subdifferential of the proper, convex and lower-semicontinuous only and the other part facilitates the problem decomposition. In particular, if the nonlinear term in the mixed variational inequality is the indicator function of a closed convex set in the Hilbert space, then these splitting (forward-backward) methods reduce to the projection and extragradient methods for solving the variational inequalities. It has been established [9, 10, 12, 29, 38, 39, 42, 60, 62, 63, 65] that Noor iterations and their modified form, perform better than two-step (Ishikawa iteration) and one step method Mann iteration. In recent years, considerable interest has been shown in developing various extensions and generalizations of Noor iterations, both for their own sake and for their applications. For novel applications, modifications and generalizations of the Noor iterations. These methods include Mann iteration, Ishikawa iteration, modified forward-backward splitting methods of Tseng [67] and Noor [38,42] as special cases. Noor iterations have been modified and generalized in different directions to explore their applications in fractal, chaos, images, signal recovery, polynomiography, fixed point theory, compress programming, nonlinear equations, compressive sensing and image in painting, see [2–5,7–9,14,19,20,26, 29,52,53,60–63,65] and the references therein.

Dupuis and Nagurney [17] introduced and studied the projected dynamical systems associated with variational inequalities using the equivalent fixed point formulation. The novel feature of the projected dynamical system is that the its set of stationary points corresponds to the set of the corresponding set of the solutions of the variational inequality problem. Thus the equilibrium and nonlinear programming problems, which can be formulated in the setting of the variational inequalities, can now be studied in the more general framework of the dynamical systems. It has been shown [17,21,22,28,40,41,49,50,54,56,68,69] that these dynamical systems are useful in developing efficient and powerful numerical techniques for solving variational inequalities.

Motivated and inspired by ongoing research in these fascinations areas, we consider a dynamical system coupled with second order boundary value problems associated with variational inequalities. In this paper, we establish that the second boundary value problems can be exploited to suggest and analyzed multi step methods for finding the approximate solutions of variational inequalities and related optimization problem. This is a new approach. Using the finite difference schemes, we suggest and analyzed some new multi step iterative methods for solving variational inequalities. Some special cases are also pointed as potential applications of the obtained results. These multi step methods include Mann iteration, Ishikawa iterations and Noor iterations as special cases. We have only considered theoretical aspects of the suggested methods. It is an interesting problem to implement these methods and to illustrate the their efficiency. Comparison with other methods need further research efforts. The ideas and techniques of this paper may be extended for other classes of mixed quasi variational inequalities and related optimization problems.

2 Basic Definitions and Results

Let Ω be a set in a real Hilbert space $\mathcal{H}$ with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$. Let $\mathcal{T}, g : \mathcal{H} \longrightarrow \mathcal{H}$ be nonlinear operators and let $\phi : \mathcal{H} \longrightarrow \mathcal{H}$ be a lower semi-continuous function.

We consider the problem of finding $\mu \in \mathcal{H}$, such that

$$\langle\, \mathcal{T}\mu + \mu - g(\mu), \nu - \mu\rangle + \phi(\nu) - \phi(\mu) \geq 0, \quad \forall \nu \in \mathcal{H} \tag{2.1}$$

which is called the mixed general variational inequality, introduced and studied by Noor and Noor [50,51]. It has been shown [50] that the optimality conditions of sum of two differentiable nonconvex functions can be characterized via the general variational inequalities of the type (2.1). Let $F : H \longrightarrow R$ be a differentiable convex function and ϕ be a lower semi-continuous convex function. If $\mathcal{T} = \nabla F$ and $g = I$, then problem (2.1) is equivalent to finding $\mu \in H$ such that

$$0 \in \nabla F(\mu) + \partial\phi(\mu). \tag{2.2}$$

Problem (2.2) is nothing else than the convex optimization optimization problem:

$$\min_{\mu \in H}\{J(\mu) + \phi(\mu)\}.$$

Special cases

We now discuss some special cases of general variational inequalities (2.1).

1. If ϕ is the indicator function of a closed convex set $\Omega \subseteq \mathcal{H}$, then problem (2.1) reduces to finding $\mu \in \Omega$ such that

$$\langle \mathcal{T}\mu + \mu - g(\mu), \nu - \mu \rangle \geq 0, \quad \forall \nu \in \Omega, \tag{2.3}$$

which is called the general variational inequality introduced and studied by Noor and Noor [51]. It has been shown [38] that the optimality conditions of the differentiable nonconvex functions can be characterized via the general variational inequalities of the type (2.3).

2. If $g = I$, then the problem (2.1) reduces to finding $\mu \in \mathcal{H}$, such that

$$\langle \mathcal{T}\mu, \nu - \mu \rangle + \phi(\nu) - \phi(\mu) \geq 0, \quad \forall\, \nu \in \mathcal{H}, \tag{2.4}$$

is called the mixed variational inequality, introduced by Lions and Stampacchia [64]. It has been shown a wide class of obstacle boundary value and initial value problems can be studied in the general framework of variational inequalities. For the applications, motivation, numerical methods, sensitivity analysis, dynamical system, merit functions and other aspects of variational inequalities, see [6, 7, 9–12, 17–19, 21–24, 27–29, 31–51, 54–59, 59–65] and the references therein. For example, the mixed variational inequality (2.4) characterizes the Signorini problem with non-local friction. If S is an open bounded domain in R^n with regular boundary ∂S, representing the interior of an elastic body subject to external forces and if a part of the boundary may come into contact with a rigid foundation, then (2.4) is simply a statement of the virtual work for an elastic body restrained by friction forces, assuming that a non-local law of friction holds. The strain energy of the body corresponding to an admissible displacement ν is $\langle T\nu, \nu\rangle$. Thus $\langle T\mu, \nu - \mu\rangle, \forall \mu, \nu \in H$ is the work produced by the stresses through strains caused by the virtual displacement $\nu - \mu$. The friction forces are represented by the function $\phi(.)$. Similar problems arise in the study of the fluid flow through porous media. For the physical and mathematical formulation of the mixed variational inequalities of type (2.4), see [23].

3. If $\mu = g(\mu)$, then problem (2.1) is equivalent to finding $\mu \in \mathcal{H}$

$$\langle T(g(\mu), \nu - g(\mu)\rangle + \phi(\nu) - \phi(g(\mu)) \geq 0, \quad \forall \nu \in \mathcal{H}, \tag{2.5}$$

which is called the mixed general variational inequalities. Variational inequality of the type (2.5) arises as a minimum of the sum of two differentiable nonconvex functions.

4. If $g = I$ and $\Omega^* = \{\mu \in \mathcal{H} : \langle \mu, \nu \rangle \geq 0, \forall \nu \in \Omega\}$ is a polar(dual) cone, then problem (2.3) is equivalent to finding $\mu \in \mathcal{H}$ such that

$$\mu \in \Omega, \quad \mathcal{T}\mu \in \Omega^*, \quad \langle \mathcal{T}\mu, \mu \rangle = 0, \tag{2.6}$$

which is called the general complementarity problem.

For the applications, motivations, generalization, numerical methods and other aspects of the complementarity problems in engineering and applied sciences, see [15,32,34,41,46,59] and the references therein.

5. If $\Omega = \mathcal{H}$, then problem (2.3) collapses to finding $\mu \in \mathcal{H}$ such that

$$\langle \rho \mathcal{T}\mu + \mu - g(\mu), \nu - \mu \rangle = 0, \quad \forall \nu \in \mathcal{H}.$$

Consequently, it follows that $\mu \in \mathcal{H}$ satisfies

$$\mu = g(\mu) - \rho \mathcal{T}\mu, \tag{2.7}$$

which is called the general equation and appears to be a new one.

For a different and appropriate choice of the operators and spaces, one can obtain several known and new classes of variational inequalities and related problems. This clearly shows that the problem (2.1) considered in this paper is more general and unifying one.

We need the following well-known definitions and results in obtaining our results.

