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Applications of Fractional Calculus and Borel Distribution Series for Multivalent Functions on Complex Hilbert Space

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Abstract

In this paper, we introduce applications of fractional calculus techniques for a family of multivalent analytic functions defined by the Borel distribution on Hilbert space. We derive several interesting properties, including coefficient estimates, extreme points, and convex combinations.

1. Introduction

Let H be a complex Hilbert space. Let T be a linear operator on H. For a complex analytic function f on the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$, we denoted $f(T)$, the operator on H defined by the usual Riesz-Dunford integral [2]

$$f(T) = \frac{1}{2\pi i} \int_{c} f(z)(zI - T)^{-1}\, dz\,,$$

where I is the identity operator on H, c is a positively oriented simple closed rectifiable contour lying in U and containing the spectrum $\sigma(T)$ of T in its interior domain [3]. Also $f(T)$ can be defined by the series

$$f(T) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} T^n\,,$$

which converges in the norm topology [4].

Denote by $W(p, m)$ the family of functions of the form:

$$f(z) = z^p + \sum_{n=p+m}^{\infty} a_n z^n \qquad (p,m \in \mathbb{N} = \{1,2,\ldots\}), \tag{1.1}$$

which are analytic and multivalent in the open unit disk U.

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Let $J(p\,,m)$ denote the subfamily of $W(p\,,m)$ consisting of functions of the form:

$$f(z) = z^p - \sum_{n=p+m}^{\infty} a_n z^n \qquad (a_n \geq 0\ p,m \in \ \mathbb{N} = \{1,2,\ldots\}), \tag{1.2}$$

A function $f \in W(p,m)$ is said to be multivalent starlike of order $\alpha(0 \leq \alpha < p)$ if it satisfies the condition:

$$Re\left\{\frac{zf_{,}(z)}{f(z)}\right\} > \alpha \qquad (z \in U)\,,$$

and is said to be multivalent convex of order $\alpha(0 \leq \alpha < p)$ if it satisfies the condition :

$$Re\left\{1 + \frac{zf_{,,}(z)}{f_{,}(z)}\right\} > \alpha \qquad (z \in U).$$

Denote by $S_m^*\,(p,\alpha)$ and $C_m(p,\alpha)$ the classes of multivalent starlike and multivalent convex functions of order α, respectively, which were introduced and studied by Owa [9]. It is known that (see [6] and [9])

$$f \in C_m\,(p,\alpha) \text{ if and only if } \frac{zf_{,}(z)}{p} \in S_m^*\,(p,\alpha).$$

The classes $S_m^* = S^*(p,\alpha)$ *and* $C_1\,(p,\alpha) = C(p,\alpha)$ were studied by Own [8].

A discrete random variable x is said to have a Borel distribution if it takes the values 1,2,3,... with the probabilities $\frac{e^{-\theta}}{1!}, \frac{2\theta e^{-2\theta}}{2!}, \frac{9\theta^2 e^{-3\theta}}{3!}$,... respectively, where θ is called the parameter.

Very recently, Wanas and Khuttar [12] introduced the Borel distribution (BD) whose probability mass function is

$$P(x = r) = \frac{(\theta r)^{r-1} e^{-\theta r}}{r!}, \ \ r = 1,2,3,\ldots\ .$$

Wanas and Khuttar [12] introduced a series whose coefficients are probabilities of the Borel distribution (BD)

$$\mathcal{N}_p(\theta;z) = z^p + \sum_{n=p+m}^{\infty} \frac{(\theta(n-p))^{n-p-1} e^{-\theta(n-p)}}{(n-p)!} z^n = z^p + \sum_{n=p+m}^{\infty} \Phi_{n,p}(\theta)\, z^n,$$

where $0 < \theta \leq 1$ and

$$\Phi_{n,p}(\theta) = \frac{(\theta(n-p))^{n-p-1} e^{-\theta(n-p)}}{(n-p)!}.$$

We consider a linear operator $\mathcal{M}(p,\theta)f\colon J(p\,,m) \longrightarrow J(p\,,m)$ defined by the convolution or Hadamard product

$$\mathcal{M}(p,\theta)f(z) = \mathcal{N}_p(\theta;z) * f(z) = z^p - \sum_{n=p+m}^{\infty} \frac{(\theta(n-p))^{n-p-1} e^{-\theta(n-p)}}{(n-p)!} a_n z^n\,, \tag{1.3}$$

where $a_n \geq 0$, $0 < \theta \leq 1$ and $z \in U$.

