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On a New Generalization of the Lax-Milgram Lemma

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Abstract

We consider a new generalization of the celebrated Lax-Milgram Lemma, which is called the harmonic Lax-Milgram Lemma. Some special cases are discussed. New concepts are introduced. The auxiliary principle approach is applied to discuss the existence of the solution as well as to propose some iterative schemes for computing the approximate solution of harmonic Max-Milgram Lemma. Convergence analysis of the proposed methods is considered under some mild conditions. Ideas and techniques of this paper may stimulate further research.

1 Introduction

Riesz [22] and Frechet [5] proved that a linear continuous functional can be represented by the inner product independently. This result is known as Riesz-Frechet representation theorem. It have been observed that this representation theorem is the optimum criteria of the quadratic differentiable functional on the inner product spaces. Lax and Milgram [8] proved that a linear continuous functional can be representation by an arbitrary bifunction under suitable conditions.. This representation is known as the Lax-Milgram Lemma and plays a significant role in the development of various branches of mathematical and engineering sciences. Motivated and inspired by ongoing research in this interesting field, we consider a harmonic Lax-Milgram lemma, which contains the Lax-Milgram Lemma [8] and Riesz-Frechet representation theorem as special cases. For the applications and generalizations of the Lax-Milgram Lemma, see [3–9, 11, 13, 17, 19] and the references therein.

Convexity theory contains a wealth of novel ideas and innovative techniques, which have played the

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significant role in the development of almost all the branches of pure and applied sciences such as fixed point, variational inequalities and optimizations. Several new generalizations and extensions of the convex functions and convex sets have been introduced and studied to tackle unrelated complicated and complex problems in a unified manner.

Anderson et al. [3] have investigated several aspects of the harmonic convex sets and harmonic convex functions, which can viewed as important generalizations of the convex functions and convex sets. The harmonic means have novel applications in electrical circuits theory. It is known that the total resistance of a set of parallel resistors is obtained by adding up the reciprocals of the individual resistance values, and then taking the reciprocal of their total. More precisely, if u and v are the resistances of two parallel resistors, then the total resistance is computed by the formula:

$$\left(\frac{1}{u}+\frac{1}{v}\right)^{-1}=\frac{uv}{u+v},$$

which is half the harmonic means. Al-Azemi et al. [1] studied the Asian options with harmonic average, which can be viewed a new direction in the study of the risk analysis and financial mathematics. Noor et al. [14] used the harmonic mean to suggest some iterative methods for solving nonlinear equations. Noor et al. [14] have shown that the minimum of the differentiable harmonic convex functions on the harmonic convex set can be characterized by a class of variational inequalities. For more details, see [14–21] and the references therein. We use the auxiliary principle technique, which is mainly due to Lions and Stampachia [9] and Glowinski et al. [7], to discuss various aspects of the general Lax-Milgram Lemma. Noor [12] and Noor et al. [14–21] have shown that the auxiliary principle technique can be used to suggest some iterative methods for solving the boundary value problems and various classes of variational inequalities. In Section 2, we introduce the harmonic Lax-Milgram Lemma and discuss its applications. The auxiliary principle technique is used to discuss the existence of a unique solution as well as to suggest some iterative methods for the boundary value problems. Convergence analysis of the proposed method is also considered under some mild conditions.

2 Formulations and Basic Facts

Let H be a Hilbert space, whose norm and inner product are denoted by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ respectively. For a given operator N and continuous functional f, we consider the problem of finding $u \in H$ such that

$$\left\langle N\left(\frac{2uv}{u+v}\right), v\right\rangle = \langle f, v\rangle, \qquad \forall v \in H, \tag{2.1}$$

which is called the harmonic Lax-Milgram Lemma. Equivalently the problem (2.1) equivalent to finding $u \in H$, we

$$\left\langle N\left(\frac{2uv}{u+v}\right), v-u\right\rangle = \langle f, v-u\rangle, \qquad \forall v \in H, \tag{2.2}$$

which is known as the weak formulation of the boundary value problems and appears to be a new ones. This is also called the weak formulation of the harmonic boundary value problems. A wide class of problems arising in pure and applied sciences can be studied via equations (2.1) and (2.2).