Definition 2.1. *Let $\mathcal{T} : \mathcal{H} \longrightarrow \mathcal{H}$ be a given mapping.*

i. *The mapping $\mathcal{T}$ is called strongly monotone, if there exists a constant $\alpha \geq 0$ such that*

$$\left\langle \mathcal{T}\mu - \mathcal{T}\nu, \mu - \nu \right\rangle \geq \alpha \|\mu - \nu\|^2, \qquad \forall \mu, \nu \in \mathcal{H}.$$

ii. *The mapping $\mathcal{T}$ is called monotone, if*

$$\left\langle \mathcal{T}\mu - \mathcal{T}\nu, \mu - \nu \right\rangle \geq 0, \quad \forall \mu, \nu \in \mathcal{H}.$$

iii. *The mapping $\mathcal{T}$ is called $\eta-$Lipschitz continuous, if there exists a constant $\eta > 0$ such that*

$$\| \mathcal{T}\mu - \mathcal{T}\nu \| \leq \eta \|\mu - \nu\|, \qquad \forall \mu, \nu \in \mathcal{H}.$$

Definition 2.2. *[11] If T is a maximal monotone operator on H, then, for a constant $\rho > 0$, the resolvent operator associated with T is defined by*

$$J_T(\mu) = (I + \rho T)^{-1}(\mu), \quad \forall \mu \in H,$$

where I is the identity operator. It is known that a monotone operator T is maximal monotone, if and only if, its resolvent operator J_T is defined everywhere. Furthermore, the resolvent operator J_T is nonexpansive, that is,

$$\|J_T(\mu) - J_T(\nu)\| \leq \|\mu - \nu\|, \quad \forall \mu, \nu \in H.$$

Remark 2.1. *Since the subdifferential $\partial\phi$ of a proper, convex and lower-semicontinuous $\phi : H \longrightarrow R \cup \{+\infty\}$ is a maximal monotone operator, we define by*

$$J_\varphi \equiv (I + \rho\partial\phi)^{-1},$$

the resolvent operator associated with $\partial\phi$ and $\rho > 0$ is a constant.

We also need the following result, known as the resolvent Lemma(best approximation) Lemma, which plays a crucial part in establishing the equivalence between the mixed variational inequalities and the fixed point problem. This result can be used in the analysing the convergence analysis of the projective implicit and explicit methods for solving the mixed variational inequalities and related optimization problems.

Lemma 2.1. *[11] For a given $z \in H$, $\mu \in H$ satisfies the inequality*

$$\langle \mu - z, \nu - \mu \rangle + \rho\phi(\nu) - \rho\phi(\mu) \geq 0, \quad \forall \nu \in H, \tag{2.8}$$

if and only if

$$\mu = J_\phi(z),$$

where J_ϕ is the resolvent operator.

It is well known that the resolvent operator J_ϕ is nonexpansive, that is,

$$\|J_\phi(\mu) - J_\phi(\nu)\| \leq \|\mu - \nu\|, \forall \mu, \nu \in H.$$

This property of the resolvent operator plays an important part in the derivation of our main results.

3 Main Results

In this section, we consider the projected dynamical system associated with the general variational inequalities. The innovative and novel feature of a projected dynamical system is that its set of stationary points corresponds to the set of solutions of the corresponding variational inequality problem. Equilibrium and nonlinear problems arising in various branches in pure and applied sciences can now be studied in the more general setting of dynamical systems. It has been shown [17,21,22,28,40,41,49,50,54,56,68,69] that the dynamical systems are useful in developing some efficient numerical techniques for solving variational inequalities and related optimization problems. In recent years, much attention has been given to study the globally asymptotic stability of these projected dynamical systems. We use this equivalent fixed point formulation to suggest and analyze the resolvent dynamical system associated with the general variational inequalities (2.1).

$$\frac{d\mu}{dt} = \lambda\{J_\phi[g(\mu) - \rho T\mu] - \mu\}, \quad \mu(t_0) = \mu_0 \in \mathcal{H}, \tag{3.1}$$

where λ is a parameter. The system of type (3.1) is called the resolvent general dynamical system. Here the right hand side is related to the projection operator and is discontinuous on the boundary. It is clear from the definition that the solution to (3.1) always stays in the constraint set. This implies that the qualitative results such as the existence, uniqueness and continuous dependence of the solution on the given data can be studied.

The equilibrium points of the dynamical system (3.1) are naturally defined as follows.

Definition 3.1. *An element $\mu \in \mathcal{H}, g$ is an equilibrium point of the dynamical system (3.1), if $\frac{d\mu}{dt} = 0$, that is,*

$$J_\phi[g(\mu) - \rho T\mu] - \mu = 0.$$

Thus it is clear that $\mu \in \mathcal{H}$ is a solution of the general variational inequality (2.1), if and only if, $\mu \in \mathcal{H}$ is an equilibrium point.

Definition 3.2. *The dynamical system is said to converge to the solution set S^* of (3.1), if , irrespective of the initial point, the trajectory of the dynamical system satisfies*

$$\lim_{t\to\infty} dist(\mu(t), S^*) = 0, \tag{3.2}$$

where

$$dist(\mu, S^*) = inf_{\nu \in S^*} \|\mu - \nu\|.$$

It is easy to see, if the set S^* has a unique point μ^*, then (3.2) implies that

$$\lim_{t\to\infty} \mu(t) = \mu^*.$$

If the dynamical system is still stable at μ^* in the Lyapunov sense, then the dynamical system is globally asymptotically stable at μ^*.

Definition 3.3. *The dynamical system is said to be globally exponentially stable with degree η at μ^*, if, irrespective of the initial point, the trajectory of the system satisfies*

$$\|\mu(t) - \mu^*\| \leq \eta_1 \|\mu(t_0) - \mu^*\| exp(-\eta(t - t_0)), \quad \forall t \geq t_0,$$

where μ_1 and η are positive constants independent of the initial point.

It is clear that the globally exponentially stability is necessarily globally asymptotically stable and the dynamical system converges arbitrarily fast.

Lemma 3.1. *(Gronwall Lemma) [17, 28] Let $\hat{\mu}$ and $\hat{\nu}$ be real-valued nonnegative continuous functions with domain $\{t : t \leq t_0\}$ and let $\alpha(t) = \alpha_0(|t - t_0|)$, where α_0 is a monotone increasing function. If for $t \geq t_0$,*

$$\hat{\mu} \leq \alpha(t) + \int_{t_0}^{t} \hat{\mu}(s)\hat{\nu}(s)ds,$$

then

$$\hat{\mu}(s) \leq \alpha(t) exp\{\int_{t_0}^{t} \hat{\nu}(s)ds\}.$$

We now show that the trajectory of the solution of the general dynamical system (3.1) converges to the unique solution of the general variational inequality (2.1). The analysis is in the spirit of Noor [41] and Xia and Wang [68, 69].