If $f(T)$ is defined by (1.2), we also have

$$\mathcal{M}(p,\theta)f(T) = \mathcal{N}_p(\theta;T) * f(T) = T^p - \sum_{n=p+m}^{\infty} \frac{(\theta(n-p))^{n-p-1}e^{-\theta(n-p)}}{(n-p)!} a_n T^n , \tag{1.4}$$

Definition 1.1 [10]**.** The fractional integral operator of order λ ($\lambda > 0$) is defined by

$$D_T^{-\lambda} f(T) = \frac{1}{\Gamma(\lambda)} \int_0^1 \frac{T^\lambda f(tT)}{(1-t)^{1-\lambda}} dt ,$$

where f is analytic function in a simple connected region of z-plane containing the origin.

Definition 1.2 [10]**.**The fractional derivative for operator of order $\lambda(0 \leq \lambda < 1)$ is defined by

$$D_T^{\lambda} f(T) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dT} \int_0^1 \frac{T^{1-\lambda} f(tT)}{(1-t)^{\lambda}} dt,$$

where f is analytic in a simply connected region of the z-plane containing the origin.

For $f \in J(p,m)$, from Definitions 1.1 and 1.2 by applying a simple calculation, we get

$$D_T^{-\lambda} f(T) = \frac{\Gamma(p+1)}{\Gamma(p+\lambda+1)} T^{p+\lambda} - \sum_{n=p+m}^{\infty} \frac{\Gamma(n+1)}{\Gamma(n+\lambda+1)} a_n T^{n+\lambda} \tag{1.5}$$

and

$$D_T^{\lambda} f(T) = \frac{\Gamma(p+1)}{\Gamma(p-\lambda+1)} T^{p-\lambda} - \sum_{n=p+m}^{\infty} \frac{\Gamma(n+1)}{\Gamma(n-\lambda+1)} a_n T^{n-\lambda} \tag{1.6}$$

The operator on Hilbert space were considered recently by several researcher, can refer, for example, to [1,5,7,10,11,12,13,14,15,16,17,18].

2. Coefficient Estimates

In this section, we introduce the family $E_p^m(\theta,\gamma,\mu,\eta,T)$ and obtain coefficient estimates for the function f in this family.

Definition 2.1. A function $f \in J(p,m)$ is said to be in the family $E_p^m(\theta,\gamma,\mu,\eta,T)$ if and only if satisfies the inequality:

$$\begin{aligned} &\left\| T^2(\mathcal{M}(p,\theta)f(T))_{\prime\prime} - p\big(T(\mathcal{M}(p,\theta)f(T))_{\prime} - \mathcal{M}(p,\theta)f(T)\big) \right\| \\ &\qquad < \eta \left\| (p-2\gamma\mu)\big(T(\mathcal{M}(p,\theta)f(T))_{\prime} - \mathcal{M}(p,\theta)f(T)\big) + (2\gamma-1)T^2\,(\mathcal{M}(p,\theta)f(T))_{\prime\prime} \right\|, \end{aligned} \tag{2.1}$$

where $\frac{1}{2} \leq \gamma \leq 1$, $0 \leq \mu < \frac{1}{2}$, $0 < \eta \leq 1$, $0 < \theta \leq 1$ and for all operator T with $\|T\| < 1$ and $T \neq 0$ ($\emptyset$ denote the zero operator on H).

