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Applications of (M,N)-Lucas Polynomials for a Certain Family of Bi-Univalent Functions Associating λ-Pseudo-Starlike Functions with Sakaguchi Type Functions

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Abstract

In this article, we use the (M,N)-Lucas Polynomials to determinate upper bounds for the Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions belongs to a certain family of holomorphic and bi-univalent functions associating λ-pseudo-starlike functions with Sakaguchi type functions defined in the open unit disk $\mathbb{D}$. Also, we discuss Fekete-Szegö problem for functions belongs to this family.

1 Introduction

The Lucas Polynomials plays an important role in a diversity of disciplines in the mathematical, statistical, physical and engineering sciences (see, for example [8, 12, 31]).

Let $\mathcal{A}$ stand for the collection of functions f that are holomorphic in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ that have the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

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Further, let S indicate the sub-collection of the set $\mathcal{A}$ containing of functions in $\mathbb{D}$ satisfying (1.1) which are univalent in $\mathbb{D}$. It is known that (see [7]), every function $f \in S$ has an inverse f^{-1} defined by $f^{-1}(f(z)) = z, (z \in \mathbb{D})$ and $f(f^{-1}(w)) = w$, $(|w| < r_0(f), r_0(f) \geq \frac{1}{4})$, where

$$f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots . \tag{1.2}$$

A function $f \in \mathcal{A}$ is called bi-univalent in $\mathbb{D}$ if both f and f^{-1} are univalent in $\mathbb{D}$, let we name by the notation Σ the set of bi-univalent functions in $\mathbb{D}$ satisfying (1.1). In fact, Srivastava et al. [21] have actually revived the study of holomorphic and bi-univalent functions in recent years, it was followed by such works as those by Caglar et al. [6], Orhan et al. [13], Altinkaya and Yalçin [1], Srivastava and Wanas [22] and others (see, for example [2, 3, 5, 10, 16–20, 23–30]).

Recently, Frasin [9] introduced and studied the family $S(\gamma, m, n)$ consisting of functions $f \in \mathcal{A}$ which satisfy the condition

$$Re\left\{\frac{(m-n)zf'(z)}{f(mz)-f(nz)}\right\} > \gamma,$$

for some $0 \leq \gamma < 1$, $m, n \in \mathbb{C}$ with $m \neq n$, $|n| \leq 1$ and for all $z \in \mathbb{D}$. We note that the family $S(\gamma, 1, n)$ was given by Owa et al. [14] while the family $S(\gamma, 1, -1) \equiv S_s(\gamma)$ was considered by Sakaguchi [15] and is called Sakaguchi function of order γ. Also, $S(0, 1, -1) \equiv S_s$ is the family of starlike functions with respect to symmetrical points in $\mathbb{D}$ and $S(\gamma, 1, 0) \equiv S^*(\gamma)$ which is the family of starlike functions of order γ $(0 \leq \gamma < 1)$.

In [4] Babalola defined the family $\mathcal{L}_\lambda(\gamma)$ of λ-pseudo-starlike functions of order γ which are the function $f \in \mathcal{A}$ such that

$$Re\left\{\frac{z\left(f'(z)\right)^\lambda}{f(z)}\right\} > \gamma,$$

where $0 \leq \gamma < 1$, $\lambda \geq 1$, and $z \in \mathbb{D}$. In particular, Babalola [4] showed that all λ-pseudo-starlike functions are Bazilevic of type $1 - \frac{1}{\lambda}$ and order $\gamma^{\frac{1}{\lambda}}$ and are univalent in $\mathbb{D}$. It is observed that for $\lambda = 1$, we have the family of starlike functions.

With a view to recalling the principle of subordination between holomorphic functions, let the functions f and g be holomorphic in $\mathbb{D}$, we say that the function f is subordinate to g, if there exists a Schwarz function ω holomorphic in $\mathbb{D}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ $(z \in \mathbb{D})$ such that $f(z) = g(\omega(z))$. This subordination is indicated by $f \prec g$ or $f(z) \prec g(z)$ $(z \in \mathbb{D})$.