Theorem 3.1. *Let the operators $T, g : H \longrightarrow H$ be both Lipschitz continuous with constants $\beta > 0$ and $\mu > 0$ respectively. Then, for each $\mu_0 \in \mathcal{H}$, there exists a unique continuous solution $\mu(t)$ of the dynamical system (3.1) with $\mu(t_0) = \mu_0$ over $[t_0, \infty)$.*

Proof. Let

$$G(\mu) = \lambda\{J_\phi[g(\mu) - \rho T\mu] - \mu\}.$$

where $\lambda > 0$ is a constant and $G(\mu) = \frac{d\mu}{dt}$. $\forall \mu, \nu \in \mathcal{H}$, we have

$$\begin{aligned} \|G(\mu) - G(\nu)\| &\leq \lambda\{\|J_\phi[g(\mu) - \rho T\mu] - J_\phi[g(\nu) - \rho T\nu]\| + \|\mu - \nu\|\} \\ &\leq \lambda\|\mu - \nu\| + \lambda\|g(\mu) - g(\nu)\| + \lambda\rho\|T\mu - T\nu\| \\ &\leq \lambda\{1 + \mu + \beta\rho\}\|\mu - \nu\|. \end{aligned}$$

This implies that the operator $G(\mu)$ is a Lipschitz continuous in $\mathcal{H}$, and for each $\mu_0 \in \mathcal{H}$, there exists a unique and continuous solution $\mu(t)$ of the dynamical system (3.1), defined on an interval $t_0 \leq t < T_1$ with the initial condition $\mu(t_0) = \mu_0$. Let $[t_0, T_1)$ be its maximal interval of existence. Then we have to show that $T_1 = \infty$. Consider, for any $\mu \in \mathcal{H}$,

$$\begin{aligned} \|G(\mu)\| = \|\frac{d\mu}{dt}\| &= \lambda\|J_\phi[g(\mu) - \rho T\mu] - \mu\| \\ &\leq \lambda\{\|J_\phi[g(\mu) - \rho T\mu] - J_\phi[0]\| + \|J_\phi[0] - \mu\|\} \\ &\leq \lambda\{\rho\|T\mu\| + \|J_\phi[u] - J_\phi[0]\| + \|J_\phi[0] - \mu\|\} \\ &\leq \lambda\{(\rho\beta + 1 + \eta)\|\mu\| + \|J_\phi[0]\|\}. \end{aligned}$$

Then

$$\begin{aligned} \|\mu(t)\| &\leq \|\mu_0\| + \int_{t_0}^{t} \|T\mu(s)\|ds \\ &\leq (\|\mu_0\| + k_1(t - t_0)) + k_2 \int_{t_0}^{t} \|\mu(s)\|ds, \end{aligned}$$

where $k_1 = \lambda\|J_\phi[0]\|$ and $k_2 = \lambda(\rho\beta + 1 + \mu)$. Hence by the Gronwall Lemma 3.1, we have

$$\|\mu(t)\| \leq \{\|\mu_0\| + k_1(t - t_0)\}e^{k_2(t-t_0)}, \quad t \in [t_0, T_1).$$

This shows that the solution is bounded on $[t_0, T_1)$. So $T_1 = \infty$. □

Theorem 3.2. *Let the operators $T, g : \mathcal{H} \longrightarrow H$ be Lipschitz continuous with constants $\beta > 0$ and $\mu > 0$ respectively. If the operator $g : \mathcal{H} \longrightarrow \mathcal{H}$ is strongly monotone with constant $\gamma > 0$ and $\lambda > 0$, then the dynamical system (3.1) converges globally exponentially to the unique solution of the mixed general variational inequality (2.1).*

Proof. Since the operators T, g are both Lipschitz continuous, it follows from Theorem 3.1 that the dynamical system (3.1) has unique solution $\mu(t)$ over $[t_0, T_1)$ for any fixed $\mu_0 \in \mathcal{H}$. Let $\mu(t)$ be a solution of the initial value problem (3.1). For a given $\mu^* \in \mathcal{H}$ satisfying (2.1), consider the Lyapunov function

$$L(\mu) = \lambda\|\mu(t) - \mu^*\|^2, \quad \mu(t) \in \mathcal{H}. \tag{3.3}$$

From (3.1) and (3.3), we have

$$\begin{aligned} \frac{dL}{dt} &= 2\lambda\langle\mu(t) - \mu^*, J_\phi[g(\mu(t)) - \rho T\mu(t)] - \mu(t)\rangle \\ &= -2\lambda\langle\mu(t) - \mu^*, \mu(t) - \mu^*\rangle \\ &\quad +2\lambda\langle\mu(t) - \mu^*, J_\phi[g(\mu(t)) - \rho T\mu(t)] - \mu^*\rangle \\ &\leq -2\lambda\|\mu(t) - \mu^*\|^2 \\ &\quad +2\lambda\langle\mu(t) - \mu^*, J_\phi[g(\mu(t)) - \rho T\mu(t)] - \mu^*\rangle, \end{aligned} \tag{3.4}$$

where $u^* \in \mathcal{H}$ is a solution of (2.1). Thus

$$\mu^* = J_\phi[g(\mu^*) - \rho T\mu^*].$$

Using the Lipschitz continuity of the operators T, g, we have

$$\begin{aligned} \|J_\phi[g(\mu) - \rho T\mu] - J_\phi[g(\mu^*) - \rho T\mu^*]\| &\leq \|g(\mu) - g(\mu^*) - \rho(T\mu - T\mu^*)\| \\ &\leq (\mu + \rho\beta)\|\mu - \mu^*\|. \end{aligned} \tag{3.5}$$

From (3.4) and (3.5), we have

$$\frac{d}{dt}\|\mu(t) - \mu^*\| \leq 2\alpha\lambda\|\mu(t) - \mu^*\|,$$

where

$$\alpha = \mu + \rho\beta\lambda.$$

Thus, for $\lambda = -\lambda_1$, where λ_1 is a positive constant, we have

$$\|\mu(t) - \mu^*\| \leq \|\mu(t_0) - \mu^*\| e^{-\alpha\lambda_1(t-t_0)},$$

which shows that the trajectory of the solution of the dynamical system (3.1) converges globally exponentially to the unique solution of the general variational inequality (2.1). □

We use the projected dynamical system (3.1) to suggest some iterative for solving variational inequalities (2.1). These methods can be viewed in the sense of Korpelevich [11] and Noor [18, 19] involving the double projection operator.

For simplicity, we take $\lambda = 1$. Thus the dynamical system (3.1) becomes

$$\frac{d\mu}{dt} + \mu = J_\phi[g(\mu) - \rho T\mu], \quad \mu(t_0) = \alpha. \tag{3.6}$$

We construct the implicit iterative method using the forward difference scheme. Discretizing (3.1), we have

$$\frac{\mu_{n+1} - \mu_n}{h} + \mu_{n+1} = J_\phi[g(\mu_{n+1}) - \rho T\mu_{n+1}], \tag{3.7}$$

where $h > 0$ is the step size. Now, we can suggest the following implicit iterative method for solving the variational inequality (2.1).

Algorithm 3.1. *For a given $\mu_0 \in \mathcal{H}$, compute μ_{n+1} by the iterative scheme*

$$\mu_{n+1} = J_\phi\left[g(\mu_{n+1}) - \rho T\mu_{n+1} - \frac{\mu_{n+1} - \mu_n}{h}\right], \quad n = 0, 1, 2, \ldots.$$

This is an implicit method and is quite different from the implicit method of [5]. Using Lemma 2.1, Algorithm 3.1 can be rewritten in the equivalent form as:

Algorithm 3.2. *For a given $\mu_0 \in \mathcal{H}$, compute μ_{n+1} by the iterative scheme*

$$\langle \rho T\mu_{n+1} + \mu_{n+1} - g(\mu_{n+1}) + \frac{\mu_{n+1} - \mu_n}{h}, \nu - \mu_{n+1}\rangle + \phi(\nu) - \phi(\mu_{n+1}) \geq 0, \quad \forall \nu \in \mathcal{H}. \tag{3.8}$$

We now study the convergence analysis of algorithm 3.1

Theorem 3.3. *Let $\mu \in \mathcal{H}$ be a solution of mixed general variational inequality (2.1). Let μ_{n+1} be the approximate solution obtained from (3). If T is pseudo g-monotone, then*

$$\|\mu - \mu_{n+1}\|^2 \leq \|\mu - \mu_n\|^2 - \|\mu_n - \mu_{n+1}\|^2. \tag{3.9}$$

Proof. Let $\mu \in \mathcal{H}$ be a solution of (2.1). Then

$$\langle \rho T\nu + \nu - g(\nu), \nu - \mu\rangle + \phi(\nu) - \phi(\mu) \geq 0, \quad \forall \nu \in \mathcal{H}, \tag{3.10}$$

since T is a pseudo g-monotone operator.