Theorem 2.1. *Let* $f \in J(p,m)$ *be defined by* (1.2). *Then* $f \in E_p^m(\theta,\gamma,\mu,\eta,T)$ *for all* $T \neq \emptyset$ *if and only if*

$$\sum_{n=p+m}^{\infty} \frac{(n-1)(\theta(n-p))^{n-p-1}e^{-\theta(n-p)}[2\gamma\eta(n-\mu)+(p-n)(1+\eta)]}{(n-p)!} a_n \leq 2\gamma\eta(p-1)(p-\mu),$$

where $\frac{1}{2} \leq \gamma \leq 1,\ 0 \leq \mu < \frac{1}{2},\ 0 < \eta \leq 1,\ 0 < \theta \leq 1.$

The result is sharp for the function f *given by*

$$f(z) = z^p - \frac{2\gamma\eta(p-1)(p-\mu)(n-p)!}{(n-1)(\theta(n-p))^{n-p-1}e^{-\theta(n-p)}[2\gamma\eta(n-\mu)+(p-n)(1+\eta)]} z^n, \quad n \geq 2. \tag{2.2}$$

Proof. Suppose that the inequality (2.1) holds. Then, we have

$$\left\|T^2(\mathcal{M}(p,\theta)f(T))'' - p\big(T(\mathcal{M}(p,\theta)f(T))' - \mathcal{M}(p,\theta)f(T)\big)\right\|$$
$$-\eta\left\|(p-2\gamma\mu)\big(T(\mathcal{M}(p,\theta)f(T))' - \mathcal{M}(p,\theta)f(T)\big) + (2\gamma-1)T^2(\mathcal{M}(p,\theta)f(T))''\right\|$$

$$= \left\| - \sum_{n=p+m}^{\infty} \frac{(n-1)(p-n)(\theta(n-p))^{n-p-1}e^{-\theta(n-p)}}{(n-p)!} a_n T^2 \right\|$$

$$-\eta\left\| 2\gamma(p-1)(p-\mu)T^p - \sum_{n=p+m}^{\infty} \frac{(n-1)(\theta(n-p))^{n-p-1}e^{-\theta(n-p)}[2\gamma(n-\mu)+(p-n)]}{(n-p)!} a_n T^n \right\|$$

$$\leq \sum_{n=p+m}^{\infty} \frac{(n-1)(p-n)(\theta(n-p))^{n-p-1}e^{-\theta(n-p)}}{(n-p)!} a_n \|T\|^2 - 2\gamma\eta(p-1)(p-\mu)\|T\|^p$$
$$+ \sum_{n=p+m}^{\infty} \frac{\eta(n-1)(p-n)(\theta(n-p))^{n-p-1}e^{-\theta(n-p)}}{(n-p)!} a_n \|T\|^n$$

$$\leq \sum_{n=p+m}^{\infty} \frac{(n-1)(\theta(n-p))^{n-p-1}e^{-\theta(n-p)}[2\gamma\eta(n-\mu)+(p-n)(1+\eta)]}{(n-p)!} a_n - 2\gamma\eta(p-1)(p-\mu) \leq 0.$$

Hence $f \in E_p^m(\theta,\gamma,\mu,\eta,T)$.

To show the converse, let $f \in E_p^m(\theta,\gamma,\mu,\eta,T)$. Then

$$\left\|T^2(\mathcal{M}(p,\theta)f(T))'' - p\big(T(\mathcal{M}(p,\theta)f(T))' - \mathcal{M}(p,\theta)f(T)\big)\right\|$$
$$< \eta\left\|(p-2\gamma\mu)\big(T(\mathcal{M}(p,\theta)f(T))' - \mathcal{M}(p,\theta)f(T)\big) + (2\gamma-1)T^2(\mathcal{M}(p,\theta)f(T))''\right\|,$$

gives

$$\left\| - \sum_{n=p+m}^{\infty} \frac{(n-1)(p-n)(\theta(n-p))^{n-p-1}e^{-\theta(n-p)}}{(n-p)!} a_n T^2 \right\|$$
$$< \eta\left\| 2\gamma(p-1)(p-\mu)T^p - \sum_{n=p+m}^{\infty} \frac{(n-1)(\theta(n-p))^{n-p-1}e^{-\theta(n-p)}[2\gamma(n-\mu)+(p-n)]}{(n-p)!} a_n T^n \right\|.$$