From problems (2.1) and (2.2), on can easily obtain the problem of finding $u \in H$ such that

$$N\left(\frac{2uv}{u+v}\right) = f, \tag{2.3}$$

which is called the harmonic nonlinear equation.

Note that, for $v = u$, the problem (2.3) reduces to finding $u \in H$ such that

$$N(u) = f, \qquad \forall v \in H, \tag{2.4}$$

is the usual nonlinear equation.

If $a\left(\frac{2uv}{v+u}, v-u\right) = \left\langle N\left(\frac{2uv}{v+u}\right), v-u\right\rangle$, where $a(\cdot,\cdot) : H \times H \to H$ is a bifunction, then the problem (2.2) is equivalent to finding $u \in H$ such that

$$a\left(\frac{2uv}{v+u}, v-u\right) = \langle f, v-u\rangle, \qquad \forall v \in H, \tag{2.5}$$

which is called the harmonic Lax-Milgram Lemma. This result has been used to discuss the existence of a unique solution of the boundary value problems. This result can have tantamount significance in the study of function spaces and partial differential equations.

If $N\left(\frac{2uv}{u+v}\right) = \frac{2uv}{u+v}$, $\forall v \in H$, then the problem (2.1) is equivalent to finding $u \in H$ such that

$$\left\langle \frac{2uv}{u+v}, v\right\rangle = \langle f, v\rangle, \qquad \forall v \in H, \tag{2.6}$$

which can be viewed as the Riesz-Frechet representation for the continuous functionals with respect to harmonic inner product.

We now introduce some new concepts in harmonic convex analysis.

Definition 2.1. *A set $\Omega \subseteq H$ is said to be a harmonic-like convex set, if*

$$\frac{2uv}{u+v} + t(v-u) \in \Omega, \qquad \forall u, v \in \Omega, t \in [0,1].$$

Note that, for $t = 0$, $\frac{2uv}{u+v} \in \Omega$, $t = 1$, $\left(\frac{2uv}{u+v} + v - u\right) \in \Omega$.

Definition 2.2. *A function f on the harmonic-like convex set Ω is said to be harmonic-like convex function, if*

$$f\left(\frac{2uv}{u+v}\right)+t(v-u)\leq f(u)+t(f(v)-f(u)),\qquad \forall u,v\in\Omega,\ t\in[0,1].$$

We remark that, if $v\to u$, then the harmonic-like convex set become the convex set and the harmonic-like convex functions reduces to the convex function.

Using the technique of Noor [12], one easily prove that the minimum of the differentiable harmonic-like convex function can be discussed.

Theorem 2.3. *Let f be a differentiable harmonic-like convex function. Then $u\in H$ is the minimum of the harmonic-like convex function, if and only if, $u\in H$ satisfies*

$$f'\left(\frac{2uv}{u+v}\right)=0,\qquad \forall v\in H, \tag{2.7}$$

which is special case of the nonlinear harmonic equation (2.3).

Definition 2.4. *An operator $N(\cdot)$ is said to be*

(I). strongly harmonic-like monotone, if there exists a constant $\alpha\geq 0$ such that

$$\left\|u-v-\rho\left(N\left(\frac{2u\eta}{u+\eta}\right)-N\left(\frac{2v\zeta}{v+\zeta}\right)\right)\right\|\geq\alpha\|u-v\|^2,\quad \forall u,v,\eta,\zeta\in H.$$

(II). harmonic-like Lipchitz continuous, if there exists a constant $\beta\geq 0$ such that

$$\left\|N\left(\frac{2u\eta}{u+\eta}\right)-N\left(\frac{2v\zeta}{v+\zeta}\right)\right\|\leq\beta\|u-v\|,\quad \forall u,v,\eta,\zeta\in H.$$

We remark that, for $u=\eta$ and $v=\zeta$, the definition 2.4 reduces to the classical strongly monotonicity and Lipchitz continuity of the operator.