Set $\nu = \mu_{n+1}$ in (3.10), to have

$$\langle \rho T\mu_{n+1} + \mu_{n+1} - g(\mu_{n+1}), \mu_{n+1} - \mu\rangle + \phi(\mu_{n+1}) - \phi(\mu) \geq 0. \tag{3.11}$$

Take $\nu = \mu$ in equation (3.8), we have

$$\langle \rho T\mu_{n+1} + u_{n+1} - g(u_{n+1}) + \frac{\mu_{n+1} - \mu_n}{h}, \mu - \mu_{n+1}\rangle + \phi(\mu) - \phi(un+1) \geq 0. \tag{3.12}$$

From (3.11) and (3.12), we have

$$\langle \mu_{n+1} - \mu_n, \mu - \mu_{n+1}\rangle \geq 0. \tag{3.13}$$

From (3.13) and using $2\langle a, b\rangle = \|a+b\|^2 - \|a\|^2 - \|b\|^2, \quad \forall a, b \in \mathcal{H}$, we obtain

$$\|\mu_{n+1} - \mu\|^2 \leq \|\mu - \mu_n\|^2 - \|\mu_{n+1} - \mu_n\|^2, \tag{3.14}$$

the required result. □

Theorem 3.4. *Let $\mu \in \mathcal{H}$ be the solution of mixed general variational inequality (2.1). Let μ_{n+1} be the approximate solution obtained from (3). If T is a pseudo g-monotone operator, then μ_{n+1} converges to $\mu \in \mathcal{H}$ satisfying (2.1).*

Proof. Let T be a pseudo g-monotone operator. Then, from (3.23), it follows the sequence $\{\mu_i\}_{i=1}^{\infty}$ is a bounded sequence and

$$\sum_{i=1}^{\infty} \|\mu_n - \mu_{n+1}\|^2 \leq \|\mu - \mu_0\|^2,$$

which implies that

$$\lim_{n\to\infty} \|\mu_{n+1} - \mu_n\|^2 = 0. \tag{3.15}$$

Since sequence $\{u_i\}_{i=1}^{\infty}$ is bounded, so there exists a cluster point $\hat{\mu}$ to which the subsequence $\{u_{ik}\}_{k=1}^{\infty}$ converges. Taking limit in (3.8) and using (3.15), it follows that $\hat{\mu} \in \mathcal{H}$ satisfies

$$\langle T\hat{\mu} + \hat{\mu} - g(\hat{\mu}), \nu - \hat{\mu}\rangle + \phi(\nu) - \phi(\hat{u}) \geq 0, \quad \forall \nu \in \mathcal{H},$$

and

$$\|\mu_{n+1} - \mu\|^2 \leq \|\mu - \mu_n\|^2.$$

Using this inequality, one can show that the cluster point $\hat{u}$ is unique and

$$\lim_{n\to\infty} \mu_{n+1} = \hat{\mu}.$$

□

We now suggest an other implicit iterative method for solving (2.1). Discretizing (3.1), we have

$$\frac{\mu_{n+1} - \mu_n}{h} + \mu_n = J_\phi[g(\mu_{n+1}) - \rho T\mu_{n+1}], \tag{3.16}$$

where h is the step size.

For $h = 1$, this formulation enable us to suggest the following iterative method.

Algorithm 3.3. *For a given $\mu_0 \in \mathcal{H}$, compute μ_{n+1} by the iterative scheme*

$$\mu_{n+1} = J_\phi\Big[g(\mu_{n+1}) - \rho T\mu_{n+1}\Big], \quad n = 0, 1, 2, \ldots.$$

Using Lemma 2.1, Algorithm 3.3 can be rewritten in the equivalent form as:

Algorithm 3.4. *For a given $\mu_0 \in \mathcal{H}$, compute μ_{n+1} by the iterative scheme*

$$\langle \rho T\mu_{n+1} + \mu_{n+1} - g(\mu_{n+1}), \nu - \mu_{n+1}\rangle + \phi(\nu) - \phi(\mu_{n+1}) \geq 0, \quad \forall \nu \in \mathcal{H}. \tag{3.17}$$

Remark 3.1. *For appropriate and suitable choice of the discretizing (3.1), one can suggest and analyze a wide class of iterative methods for solving mixed general variational inequalities. This is an interesting problem for future research.*

We now introduce the second order dynamical system associated with the variational inequality (2.1), which is the main aim of this paper. To be more precise, we consider the problem of finding $\mu \in \mathtt{H}$ such that

$$\gamma\ddot{\mu} + \dot{\mu} = \lambda\big\{J_\phi[g(\mu) - \rho\mathcal{T}\mu] - \mu\big\}, \quad \mu(a) = \alpha, \quad \mu(b) = \beta, \tag{3.18}$$

where $\gamma > 0, \lambda > 0$ and $\rho > 0$ are constants. We would like to emphasize that the problem (3.18) is indeed a second order boundary value problem.

The equilibrium point of the dynamical system (3.18) is naturally defined as follows.

Definition 3.4. *An element $\mu \in \mathcal{H}$ is an equilibrium point of the dynamical system (3.18), if $\gamma\frac{d^2u}{dx^2} + \frac{du}{dx} =$ 0, that is,*

$$\mu = J_\phi[g(\mu) - \rho\mathcal{T}\mu].$$

This implies that

$$\mu = J_\phi\big[g(\mu) - \rho\mathcal{T}\mu + \gamma\frac{d^2\mu}{dx^2} + \frac{d\mu}{dx}\big]. \tag{3.19}$$

Thus it is clear that $\mu \in \mathcal{H}$ is a solution of the variational inequality (2.1), if and only if, $\mu \in \mathcal{H}$ is an equilibrium point.

For simplicity, we take $\lambda = 1$. Thus the problem (3.18) is equivalent to finding $\mu \in \phi$ such that

$$\gamma\ddot{\mu} + \dot{\mu} + \mu = J_\phi\big[g(\mu) - \rho\mathcal{T}\mu\big], \quad \mu(a) = \alpha, \quad \mu(b) = \beta. \tag{3.20}$$

The problem (3.20) is called the second dynamical system, which is a second order boundary value problem. This interlink among various areas is fruitful from numerical analysis in developing implementable numerical methods for finding the approximate solutions of the variational inequalities. Consequently, we can explore the ideas and techniques of the differential equations to suggest and propose hybrid proximal point methods for solving the variational inequalities and related optimization problems. We discretize the second-order dynamical systems (3.20) using central finite difference and backward difference schemes to have

$$\gamma\frac{\mu_{n+1} - 2\mu_n + \mu_{n-1}}{h^2} + \frac{\mu_n - \mu_{n-1}}{h} + \mu_n = J_\phi[g(\mu_n) - \rho(\mathcal{T}\mu_{n+1})], \tag{3.21}$$

where h is the step size.

If $\gamma = 1, h = 1$, then from equation (3.21), we have

Algorithm 3.1. *For a given $\mu_0 \in \mathcal{H}$, compute μ_{n+1} by the iterative scheme*

$$\mu_{n+1} = J_\phi[g(\mu_n) - \rho\mathcal{T}\mu_{n+1}].$$

Algorithm 3.1 is an implicit method. To implement the implicit method, we use the predictor-corrector technique to suggest the method.

Algorithm 3.2. *For given $\mu_0, \mu_1 \in \mathcal{H}$, compute μ_{n+1} by the iterative scheme*

$$\begin{aligned} y_n &= J_\phi[g(\mu_n) - \rho\mathcal{T}\mu_n] \\ \mu_{n+1} &= J_\phi[g(\mu_n) - \rho\mathcal{T}y_n], \end{aligned}$$

is called the extraresolvent method of Korpelevich [25] for solving the mixed general variational inequality.

Problem (3.20) can be rewritten as

$$\gamma\ddot{\mu} + \dot{\mu} + \mu = J_\phi[g((1-\theta_n)\mu + \theta_n\mu) - \rho\mathcal{T}((1-\theta_n)\mu + \theta_n\mu)], \quad \mu(a) = \alpha, \mu(b) = \beta, \tag{3.22}$$

where $\gamma > 0, \theta_n$ and $\rho > 0$ are constants.