Setting $T = rI$ $(0 < r < 1)$ in the above inequality, we get

$$\frac{\sum_{n=p+m}^{\infty}\frac{(n-1)(p-n)(\theta(n-p))^{n-p-1}e^{-\theta(n-p)}}{(n-p)!}a_n r^n}{2\gamma(p-1)(p-\mu)r^p-\sum_{n=p+m}^{\infty}\frac{(n-1)(\theta(n-p))^{n-p-1}e^{-\theta(n-p)}[2\gamma(n-\mu)+(p-n)]}{(n-p)!}a_n r^n} < \eta. \qquad (2.3)$$

Upon clearing denominator in (2.3) and letting $r\longrightarrow 1$, we obtain

$$\sum_{n=p+m}^{\infty}\frac{(n-1)(p-n)(\theta(n-p))^{n-p-1}e^{-\theta(n-p)}}{(n-p)!}a_n$$

$$< 2\gamma\eta(p-1)(p-\mu)r^p-\sum_{n=p+m}^{\infty}\frac{\eta(n-1)(\theta(n-p))^{n-p-1}e^{-\theta(n-p)}[2\gamma(n-\mu)+(p-n)]}{(n-p)!}a_n$$

or

$$\sum_{n=p+m}^{\infty}\frac{(n-1)(\theta(n-p))^{n-p-1}e^{-\theta(n-p)}[2\gamma\eta(n-\mu)+(p-n)(1+\eta)]}{(n-p)!}a_n \le 2\gamma\eta(p-1)(p-\mu),$$

which completes the proof .

Corollary 2.1. *If* $f\in E_p^m(\theta,\gamma,\mu,\eta,T)$, *then*

$$a_n \le \frac{2\gamma\eta(p-1)(p-\mu)(n-p)!}{(n-1)(\theta(n-p))^{n-p-1}e^{-\theta(n-p)}[2\gamma\eta(n-\mu)+(p-n)(1+\eta)]},\ \ n\ge 2. \qquad (2.4)$$

3. Applications of the Fractional Calculus

Theorem 3.1. *If* $f\in E_p^m(\theta,\gamma,\mu,\eta,T)$, *then*

$$\|D_T^{-\lambda}f(T)\| \le \frac{\Gamma(p+1)}{\Gamma(p+\lambda+1)}\|T\|^{p+\lambda}\left[1+\frac{2\gamma\eta(p-1)(p-\mu)m!\Gamma(p+m+1)\Gamma(p+\lambda+1)}{(p+m-1)(\theta m)^{m-1}e^{-\theta m}[2\gamma\eta(p+m-\mu)-m(1+\eta)]\Gamma(p+1)\Gamma(p+m+\lambda+1)}\|T\|^m\right] \quad (3.1)$$

and

$$\|D_T^{-\lambda}f(T)\| \ge \frac{\Gamma(p+1)}{\Gamma(p+\lambda+1)}\|T\|^{p+\lambda}\left[1-\frac{2\gamma\eta(p-1)(p-\mu)m!\Gamma(p+m+1)\Gamma(p+\lambda+1)}{(p+m-1)(\theta m)^{m-1}e^{-\theta m}[2\gamma\eta(p+m-\mu)-m(1+\eta)]\Gamma(p+1)\Gamma(p+m+\lambda+1)}\|T\|^m\right]. \quad (3.2)$$

The result is sharp for the function f *given by*

$$f(z)=z^p-\frac{2\gamma\eta(p-1)(p-\mu)m!}{(p+m-1)(\theta m)^{m-1}e^{-\theta m}[2\gamma\eta(p+m-\mu)-m(1+\eta)]}z^{p+m},\ \ (p,m\in\mathbb{N}). \qquad (3.3)$$