3 Main Results

In this section, we use the auxiliary principle technique, the origin of which can be traced back to Lions and Stampacchia [9] and Glowinski et al. [4], as developed by Noor [14–21]. The main of idea of this technique is to consider an arbitrary auxiliary problem related to the original problem. This way, one defines a mapping connecting the solutions of both problems. To prove the existence of a solution of the original problem, it is enough to show that this connecting mapping is a contraction which yields the solution of the original problem. Another novel feature of this approach is that this technique enables us to suggest some iterative methods for solving the boundary value problems.

Theorem 3.1. *Let the operator L be strongly monotone with constant $\alpha > 0$ and Lipschitz continuous with constant $\beta > 0$, respectively. If there exists a constant $\rho > 0$ such that*

$$0 < \rho \leq \frac{\alpha}{\beta^2}, \tag{3.1}$$

then there exists a unique solution of problem (2.2).

Proof. We now use the auxiliary principle technique to prove the existence of a solution of (2.2). To be more precise, for a given $u \in H$ satisfying (2.2), consider the problem of finding $u \in H$ such that

$$\left\langle \rho N\left(\frac{2uv}{u+v}\right), v - w \right\rangle + \langle w - u, v - w\rangle = \langle \rho f, v - w\rangle, \qquad \forall v \in H, \tag{3.2}$$

which is called the auxiliary problem, where $\rho > 0$ is a constant. It is clear that (3.2) defines a mapping w connected the both problems (2.1) and (3.2). To prove the existence of a solution of (2.2), it is enough to show that the mapping w defined by (3.2) is a contraction mapping.

Let $w_1 \neq w_2 \in H$ (corresponding to $u_1 \neq u_2 \in H$) be solutions of (3.2). Then

$$\left\langle \rho N\left(\frac{2u_1v}{u_1+v}\right), v - w_1 \right\rangle + \langle w_1 - u_1, v - w_1\rangle = \langle \rho f, v - w_1\rangle, \qquad \forall v \in H, \tag{3.3}$$

$$\left\langle \rho N\left(\frac{2u_2v}{u_2+v}\right), v - w_2 \right\rangle + \langle w_2 - u_2, v - w_2\rangle = \langle \rho f, v - w_2\rangle, \qquad \forall v \in H. \tag{3.4}$$

Taking $v = w_2$ in (3.3) and $v = w_1$ in (3.4) and adding the resultants, we have

$$\begin{aligned} \|w_1 - w_2\|^2 &= \langle w_1 - w_2, w_1 - w_2\rangle \\ &= \left\langle u_2 - u_1 - \rho\left[N\left(\frac{2u_2w_2}{u_2+w_2}\right) - N\left(\frac{2u_1w_1}{u_1+w_1}\right)\right], w_1 - w_2 \right\rangle \\ &\leq \left\| u_2 - u_1 - \rho\left[N\left(\frac{2u_2w_2}{u_2+w_2}\right) - N\left(\frac{2u_1w_1}{u_1+w_1}\right)\right]\right\| \|w_1 - w_2\|, \end{aligned} \tag{3.5}$$

from which, it follows that

$$\|w_1 - w_2\| \leq \left\| u_2 - u_1 - \rho\left[N\left(\frac{2u_2w_2}{u_2+w_2}\right) - N\left(\frac{2u_1w_1}{u_1+w_1}\right)\right]\right\|. \tag{3.6}$$

Using the strongly harmonic-like monotonicity and harmonic-like Lipchitz continuity of the operator