Discretising the system (3.22), we have

$$\begin{aligned} &\gamma\frac{\mu_{n+1} - 2\mu_n + \mu_{n-1}}{h^2} + \frac{\mu_{n+1} - \mu_n}{h} + \mu_n \\ &= J_\phi[g((1-\theta_n)\mu_n + \theta_n\mu_{n-1}) - \rho\mathcal{T}((1-\theta_n)\mu_n + \theta_n\mu_{n-1})] \end{aligned}$$

from which, for $\gamma = 0, \quad h = 1$, we have

Algorithm 3.3. *For a given $\mu_0, \mu_1 \in \mathcal{H}$, compute μ_{n+1} by the iterative scheme*

$$\mu_{n+1} = J_\phi[g((1-\theta_n)u_n + \theta_n\mu_{n-1}) - \rho\mathcal{T}((1-\theta_n)\mu_n + \theta_n\mu_{n-1})].$$

Using the predictor corrector technique, Algorithm 3.3 can be written as

Algorithm 3.4. *For a given $\mu_0, \mu_1 \in \mathcal{H}$, compute μ_{n+1} by the iterative scheme*

$$\begin{aligned} y_n &= (1-\theta_n)\mu_n + \theta_n\mu_{n-1} \\ \mu_{n+1} &= J_\phi[g(y_n) - \rho\mathcal{T}y_n], \end{aligned}$$

which is called the new two step inertial iterative method for solving the variational inequality.

We discretize the second-order dynamical systems (3.20) using central finite difference and backward difference schemes to have

$$\gamma\frac{\mu_{n+1} - 2\mu_n + \mu_{n-1}}{h^2} + \frac{\mu_n - \mu_{n-1}}{h} + \mu_{n+1} = J_\phi[g(\mu_n) - \rho\mathcal{T}\mu_{n+1}],$$

where h is the step size.

Using this discrete form, we can suggest the following an iterative method for solving the variational inequalities (2.1).

Algorithm 3.5. *For given $\mu_0, \mu_1 \in \mathcal{H}$, compute μ_{n+1} by the iterative scheme*

$$\mu_{n+1} = J_\phi[g(\mu_n) - \rho\mathcal{T}\mu_{n+1} - \frac{\gamma\mu_{n+1} - (2\gamma - h)\mu_n + (\gamma - h)\mu_{n-1}}{h^2}].$$

Algorithm 3.5 is called the inertial proximal method for solving the general variational inequalities and related optimization problems. This is a new proposed method.

We can rewrite the Algorithm 3.5 in the equivalent form as follows:

Algorithm 3.6. *For a given $\mu_0, \mu_1 \in \mathcal{H}$, compute μ_{n+1} by the iterative scheme*

$$\begin{aligned} &\langle \rho\mathcal{T}\mu_{n+1} + \frac{(\gamma + h^2)\mu_{n+1} - (2\gamma - h)\mu_n + (\gamma - h)\mu_{n-1}}{h^2} - g(\mu_n), \nu - \nu_{n+1}\rangle \\ &\quad + \rho(\phi(\nu) - \phi(\mu_{n+1})) \geq 0, \forall \nu \in \mathcal{H} \end{aligned} \tag{3.23}$$

We note that, for $\gamma = 0, \quad h = 1$, Algorithm 3.6 reduces to the following iterative method for solving variational inequalities (2.1).

Algorithm 3.7. *For given $\mu_0, \mu_1 \in \mathcal{H}$,, compute μ_{n+1} by the iterative scheme*

$$\mu_{n+1} = J_\phi[g(\mu_n) + (\mu_n - \mu_{n-1}) - \rho\mathcal{T}\mu_{n+1)}].$$

We again discretize the second-order dynamical systems (3.20) using central difference scheme and forward difference scheme to suggest the following inertial proximal method for solving (2.1).

Algorithm 3.8. *For a given $\mu_0, \mu_1 \in \mathcal{H}$, compute μ_{n+1} by the iterative scheme*

$$\mu_{n+1} = J_\phi[g(\mu_{n+1}) - \rho\mathcal{T}\mu_{n+1} - \frac{(\gamma + h)\mu_{n+1} - (2\gamma + h)\mu_n + \gamma\mu_{n-1}}{h^2}].$$

Algorithm 3.8 is quite different from other inertial proximal methods for solving the variational inequalities.

If $\gamma = 0$, then Algorithm 3.8 collapses to:

Algorithm 3.9. *For a given $\mu_0 \in \mathcal{H}$, compute μ_{n+1} by the iterative scheme*

$$\mu_{n+1} = J_\phi[g(\mu_{n+1}) - \rho\mathcal{T}\mu_{n+1} - \frac{\mu_{n+1} - \mu_n}{h}].$$

Algorithm 3.8 is an proximal method for solving the variational inequalities. Such type of proximal methods were suggested by Noor [36] using the fixed point problems.

In brief, by suitable descritization of the second-order dynamical systems (3.20), one can construct a wide class of explicit and implicit method for solving inequalities.

Rewriting the problem (3.20) in the following form

$$\gamma\ddot{\mu} + \dot{\mu} + \mu = J_\phi\big[g(\frac{\mu+\mu}{2}) - \rho\mathcal{T}(\frac{\mu+\mu}{2})\big], \tag{3.24}$$

and descretizing, taking $\lambda = 1, h = 1$, we obtain

Algorithm 3.10. *For given $\mu_0 \in \mathcal{H}$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\mu_{n+1} = J_\phi\big[g(\frac{\mu_n + \mu_{n+1}}{2}) - \rho\mathcal{T}(\frac{\mu_n + \mu_{n+1}}{2})\big],$$

which is an implicit iterative method. Using the predictor and corrector technique, we suggest the following two-step iterative method for solving the variational inequalities.

Algorithm 3.11. *For given $\mu_0 \in \mathcal{H}$, compute the approximate solution μ_{n+1} by the iterative scheme*

$$\begin{aligned} y_n &= J_\phi\big[g(\mu_n) - \rho\mathcal{T}\mu_n\big] \\ \mu_{n+1} &= J_\phi\big[g(\frac{\mu_n + y_n}{2}) - \rho\mathcal{T}(\frac{\mu_n + y_n}{2})\big]. \end{aligned}$$

Algorithm 3.11 is the two step iterative method.

Clearly Algorithm 3.10 and Algorithm 3.11 are equivalent.

It is enough to prove the convergence of Algorithm 3.10, which is the main motivation of our next result.

Theorem 3.5. *Let the operator T, g be Lipschitz continuous with constant $\beta > 0, \sigma > 0$, respectively. Let $u \in \mathcal{H}$ be solution of (2.1) and μ_{n+1} be an approximate solution obtained from Algorithm 3.10. If there exists a constant $\rho > 0$, such that*

$$\rho < \frac{1-\sigma}{\beta}, \quad \sigma < 1, \tag{3.25}$$

then the approximate solution μ_{n+1} converge to the exact solution $\mu \in \Omega$.