Proof. Let $f\in E_p^m(\theta,\gamma,\mu,\eta,T)$. By (1.5), we have

$$\frac{\Gamma(p+\lambda+1)}{\Gamma(p+1)}T^{-\lambda}D_T^{-\lambda}f(T)=T^p-\sum_{n=p+m}^{\infty}\frac{\Gamma(n+1)\Gamma(p+\lambda+1)}{\Gamma(p+1)\Gamma(n+\lambda+1)}a_nT^n.$$

Setting

$$\psi(n,\lambda) = \frac{\Gamma(n+1)\Gamma(p+\lambda+1)}{\Gamma(p+1)\Gamma(n+\lambda+1)} \ ,(n \geq p+m \ ,p,m \in \mathbb{N}).$$

We get

$$\frac{\Gamma(p+\lambda+1)}{\Gamma(p+1)} T^{-\lambda} D_T^{-\lambda} f(T) = T^p - \sum_{n=p+m}^{\infty} \psi(n,\lambda) a_n T^n \ .$$

Since for $n \geq p+m$, ψ is a decreasing function of n, then we have

$$0 < \psi(n,\lambda) \leq \psi(p+m,\lambda) = \frac{\Gamma(p+m+1)\Gamma(p+\lambda+1)}{\Gamma(p+1)\Gamma(p+m+\lambda+1)}. \tag{3.4}$$

Now, by the application of Theorem 2.1 and (3.4), we obtain

$$\left\| \frac{\Gamma(p+\lambda+1)}{\Gamma(p+1)} T^{-\lambda} D_T^{-\lambda} f(T) \right\|$$

$$\leq \|T\|^p + \sum_{n=p+m}^{\infty} \psi(n,\lambda) a_n \|T\|^n$$

$$\leq \|T\|^p + \psi(p+m,\lambda) \|T\|^{p+m} \sum_{n=p+m}^{\infty} a_n$$

$$\leq \|T\|^p + \frac{2\gamma\eta(p-1)(p-\mu)m!\Gamma(p+m+1)\Gamma(p+\lambda+1)}{(p+m-1)(\theta m)^{m-1} e^{-\theta m}[2\gamma\eta(p+m-\mu) - m(1+\eta)]\Gamma(p+1)\Gamma(p+m+\lambda+1)} \|T\|^{p+m},$$

which gives (3.1), we also have

$$\left\| \frac{\Gamma(p+\lambda+1)}{\Gamma(p+1)} T^{-\lambda} D_T^{-\lambda} f(T) \right\|$$

$$\geq \|T\|^p - \sum_{n=p+m}^{\infty} \psi(n,\lambda,\theta) a_n \|T\|^n$$

$$\geq \|T\|^p - \psi(p+m,\lambda) \|T\|^{p+m} \sum_{n=p+m}^{\infty} a_n$$

$$\geq \|T\|^p - \frac{2\gamma\eta(p-1)(p-\mu)m!\Gamma(p+m+1)\Gamma(p+\lambda+1)}{(p+m-1)(\theta m)^{m-1} e^{-\theta m}[2\gamma\eta(p+m-\mu) - m(1+\eta)]\Gamma(p+1)\Gamma(p+m+\lambda+1)} \|T\|^{p+m},$$

which gives (3.2).