L with constants $\alpha > 0$ and $\beta > 0$, we have

$$\begin{aligned}
\|w_1 - w_2\|^2 &\leq \left\| u_2 - u_1 - \rho\left[N\left(\frac{2u_2w_2}{u_2+w_2}\right) - N\left(\frac{2u_1w_1}{u_1+w_1}\right)\right]\right\|^2 \\
&= \left\langle u_2 - u_1 - \rho\left[N\left(\frac{2u_2w_2}{u_2+w_2}\right) - N\left(\frac{2u_1w_1}{u_1+w_1}\right)\right],\right. \\
&\qquad \left. u_2 - u_1 - \rho\left[N\left(\frac{2u_2w_2}{u_2+w_2}\right) - N\left(\frac{2u_1w_1}{u_1+w_1}\right)\right]\right\rangle \\
&= \langle u_1 - u_2, u_1 - u_2\rangle - 2\rho\left\langle N\left(\frac{2u_2w_2}{u_2+w_2}\right) - N\left(\frac{2u_1w_1}{u_1+w_1}\right), u_2 - u_1\right\rangle \\
&\qquad + \rho^2\left\langle N\left(\frac{2u_2w_2}{u_2+w_2}\right) - N\left(\frac{2u_1w_1}{u_1+w_1}\right), N\left(\frac{2u_2w_2}{u_2+w_2}\right) - N\left(\frac{2u_1w_1}{u_1+w_1}\right)\right\rangle \\
&\leq (1 - 2\rho\alpha + \rho^2\beta^2)\|u_1 - u_2\|^2, \qquad (3.7)
\end{aligned}$$

using the strongly harmonic-like monotonicity with constant $\alpha > 0$ and harmonic-like Lipschitz continuity with constant $\beta \geq 0$ of the harmonic operator N.

Form (3.7), we obtain

$$\begin{aligned}
\|w_1 - w_2\| &\leq \sqrt{1 - 2\rho\alpha + \rho^2\beta^2} \cdot \|u_1 - u_2\| \\
&= \vartheta(\rho)\|u_1 - u_2\| \qquad (3.8)
\end{aligned}$$

where $\vartheta(\rho) = \sqrt{1 - 2\rho\alpha + \rho^2\beta^2}$.

We have to show that $\vartheta(\rho) < 1$ It is clear that $\vartheta(\rho)$ assumes its minimum value for $\rho = \frac{\alpha}{\beta^2}$ with $\vartheta(\rho) = \sqrt{1 - \frac{\alpha}{\beta^2}}$. From (3.1), It follows that $\vartheta(\rho) < 1$ for $0 < \rho \leq \frac{\alpha}{\beta^2}$. Thus the mapping is a contraction mapping and consequently, it has a fixed point $w(u) = u \in H$ satisfying the problem (2.2). □

It is worth mentioning that, if $w(u) = u \in H$ is a solution of (2.2), then the auxiliary principle technique enables us to suggest the following iterative method for solving the problem (2.2).

Algorithm 3.2. *For a given initial value u_0, compute the approximate solution u_{n+1} by the iterative scheme*

$$\left\langle \rho N\left(\frac{2u_nv}{u_n+v}\right) + u_{n+1} - u_n, v - u_{n+1}\right\rangle = \langle \rho f, v - u_{n+1}\rangle, \qquad \forall v \in H.$$

We again use the auxiliary principle technique to suggest an implicit method for solving the problem (2.2). For a given $u \in H$ satisfying (2.2), consider the problem of finding $w \in H$ such that,

$$\left\langle \rho N\left(\frac{2vw}{v+w}\right), v - w\right\rangle + \langle w - u, v - w\rangle = \langle \rho f, v - w\rangle, \qquad \forall v \in H, \qquad (3.9)$$

which is called the auxiliary problem. We note that the auxiliary problems (3.2) and (3.9) are quite different. Clearly $w = u \in H$ is a solution of (2.1). This observation allows us to suggest the following iterative method for solving the problem (2.2).