Proof. Let $\mu \in \mathcal{H}$ be a solution of (2.1) and μ_{n+1} be the approximate solution obtained from Algorithm

3.10. Then. using the Lipschitz continuity of the operators T and g with constants β, σ, we obtain

$$\begin{aligned}
\|\mu_{n+1} - \mu\| &= \|J_\phi[g(\frac{\mu_n + \mu_{n+1}}{2}) - \rho T(\frac{\mu_n + \mu_{n+1}}{2})] - J_\phi[g(\frac{\mu + \mu}{2}) - \rho T(\frac{\mu + \mu}{2})]\| \\
&\leq \|g(\frac{\mu_n + \mu_{n+1}}{2}) - g(\frac{\mu + \mu}{2}) - \rho(T(\frac{\mu_{n+1} + \mu_n}{2}) - T(\frac{\mu + \mu}{2}))\| \\
&\leq \|g(\frac{\mu_n + \mu_{n+1}}{2}) - g(\frac{\mu + \mu}{2})\| + \rho(T(\frac{\mu_{n+1} + \mu_n}{2}) - T(\frac{\mu + \mu}{2}))\| \\
&\leq (\sigma + \rho\beta)\|(\frac{\mu_n + \mu_{n+1}}{2}) - (\frac{\mu + \mu}{2})\| \\
&\leq \frac{\sigma + \rho\beta}{2}\{\|\mu_{n+1} - \mu\| + \|\mu_n - \mu\|\},
\end{aligned}$$

from which, we obtain

$$\begin{aligned}
\|\mu_{n+1} - \mu\| &\leq \frac{\sigma + \rho\beta}{2 - \sigma - \rho\beta}\|\mu_n - \mu\| \\
&= \theta\|\mu_n - \mu\|,
\end{aligned}$$

where

$$\theta = \frac{\sigma + \rho\beta}{2 - \sigma - \rho\beta}.$$

From (3.25), it implies that $\theta < 1$. This shows that the approximate solution μ_{n+1} obtained from Algorithm 3.10 converges to the exact solution $\mu \in \mathcal{H}$ satisfying the general variational inequality (2.1). □

To implement the implicit Algorithm 3.10, one uses the predictor-corrector technique. Thus, we obtain new multi step step methods for solving variational inequalities.

Algorithm 3.12. *For given $\mu_0 \in \Omega$, compute the approximate solution u_{n+1} by the iterative schemes*

$$\begin{aligned}
y_n &= (1 - \alpha_n)\mu_n + \alpha_n J_\phi\big[\mu_n - \rho\mathcal{T}\mu_n\big] \\
w_n &= (1 - \eta_n)y_n + \eta_n J_\phi\big[g(\frac{\mu_n + y_n}{2}) - \rho\mathcal{T}(\frac{\mu_n + y_n}{2})\big] \\
\mu_{n+1} &= (1 - \beta_n)w_n + \beta_n J_\phi\big[g(\frac{w_n + y_n}{2}) - \rho\mathcal{T}(\frac{w_n + y_n}{2})\big],
\end{aligned}$$

which is a three step method, where $\alpha_n, \eta_n, \beta_n$ are constants.

Algorithm 3.13. *For given $\mu_0, \mu_1 \in \mathcal{H}$, compute the approximate solution u_{n+1} by the iterative schemes*

$$\begin{aligned}
t_n &= (1 - \theta_n)\mu_n + \theta_n\mu_{n-1} \\
y_n &= (1 - \alpha_n)t_n + \alpha_n J_\phi\big[g(\frac{\mu_n + t_n}{2}) - \rho\mathcal{T}(\frac{\mu_n - t_n}{2})\big] \\
w_n &= (1 - \beta_n)y_n + \beta_n J_\phi\big[g(\frac{\mu_n + y_n}{2}) - \rho\mathcal{T}(\frac{\mu_n + y_n}{2})\big] \\
\mu_{n+1} &= (1 - \zeta_n)w_n + \zeta_n J_\phi\big[g(\frac{w_n + y_n}{2}) - \rho\mathcal{T}(\frac{w_n + y_n}{2})\big],
\end{aligned}$$

which is a four step inertial iterative method, here $\theta_n, \alpha_n, \beta_n, \zeta_n$ are constants.

Remark 3.2. *These multi-step methods contain Mann (one-step) iteration, Ishikawa (two-step) iteration and Noor (three-step) iterations as special cases. Noor [28, 30] has proposed and suggested three step forward-backward iterative methods for finding the approximate solution of general variational inequalities using the technique of updating the solution and auxiliary principle. These tree-step methods are known as Noor iterations. Suantai et al. [65] have also considered some novel forward-backward algorithms for optimization and their applications to compressive sensing and image inpainting. We have shown that these multi step can be proposed and suggested using the dynamical systems coupled with boundary value problems, which is can be considered entirely new approach.*

Zeng et al. [70] have investigated the fractional dynamical systems associated with variational inequalities. They have investigated the criteria for the asymptotically stability of the equilibrium points. We would like to point out that our results are more general than the results of Zeng et al. [70]. These ideas and techniques may inspire the interested readers for further research in this area. We now suggest a new fractional resolvent dynamical system associated with mixed general variational inequalities.

$$D_t^{\alpha}\mu = \gamma\{-R(\mu) - \rho T J_{\phi}[g(\mu) - \rho \mathcal{T}\mu] + \rho \mathcal{T}\mu\}, \quad \mu(0) = \alpha, \quad \mu \in H, \tag{3.26}$$

where $0 < \alpha < 1$ and γ is a constant, associated with problem mixed variational inequality. For more applications and motivation, see [22].

For $\alpha = 1$, problem (3.26) reduces to finding $u \in H$ such that

$$\frac{d\mu}{dx} = \gamma\{-R(\mu) - \rho T J_{\phi}[g(\mu) - \rho \mathcal{T}u] + \rho \mathcal{T}\mu\}, \quad \mu(0) = \alpha, \quad \mu \in \mathcal{H}, \tag{3.27}$$

is called the resolvent dynamical system, which appear to be a new one. Using the technique of this section, one can investigate the asymptotically stability and other aspects.

4 Generalizations and Future Research

We would like to mention that some of the results obtained and presented in this paper can be extended for multivalued variational inequalities. To be more precise, let $C(H)$ be a family of nonempty compact subsets of H. Let $T, V : H \longrightarrow C(H)$ be the multivalued operators. For a given nonlinear bifunction $N(.,.) : H \times H \longrightarrow H$, consider the problem of finding $u \in \mathcal{H}, w \in T(\mu), y \in V(\mu)$ such that

$$\langle N(w, y) + \mu - g(\mu), \nu - \mu\rangle + \phi(\nu) - \phi(\mu) \geq 0, \quad \forall \nu \in \mathcal{H}, \tag{4.1}$$

which is called the multivalued mixed general variational inequality. We would like to mention that one can obtain various classes of variational inequalities for appropriate and suitable choices of the bifunction $N(.,.)$, and the operators.

1. For $g(\mu) = \mu$, the problem (4.1) reduces to finding $\mu \in \mathcal{H}$, such that

$$\langle N(w, y), \nu - \mu \rangle + \phi(\nu) - \phi(g(\mu)) \geq 0, \quad \forall \quad \nu \in \mathcal{H}, \tag{4.2}$$

is called the multivalued mixed general variational inequality, which appears to be a new one.

2. If $N(w, y) = T\mu$, then the problem (4.1) is equivalent to find $\mu \in \mathcal{H}$, such that

$$\langle T\mu + \mu - g(\mu), \nu - \mu \rangle + \phi(\nu) - \phi(\mu) \geq 0 \quad \forall \nu \in \mathcal{H},$$

which is the mixed general variational inequality (2.1).

Using Lemma 2.1, one can prove that the problem (4.1) is equivalent to finding $u \in \mathcal{H}$ such that

$$\mu = J_\phi[g(\mu) - \rho N(w, y)]. \tag{4.3}$$

This shows that the the problem (4.1) is equivalent to the fixed point problem (4.3). This equivalent formulation is applied to consider the second order dynamical system associated with the problem (4.1) as

$$\gamma \ddot{\mu} + \dot{\mu} + \mu = J_\phi[g(\mu) - \rho \mathcal{T}\mu], \quad \mu(a) = \alpha, \quad \mu(b) = \beta,$$

which is the second order boundary value problem and may be applied to suggest and investigate proximal point methods for solving the multivalued mixed variational inequality (4.1) applying the techniques developed in this paper. Consequently, all results obtained for the problem (2.1) continue to hold for the problem (4.1) with suitable modifications and adjustments. Since the problem (4.1) and problem (4.2) are equivalent, if the convex set is a convex cone. This implies that the dynamical system approach may be exploited to solve the complementarity problems. The development of efficient implementable numerical methods for solving the multivalued variational inequalities, random elastic traffic equilibrium problem and optimization problems requires further efforts. Despite the current research activates, very few results are available. The development of efficient implementable numerical methods for solving the general quasi variational inequalities and non optimizations problems requires further efforts.