By taking $\lambda = 1$ in Theorem 3.1, we obtain the following Corollary:

Corollary 3.1. *If* $f \in E_p^m(\theta,\gamma,\mu,\eta,T)$, *then*

$$\left\| \int_0^1 T f(tT) dt \right\| \leq \frac{\|T\|^{p+1}}{p+1} \left[1 + \frac{2\gamma\eta\left(p^2 - 1\right)(p-\mu)m!}{((p+m)^2 - 1)(\theta m)^{m-1} e^{-\theta m}[2\gamma\eta(p+m-\mu) - m(1+\eta)]} \|T\|^m \right]$$

and

$$\left\| \int_0^1 T(\,tT)dt \right\| \geq \frac{\|T\|^{p+1}}{p+1}\left[1 - \frac{2\gamma\eta\big(p^2\ -1\big)(p-\mu)m!}{((p+m)^2-1)(\theta m)^{m-1}e^{-\theta m}[2\gamma\eta(p+m-\mu)-m(1+\eta)]}\|T\|^m\right].$$

Proof. By Definition 1.1 and Theorem 3.1 for $\lambda = 1$, we have $D_T^{-\lambda} f(T) = \int_0^1 f(tT)dt$, the result is true.

Theorem 3.2. *If* $f \in E_p^m(\theta,\gamma,\mu,\eta,T)$, *then*

$$\|D_T^\lambda f(T)\| \leq \frac{\Gamma(p+1)}{\Gamma(p-\lambda+1)}\|T\|^{p-\lambda}\left[1 + \frac{2\gamma\eta(p-1)(p-\mu)m!\Gamma(p+m+1)\Gamma(p-\lambda+1)}{(p+m-1)(\theta m)^{m-1}e^{-\theta m}[2\gamma\eta(p+m-\mu)-m(1+\eta)]\Gamma(p+1)\Gamma(p+m-\lambda+1)}\|T\|^m\right] \quad (3.5)$$

and

$$\|D_T^\lambda f(T)\| \geq \frac{\Gamma(p+1)}{\Gamma(p-\lambda+1)}\|T\|^{p-\lambda}\left[1 - \frac{2\gamma\eta(p-1)(p-\mu)m!\Gamma(p+m+1)\Gamma(p-\lambda+1)}{(p+m-1)(\theta m)^{m-1}e^{-\theta m}[2\gamma\eta(p+m-\mu)-m(1+\eta)]\Gamma(p+1)\Gamma(p+m-\lambda+1)}\|T\|^m\right]. \quad (3.6)$$

The result is sharp for the function f *given by* (3.3).

Proof. Let $f \in E_p^m(\theta,\gamma,\mu,\eta,T)$. By (1.6), we have

$$\frac{\Gamma(p-\lambda+1)}{\Gamma(p+1)}T^\lambda D_T^\lambda f(T) = T^p - \sum_{n=p+m}^{\infty} \frac{\Gamma(n+1)\Gamma(p-\lambda+1)}{\Gamma(p+1)\Gamma(n-\lambda+1)} a_n T^n$$

$$= T^p - \sum_{n=p+m}^{\infty} \phi(n,\lambda) a_n T^n\ ,$$

where

$$\phi(n,\lambda) = \frac{\Gamma(n+1)\Gamma(p-\lambda+1)}{\Gamma(p+1)\Gamma(n-\lambda+1)}, \quad (n \geq p+m, p,m \in \mathbb{N}).$$

Since for $n \geq p+m$, ϕ is a decreasing function of n, thus we have

$$0 < \phi(n,\lambda) \leq \phi\ (p+m\ ,\lambda) = \frac{\Gamma(p+m+1)\Gamma(p-\lambda+1)}{\Gamma(p+1)\Gamma(p+m-\lambda+1)}\ .$$

Also, by using Theorem 2.1, we get

$$\sum_{n=p+m}^{\infty} a_n \leq \frac{2\gamma\eta(p-1)(p-\mu)m!}{(p+m-1)(\theta m)^{m-1}e^{-\theta m}[2\gamma\eta(p+m-\mu)-m(1+\eta)]}.$$