Algorithm 3.3. *For a given initial value u_0, compute the approximate solution u_{n+1} by the iterative scheme*

$$\left\langle \rho N\left(\frac{2u_{n+1}v}{u_{n+1}+v}\right) + u_{n+1} - u_n, v - u_{n+1}\right\rangle = \langle \rho f, v - u_{n+1}\rangle, \qquad \forall v \in H. \tag{3.10}$$

This is a predictor-corrector method as a predictor. Consequently, we obtain the two-step method for solving the problem (2.1).

Algorithm 3.4. *For a given initial value $u_0 \in H$, compute the approximate solution $u_{n+1} \in H$ by the iterative scheme*

$$\left\langle \rho N\left(\frac{2u_n v}{u_n+v}\right) + y_n - u_n, v - y_n\right\rangle = \langle \rho f, v - y_n\rangle, \qquad \forall v \in H,$$

$$\left\langle \rho N\left(\frac{2y_n v}{y_n+v}\right) + u_{n+1} - u_n, v - u_{n+1}\right\rangle = \langle \rho f, v - u_{n+1}\rangle, \qquad \forall v \in H,$$

which is known as two-step iterative method for solving problem (2.2).

For the convergence criteria, we need the following concept.

Definition 3.5. *An operator $N(\cdot)$ is said to be pseudo harmonic-like with respect to the functional f, if*

$$\left\langle N\left(\frac{2uv}{u+v}\right), v - u\right\rangle - \langle f, v - u\rangle \geq 0 \qquad \forall v \in H,$$
$$\Rightarrow \left\langle N\left(\frac{2uv}{u+v}\right), u - v\right\rangle - \langle f, u - v\rangle \geq 0 \qquad \forall v \in H.$$

We now consider the convergence analysis of Algorithm 3.3 and this is the main motivation of our next result.

Theorem 3.6. *Let $u \in H$ be a solution of (2.2) and let u_{n+1} be the approximate solution obtained from Algorithm 3.4. If the operator $N(\cdot)$ is pseudo harmonic-like operator, then*

$$\|u_{n+1} - u\|^2 \leq \|u_n - u\|^2 - \|u_{n+1} - u_n\|^2. \tag{3.11}$$

Proof. Let $u \in H$ be a solution of (2.2). Then

$$\left\langle N\left(\frac{2uv}{u+v}\right), u-v\right\rangle - \langle f, u-v\rangle \geq 0 \qquad \forall v \in H. \tag{3.12}$$

Since the operator N is pseudo harmonic-like monotone with respect to the functional f. Taking $v = u_{n+1}$ in (3.12) and $v = u$ in (3.10), respectively, we have

$$\left\langle N\left(\frac{2uu_{n+1}}{u+u_{n+1}}\right), u-u_{n+1}\right\rangle - \langle f, u-u_{n+1}\rangle \geq 0 \tag{3.13}$$

and

$$\left\langle N\left(\frac{2uu_{n+1}}{u+u_{n+1}}\right) + u_{n+1} - u_n, u-u_{n+1}\right\rangle - \langle f, u-u_{n+1}\rangle \geq 0. \tag{3.14}$$

From (3.13) and (3.14), we obtain

$$\langle u_{n+1} - u_n, u - u_{n+1}\rangle \geq 0.$$

From which, we have

$$\|u_{n+1} - u\|^2 \leq \|u_n - u\|^2 - \|u_{n+1} - u_n\|^2,$$

which is the required (3.14). □

Theorem 3.7. *Let $u \in H$ be a solution of (2.2) and let u_{n+1} be the approximate solution obtained from Algorithm 3.2. If all the assumptions of Theorem 3.2 holds, then*

$$\lim_{n\to\infty} u_{n+1} = u. \tag{3.15}$$

Proof. Let $u \in H$ be a solution of (2.2). From (3.11), it follows that the sequence $\{\|u - u_n\|\}$ is nonincreasing and consequently the sequence $\{u_n\}$ is bounded. Also, from (3.11), we have

$$\sum_{n=1}^{\infty} \|u_{n+1} - u\| = 0$$

which implies that

$$\|u_{n+1} - u_n\|^2 \leq \|u_0 - u\|^2 \quad \text{and} \quad \lim_{n\to\infty} \|u_{n+1} - u_n\| = 0. \tag{3.16}$$