Conclusion

In this paper, we have used the technique of the dynamical systems coupled with the second order boundary value problem to suggest some multi step inertial proximal methods for solving variational inequalities. The convergence analysis of these methods have been considered under some weaker conditions. Our method of convergence criteria is very simple as compared with other techniques.

Comparison and implementation of these new methods need further efforts. We have only discussed the theoretical aspects of the proposed iterative methods. It is an interesting problem to discuss the implementation and performance of these new methods with other methods. Applications of the fuzzy set theory, stochastic, quantum calculus, fractal, fractional and random can be found in many branches of mathematical and engineering sciences including artificial intelligence, computer science, control engineering, management science, operations research and variational inequalities. Similar methods can be suggested for stochastic, fuzzy, quantum, random and fractional variational inequalities, which is an interesting and challenging problem. Despite the recent research activates, very few results are available. The development of efficient numerical methods requires further efforts. The ideas and techniques presented in this paper may be starting point for further developments.

Contributions of the authors:

All the authors contributed equally in writing, editing, reviewing and agreed for the final version for publication.

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References

[1] Ames, W. F. (1992). *Numerical methods for partial differential equations.* Academic Press. https://doi.org/10.1142/9789814537735

[2] Ashish, Rani, M., & Chugh, R. (2012). Dynamics of antifractals in Noor orbit. *International Journal of Computer Applications, 57*(4).

[3] Ashish, Rani, M., & Chugh, R. (2014). Julia sets and Mandelbrot sets in Noor orbit. *Applied Mathematics and Computation, 228*(1), 615-631. https://doi.org/10.1016/j.amc.2013.11.077

[4] Ashish, Chugh, R., & Rani, M. (2021). *Fractals and chaos in Noor orbit: A four-step feedback approach.* LAP Lambert Academic Publishing.

[5] Ashish, Cao, J., & Noor, M. A. (2023). Stabilization of fixed points in chaotic maps using Noor orbit with applications in cardiac arrhythmia. *Journal of Applied Analysis and Computation, 13*(5), 2452-2470. https://doi.org/10.11948/20220350

[6] Barbagallo, A., & Lo Bainco, S. G. (2024). A random elastic traffic equilibrium problem via stochastic quasi-variational inequalities. *Communications in Nonlinear Science and Numerical Simulation, 131*, 107798. https://doi.org/10.1016/j.cnsns.2023.107798

[7] Bnouhachem, A. (2005). A self-adaptive method for solving general mixed variational inequalities. *Journal of Mathematical Analysis and Applications, 309*, 136-150. https://doi.org/10.1016/j.jmaa.2004.12.023

[8] Bnouhachem, A., & Hamdi, A. (2014). Parallel LQP alternating direction method for solving variational inequality problems with separable structure. *Journal of Inequalities and Applications, 2014*, ID=392. https://doi.org/10.1186/1029-242X-2014-392

[9] Bnouhachem, A., Noor, M. A., & Rassias, T. M. (2006). Three-step iterative algorithms for mixed variational inequalities. *Applied Mathematics and Computation, 183*, 436-446. https://doi.org/10.1016/j.amc.2006.05.086

[10] Bnouhachem, A., Noor, M. A., Khalfaoui, M., & Zhaohan, S. (2011). A resolvent method for solving mixed variational inequalities. *Journal of King Saud University-Science, 23*(2), 235-240. https://doi.org/10.1016/j.jksus.2010.07.015

[11] Brezis, H. (1973). *Opérateurs maximaux monotones et semigroupes de contractions dans les espaces de Hilbert.* North-Holland.

[12] Chairatsiripong, C., & Thianwan, T. (2022). Novel Noor iterations technique for solving nonlinear equations. *AIMS Mathematics, 7*(6), 10958-10976. https://doi.org/10.3934/math.2022612

[13] Cho, S. Y., Shahid, A. A., Nazeer, W., & Kang, S. M. (2006). Fixed point results for fractal generation in Noor orbit and s-convexity. *SpringerPlus, 5*, 1843.

[14] Chugh, R., Rani, M., & Ashish, R. (2012). On the convergence of logistic map in Noor orbit. *International Journal of Computer Applications, 43*(18). https://doi.org/10.5120/6200-8739

[15] Cottle, R. W., Pang, J.-S., & Stone, R. E. (2009). *The linear complementarity problem.* SIAM Publications. https://doi.org/10.1137/1.9780898719000

[16] Cristescu, G., & Lupsa, L. (2002). *Non-connected convexities and applications.* Kluwer Academic Publishers. https://doi.org/10.1007/978-1-4615-0003-2

[17] Dupuis, P., & Nagurney, A. (1993). Dynamical systems and variational inequalities. *Annals of Operations Research, 44*, 7-42. https://doi.org/10.1007/BF02073589

[18] Glowinski, R., Lions, J. L., & Tremolieres, R. (1981). *Numerical analysis of variational inequalities.* North-Holland.

[19] Glowinski, R., & Le Tallec, P. (1989). *Augmented Lagrangian and operator splitting methods in nonlinear mechanics.* SIAM. https://doi.org/10.1137/1.9781611970838

[20] Hamdi, A. (2005). A Moreau-Yosida regularization of a difference of two convex functions. *Applied Mathematics E-Notes, 5*, 164-170.

[21] Khan, A. G., Noor, M. A., & Noor, K. I. (2015). Dynamical systems for general quasi-variational inequalities. *Annals of Functional Analysis, 6*(1), 193-209. https://doi.org/10.15352/afa/06-1-14

[22] Khan, A. G., Noor, M. A., & Noor, K. I. (2019). Dynamical systems for variational inclusions involving difference operators. *Honam Mathematical Journal, 41*(1), 207-225.

[23] Kikuchi, N., & Oden, J. T. (1988). *Contact problems in elasticity.* SIAM Publishing. https://doi.org/10.1137/1.9781611970845

[24] Kinderlehrer, D., & Stampacchia, G. (2000). *An introduction to variational inequalities and their applications.* SIAM. https://doi.org/10.1137/1.9780898719451

[25] Korpelevich, G. M. (1976). The extragradient method for finding saddle points and other problems. *Ekonomika i Matematicheskie Metody, 12*, 747-756.

[26] Kwuni, Y. C., Shahid, A. A., Nazeer, W., Butt, S. I., Abbas, M., & Kang, S. M. (2019). Tricorns and multicorns in Noor orbit with s-convexity. *IEEE Access, 7.* https://doi.org/10.1109/ACCESS.2019.2928796

[27] Lions, J., & Stampacchia, G. (1967). Variational inequalities. *Communications on Pure and Applied Mathematics, 20*, 493-519. https://doi.org/10.1002/cpa.3160200302

[28] Nagurney, A., & Zhang, D. (1996). *Projected dynamical systems and variational inequalities with applications.* Kluwer Academic Publishers. https://doi.org/10.1007/978-1-4615-2301-7

[29] Nammanee, K., Noor, M. A., & Suantai, S. (2006). Convergence criteria of modified Noor iterations with errors for asymptotically nonexpansive mappings. *Journal of Mathematical Analysis and Applications, 314*, 320-334. https://doi.org/10.1016/j.jmaa.2005.03.094

[30] Niculescu, C. P., & Persson, L. E. (2018). *Convex functions and their applications.* Springer-Verlag. https://doi.org/10.1007/978-3-319-78337-6_1

[31] Noor, M. A. (1975). On variational inequalities (PhD thesis). Brunel University, London, UK.

[32] Noor, M. A. (1988). General variational inequalities. *Applied Mathematics Letters, 1*, 119-121. https://doi.org/10.1016/0893-9659(88)90054-7

[33] Noor, M. A. (1988). Quasi variational inequalities. *Applied Mathematics Letters, 1*(4), 367-370. https://doi.org/10.1016/0893-9659(88)90152-8

[34] Noor, M. A. (1988). Fixed point approach for complementarity problems. *Journal of Mathematical Analysis and Applications, 33*, 437-448. https://doi.org/10.1016/0022-247X(88)90413-1

[35] Noor, M. A. (1997). Some recent advances in variational inequalities: Part II. Other concepts. *New Zealand Journal of Mathematics, 26*, 229-255.