Thus

$$\left\| \frac{\Gamma(p-\lambda+1)}{\Gamma(p+1)} T^\lambda D_T^\lambda f(T) \right\|$$

$$\leq \|T\|^p + \phi(p+m,\lambda)\|T\|^{p+m} \sum_{n=p+m}^{\infty} a_n$$

$$\leq \|T\|^p + \frac{2\gamma\eta(p-1)(p-\mu)m!\Gamma(p+m+1)\Gamma(p-\lambda+1)}{(p+m-1)(\theta m)^{m-1}e^{-\theta m}[2\gamma\eta(p+m-\mu)-m(1+\eta)]\Gamma(p+1)\Gamma(p+m-\lambda+1)}\|T\|^{p+m}\ .$$

Then

$$\|D_T^{\lambda} f(T)\|$$

$$\leq \frac{\Gamma(p+1)}{\Gamma(p-\lambda+1)}\|T\|^{p-\lambda}\left[1+\frac{2\gamma\eta(p-1)(p-\mu)m!\Gamma(p+m+1)\Gamma(p-\lambda+1)}{(p+m-1)(\theta m)^{m-1}e^{-\theta m}[2\gamma\eta(p+m-\mu)-m(1+\eta)]\Gamma(p+1)\Gamma(p+m-\lambda+1)}\|T\|^{m}\right].$$

And by the same way, we obtain

$$\|D_T^{\lambda} f(T)\|$$

$$\geq \frac{\Gamma(p+1)}{\Gamma(p-\lambda+1)}\|T\|^{p-\lambda}\left[1-\frac{2\gamma\eta(p-1)(p-\mu)m!\Gamma(p+m+1)\Gamma(p-\lambda+1)}{(p+m-1)(\theta m)^{m-1}e^{-\theta m}[2\gamma\eta(p+m-\mu)-m(1+\eta)]\Gamma(p+1)\Gamma(p+m-\lambda+1)}\|T\|^{m}\right].$$

4. Extreme Points

We obtain here the extreme points of the class $E_p^m(\theta,\gamma,\mu,\eta,T)$.

Theorem 4.1. *Let* $f_p(z) = z^p$ *and*

$$f_n(z) = z^p - \frac{2\gamma\eta(p-1)(p-\mu)(n-p)!}{(n-1)(\theta(n-p))^{n-p-1}e^{-\theta(n-p)}[2\gamma\eta(n-\mu)+(p-n)(1+\eta)]}z^n, \;\; n \geq p+m\,.$$

Then $f \in E_p^m(\theta,\gamma,\mu,\eta,T)$ *if and only it can be expressed in the form*:

$$f(z) = \delta_p\, z^p + \sum_{n=p+m}^{\infty} \delta_n f_n(z), \tag{4.1}$$

where $(\delta_p \geq 0,\ \delta_n \geq 0,\ n \geq p+m)$ *and*

$$\delta_p + \sum_{n=p+m}^{\infty} \delta_n = 1\,.$$

Proof. Suppose that f is expressed in the form (4.1). Then, we have

$$f(z) = \delta_p z^p + \sum_{n=p+m}^{\infty} \delta_n \left[z^p - \frac{2\gamma\eta(p-1)(p-\mu)(n-p)!}{(n-1)(\theta(n-p))^{n-p-1}e^{-\theta(n-p)}[2\gamma\eta(n-\mu)+(p-n)(1+\eta)]}z^n\right]$$

$$= z^p - \sum_{n=p+m}^{\infty} \frac{2\gamma\eta(p-1)(p-\mu)(n-p)!}{(n-1)(\theta(n-p))^{n-p-1}e^{-\theta(n-p)}[2\gamma\eta(n-\mu)+(p-n)(1+\eta)]}\delta_n z^n\,.$$

Hence

$$\sum_{n=p+m}^{\infty} \frac{(n-1)(\theta(n-p))^{n-p-1}e^{-\theta(n-p)}[2\gamma\eta(n-\mu)+(p-n)(1+\eta)]}{2\gamma\eta(p-1)(p-\mu)(n-p)!} \times$$

$$\times \frac{2\gamma\eta(p-1)(p-\mu)(n-p)!}{(n-1)(\theta(n-p))^{n-p-1}e^{-\theta(n-p)}[2\gamma\eta(n-\mu)+(p-n)(1+\eta)]}\delta_n$$

$$= \sum_{n=p+m}^{\infty} \delta_n = 1-\delta_p \leq 1.$$

Then $f \in E_p^m(\theta,\gamma,\mu,\eta,T)$.