Let $\hat{u}$ be a cluster point of $\{u_n\}$ and the subsequences $\{u_{n_j}\}$ of the sequence $\{u_n\}$ converge to $\hat{u}$. Replacing u_n by $\{u_{n_j}\}$ in (3.10), taking the limit as $\lim\limits_{n\to\infty} n_j \to \infty$ and using (3.16), we have

$$\left\langle N\left(\frac{2v\hat{u}}{v+\hat{u}}\right), v-\hat{u}\right\rangle = \langle f, v-\hat{u}\rangle, \qquad \forall v \in H,$$

which shows that $\hat{u} \in H$ satisfies (2.2) and

$$\|u_{n+1} - u_n\|^2 \leq \|u_n - \hat{u}\|^2.$$

From the above inequality, it follows that the sequence $\{u_n\}$ has exactly one cluster point $\hat{u}$ and $\lim\limits_{n\to\infty} u_n = \hat{u}$. □

We can apply the auxiliary principle technique to suggest hybrid iterative methods for solving the problem (2.2). For a given $u \in H$ satisfying (2.2), consider the problem of finding $w \in H$ such that

$$\left\langle \rho N\left(\frac{2vw}{v+w}\right), v-w \right\rangle + \langle M(w) - M(u) + \eta(w-u), v-w\rangle = \langle \rho f, v-w\rangle, \qquad \forall v \in H, \tag{3.17}$$

where M is a nonlinear arbitrary operator and $\eta \geq 0$ is an arbitrary parameter.

For $M = I$ the identity operator and $\eta = 0$ the auxiliary problem is exactly the auxiliary problem (3.9). For suitable choice of the operator M and the parameter, one can obtain some new auxiliary problems associated with problem (2.1). It is obvious that $w = u \in H$ is solution of the problem (2.2). This observation is used to suggest the general hybrid iterative methods for solving the problem (2.2), which contain some new inertial iterative methods.

Algorithm 3.8. *For given $u_0, u_1 \in H$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\left\langle \rho N\left(\frac{2u_{n+1}v}{u_{n+1}+v}\right) + M(u_{n+1}) - M(u_n) + \eta(u_n - u_{n-1}), v - u_{n+1} \right\rangle = \langle \rho f, v - u_{n+1}\rangle, \qquad \forall v \in H.$$

Algorithm 3.8 is called the hybrid inertial iterative method, which contains Algorithm 3.3 and the following inertial iterative method.

Algorithm 3.9. *For given $u_0, u_1 \in H$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\left\langle \rho N\left(\frac{2u_{n+1}v}{u_{n+1}+v}\right) + \eta(u_n - u_{n-1}), v - u_{n+1} \right\rangle = \langle \rho f, v - u_{n+1}\rangle, \qquad \forall v \in H.$$

Algorithm 3.9 appears to be a new one. Using the technique of Noor, one can study the convergence criteria of Algorithm 3.9. For the applications and convergence analysis of the inertial type methods, See [1–3] and the references therein.

4 Conclusion

In this paper, we have introduced a new generalization of the Lax-Milgram Lemma.,Several special cases are discussed as applications. In particular, this new class contains Lax-Milgram Lemma and Riesz-Frechet theorem as special cases. The auxiliary principle technique is used to study the existence of the solution of the nonlinear harmonic problems. Some new iterative methods are considered. Convergence analysis of these iterative methods is investigated under suitable conditions. We would like emphasize that the results obtained in this paper may motivate and bring a number of novel, potential applications, extensions and interesting new topics for further study.

Contributions of the authors:

All the authors contributed equally in writing, editing, reviewing and agreed for the final version for publication.

Data availability:

Data sharing not applicable to this article as no data sets were generated or analyzed during the current study.

Conflict of Interest:

All authors have no conflict of interest.

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