[36] Noor, M. A. (1977). Generalized mixed variational inequalities and resolvent equations. *Positivity, 1*, 145-154.

[37] Noor, M. A. (2000). Variational inequalities for fuzzy mappings (III). *Fuzzy Sets and Systems, 110*(1), 101-108. https://doi.org/10.1016/S0165-0114(98)00131-6

[38] Noor, M. A. (2000). New approximation schemes for general variational inequalities. *Journal of Mathematical Analysis and Applications, 251*, 217-230. https://doi.org/10.1006/jmaa.2000.7042

[39] Noor, M. A. (2001). Three-step iterative algorithms for multivalued quasi variational inclusions. *Journal of Mathematical Analysis and Applications, 255*(2), 589-604. https://doi.org/10.1006/jmaa.2000.7298

[40] Noor, M. A. (2002). Resolvent dynamical systems for mixed variational inequalities. *Korean Journal of Computational and Applied Mathematics, 9*(1), 15-26. https://doi.org/10.1007/BF03012337

[41] Noor, M. A. (2004). Some developments in general variational inequalities. *Applied Mathematics and Computation, 152*, 199-277. https://doi.org/10.1016/S0096-3003(03)00558-7

[42] Noor, M. A. (2001). Three-step iterative algorithms for multivalued quasi variational inclusions. *Journal of Mathematical Analysis, Applications, 255*(2), 589-604. https://doi.org/10.1006/jmaa.2000.7298

[43] Noor, M. A. (2003). Mixed quasi variational inequalities. *Applied Mathematics and Computation, 146*, 553-578. https://doi.org/10.1016/S0096-3003(02)00605-7

[44] Noor, M. A. (2009). Extended general variational inequalities. *Applied Mathematics Letters, 22*, 182-185. https://doi.org/10.1016/j.aml.2008.03.007

[45] Noor, M. A. (2008). Differentiable non-convex functions and general variational inequalities. *Applied Mathematics and Computation, 199*(2), 623-630. https://doi.org/10.1016/j.amc.2007.10.023

[46] Noor, M. A., & Al-Said, E. A. (1999). Change of variable method for generalized complementarity problems. *Journal of Optimization Theory and Applications, 100*(2), 389-395. https://doi.org/10.1023/A:1021790404792

[47] Noor, M. A. (2010). Some iterative schemes for general mixed variational inequalities. *Journal of Applied Mathematics and Computation, 34*(1), 57-70. https://doi.org/10.1007/s12190-009-0306-x

[48] Noor, M. A., & Noor, K. I. (2008). Proximal-resolvent methods for mixed variational inequalities. *CUBO Mathematics Journal, 10*(03), 1-12.

[49] Noor, M. A., & Noor, K. I. (2022). Dynamical system technique for solving quasi variational inequalities. *U.P.B. Science Bulletin, Series A, 84*(4), 55-66.

[50] Noor, M. A., & Noor, K. I. (2022). Some new trends in mixed variational inequalities. *Journal of Advanced Mathematics Studies, 15*(2), 105-140.

[51] Noor, M. A., & Noor, K. I. (2023). Iterative schemes for solving general variational inequalities. *Differential Equations and Applications, 15*(2), 113-134. https://doi.org/10.7153/dea-2023-15-07

[52] Noor, M. A., & Noor, K. I. (2024). Some novel aspects and applications of Noor iterations and Noor orbits. *Journal of Advanced Mathematics Studies, 17*(3), 274-284.

[53] Noor, K. I., Noor, M. A., & Kankum, K. (2025). On a new generalization of the Lax-Milgram Lemma. *Earthline Journal of Mathematical Sciences, 15*(1), 23-34. https://doi.org/10.34198/ejms.15125.023034

[54] Noor, M. A., Noor, K. I., & Khan, A. G. (2015). Dynamical systems for quasi variational inequalities. *Annals of Functional Analysis, 6*(1), 193-209. https://doi.org/10.15352/afa/06-1-14

[55] Noor, M. A., Noor, K. I., & Bnouhachem, A. (2021). Some new iterative methods for variational inequalities. *Canadian Journal of Applied Mathematics, 3*(1), 1-17.

[56] Noor, M. A., Noor, K. I., & Rassias, M. T. (2020). New trends in general variational inequalities. *Acta Applicandae Mathematicae, 170*(1), 981-1046. https://doi.org/10.1007/s10440-020-00366-2

[57] Noor, M. A., Noor, K. I., & Rassias, T. M. (1993). Some aspects of variational inequalities. *Journal of Computational and Applied Mathematics, 47*, 285-312. https://doi.org/10.1016/0377-0427(93)90058-J

[58] Noor, M. A., Noor, K. I., & Yaqoob, H. (2010). On general mixed variational inequalities. *Acta Applicandae Mathematicae, 110*, 227-246. https://doi.org/10.1007/s10440-008-9402-4

[59] Noor, M. A., & Oettli, W. (1994). On general nonlinear complementarity problems and quasi equilibria. *Le Mathematiche, 49*, 31-31.

[60] Paimsang, S., Yambangwai, D., & Thainwan, T. (2024). A novel Noor iterative method of operators with property (E) as concerns convex programming applicable in signal recovery and polynomiography. *Mathematical Methods in the Applied Sciences*, 1-18. https://doi.org/10.1002/mma.10083

[61] Patriksson, M. (1998). *Nonlinear programming and variational inequalities: A unified approach*. Kluwer Academic Publishers. https://doi.org/10.1007/978-1-4757-2991-7

[62] Phuengrattana, W., & Suantai, S. (2011). On the rate of convergence of Mann, Ishikawa, Noor and SP-iterations for continuous functions on an arbitrary interval. *Journal of Computational and Applied Mathematics, 235*(9), 3006-3014. https://doi.org/10.1016/j.cam.2010.12.022

[63] Rattanaseeha, K., Imnang, S., Inkrong, P., & Thianwan, T. (2023). Novel Noor iterative methods for mixed-type asymptotically nonexpansive mappings from the perspective of convex programming in hyperbolic spaces. *International Journal of Innovative Computing Information and Control, 19*(6), 1717-1734.

[64] Stampacchia, G. (1964). Formes bilineaires coercitives sur les ensembles convexes. *Comptes Rendus de l'Académie des Sciences de Paris, 258*, 4413-4416.

[65] Suantai, S., Noor, M. A., Kankum, K., & Cholamjiak, P. (2021). Novel forward-backward algorithms for optimization and applications to compressive sensing and image inpainting. *Advances in Difference Equations, 2021*(265). https://doi.org/10.1186/s13662-021-03422-9

[66] Ullah, S., & Noor, M. A. (2020). An efficient method for solving new general mixed variational inequality. *Journal of Inequalities and Special Functions, 11*(3), 1-9.

[67] Tseng, P. (2000). A modified forward-backward splitting method for maximal monotone mappings. *SIAM Journal on Control and Optimization, 38*, 431-446. https://doi.org/10.1137/S0363012998338806

[68] Xia, Y. S., & Wang, J. (2000). A recurrent neural network for solving linear projection equations. *Neural Networks, 13*, 337-350. https://doi.org/10.1016/S0893-6080(00)00019-8

[69] Xia, Y. S., & Wang, J. (2000). On the stability of globally projected dynamical systems. *Journal of Optimization Theory and Applications, 106*, 129-150. https://doi.org/10.1023/A:1004611224835

[70] Zeng-Bao, W., & Yun-Zhi, Z. (2014). Global fractional-order projective dynamical systems. *Communications in Nonlinear Science and Numerical Simulation, 19*, 2811-2819. https://doi.org/10.1016/j.cnsns.2014.01.007