Conversely, suppose that $f \in E_p^m(\theta,\gamma,\mu,\eta,T)$, we may set

$$\delta_n = \frac{(n-1)(\theta(n-p))^{n-p-1}e^{-\theta(n-p)}[2\gamma\eta(n-\mu)+(p-n)(1+\eta)]}{2\gamma\eta(p-1)(p-\mu)(n-p)!}a_n ,$$

where a_n is given by (2.4). Then

$$f(z) = z^p - \sum_{n=p+m}^{\infty} a_n z^n$$

$$= z^p - \sum_{n=p+m}^{\infty} \frac{2\gamma\eta(p-1)(p-\mu)(n-p)!}{(n-1)(\theta(n-p))^{n-p-1}e^{-\theta(n-p)}[2\gamma\eta(n-\mu)+(p-n)(1+\eta)]}\delta_n z^n$$

$$= z^p - \sum_{n=p+m}^{\infty} (z^p - f_n(z))\delta_n$$

$$= \left(1 - \sum_{n=p+m}^{\infty} \delta_n\right)z^p + \sum_{n=p+m}^{\infty} \delta_n f_n(z)$$

$$= \delta_p z^p + \sum_{n=p+m}^{\infty} \delta_n f_n(z) .$$

This completes the proof of the theorem.

5. Convex Combination

Theorem 5.1. *The class* $E_p^m(\theta,\gamma,\mu,\eta,T)$ *is closed under convex combinations.*

Proof. For $j = 1,2,...$, let $f_j \in E_p^m(\theta,\gamma,\mu,\eta,T)$, where f_j is given by

$$f_j(z) = z^p - \sum_{n=p+m}^{\infty} a_{n,j} z^n .$$

Then by (2.1), we have

$$\sum_{n=p+m}^{\infty} (n-1)(\theta(n-p))^{n-p-1}e^{-\theta(n-p)}[2\gamma\eta(n-\mu)+(p-n)(1+\eta)]a_{n,j}$$

$$\leq 2\gamma\eta(p-1)(p-\mu)(n-p)!. \tag{5.1}$$

For $\sum_{j=1}^{\infty} \alpha_j = 1,\ 0 \leq \alpha_j \leq 1$, the convex combination of f_j maybe written as

$$\sum_{j=1}^{\infty} \alpha_j f_j(z) = z^p - \sum_{n=p+m}^{\infty} \left(\sum_{j=1}^{\infty} \alpha_j\ a_{n,j}\right) z^n .$$

Thus, by (5.1), we get

$$\sum_{n=p+m}^{\infty}(n-1)(\theta(n-p))^{n-p-1}e^{-\theta(n-p)}[2\gamma\eta(n-\mu)+(p-n)(1+\eta)]\left(\sum_{j=1}^{\infty}\alpha_j\, a_{n,j}\right)$$

$$=\sum_{j=1}^{\infty}\alpha_j\left(\sum_{n=p+m}^{\infty}(n-1)(\theta(n-p))^{n-p-1}e^{-\theta(n-p)}[2\gamma\eta(n-\mu)+(p-n)(1+\eta)]a_{n,j}\right)$$

$$\leq\sum_{j=1}^{\infty}\alpha_j\,(2\gamma\eta(p-1)(p-\mu)(n-p)!)$$

$$=2\gamma\eta(p-1)(p-\mu)(n-p)!\,.$$

Therefore

$$\sum_{j=1}^{\infty}\alpha_j f_j\,(z)\in E_p^m(\theta,\gamma,\mu,\eta,T)\,.$$

Corollary 5.1. *The family* $E_p^m(\theta,\gamma,\mu,\eta,T)$ *is a convex set.*

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