For the polynomials $M(x)$ and $N(x)$ with real coefficients, the (M,N)-Lucas Polynomials $L_{M,N,k}(x)$ are defined by the following recurrence relation (see [11]):

$$L_{M,N,k}(x) = M(x)L_{M,N,k-1}(x) + N(x)L_{M,N,k-2}(x) \quad (k \geq 2),$$

with

$$L_{M,N,0}(x) = 2, \quad L_{M,N,1}(x) = M(x) \quad \text{and} \quad L_{M,N,2}(x) = M^2(x) + 2N(x). \tag{1.3}$$

The generating function of the (M,N)-Lucas Polynomial $L_{M,N,k}(x)$ (see [11]) is given by

$$T_{\{L_{M,N,k}(x)\}}(z) = \sum_{k=2}^{\infty} L_{M,N,k}(x) z^k = \frac{2 - M(x)z}{1 - M(x)z - N(x)z^2}.$$

2 Main Results

We begin this section by defining the family $\mathcal{G}_\Sigma(\delta, \lambda, m, n; x)$ as follows:

Definition 2.1. *A function $f \in \Sigma$ is named in the family $\mathcal{G}_\Sigma(\delta, \lambda, m, n; x)$ if it fulfills the conditions:*

$$(1-\delta)\frac{(m-n)z\left(f'(z)\right)^\lambda}{f(mz) - f(nz)} + \delta\frac{(m-n)\left((zf'(z))'\right)^\lambda}{\left(f(mz) - f(nz)\right)'} \prec T_{\{L_{M,N,k}(x)\}}(z) - 1$$

and

$$(1-\delta)\frac{(m-n)w\left(g'(w)\right)^\lambda}{g(mw) - g(nw)} + \delta\frac{(m-n)\left((wg'(w))'\right)^\lambda}{\left(g(mw) - g(nw)\right)'} \prec T_{\{L_{M,N,k}(x)\}}(w) - 1,$$

where $0 \le \delta \le 1$, $\lambda \ge 1$, $m, n \in \mathbb{C}$, $m \ne n$, $|n| \le 1$, $z, w \in \mathbb{D}$ and $g = f^{-1}$ is given by(1.2).

In particular, if we choose $\delta = 0$, $m = 1$ and $n = -1$ in Definition 2.1, the family $\mathcal{G}_\Sigma(\delta, \lambda, m, n; x)$ reduces to the family $\mathcal{LS}^*_\Sigma(\lambda; x)$ of λ-pseudo bi-starlike functions with respect to symmetrical points which satisfying the following subordinations:

$$\frac{2z\left(f'(z)\right)^\lambda}{f(z) - f(-z)} \prec T_{\{L_{M,N,k}(x)\}}(z) - 1$$

and

$$\frac{2w\left(g'(w)\right)^\lambda}{g(w) - g(-w)} \prec T_{\{L_{M,N,k}(x)\}}(w) - 1.$$

And if we choose $\delta = 1$, $m = 1$ and $n = -1$ in Definition 2.1, the family $\mathcal{G}_\Sigma(\delta, \lambda, m, n; x)$ reduces to the family $\mathcal{LC}_\Sigma(\lambda; x)$ of λ-pseudo bi-convex functions with respect to symmetrical points which satisfying the following subordinations:

$$\frac{2\left((zf'(z))'\right)^\lambda}{\left(f(z) - f(-z)\right)'} \prec T_{\{L_{M,N,k}(x)\}}(z) - 1$$

and

$$\frac{2\left((wg'(w))'\right)^\lambda}{\left(g(w) - g(-w)\right)'} \prec T_{\{L_{M,N,k}(x)\}}(w) - 1.$$

Theorem 2.1. *For $0 \leq \delta \leq 1$, $\lambda \geq 1$ and $m, n \in \mathbb{C}$, $m \neq n$, $|n| \leq 1$, let $f \in \mathcal{A}$ belong to the family $\mathcal{G}_{\Sigma}(\delta, \lambda, m, n; x)$. Then*

$$|a_2| \leq \frac{|M(x)| \sqrt{|M(x)|}}{\sqrt{\left|(\Omega(\delta, \lambda, m, n) - mn)M^2(x) - (\delta+1)^2 (2\lambda - m - n)^2 (M^2(x) + 2N(x))\right|}}$$

and

$$|a_3| \leq \frac{M^2(x)}{(\delta+1)^2 (2\lambda - m - n)^2} + \frac{|M(x)|}{(2\delta+1)(3\lambda - m^2 - n^2 - mn)},$$

where

$$\Omega(\delta, \lambda, m, n) = \delta \left[(m^2 + n^2 + 4mn) - 6\lambda(m + n - \lambda) \right] + \lambda (1 - 2(m + n - \lambda)). \tag{2.1}$$

Proof. Suppose that $f \in \mathcal{G}_{\Sigma}(\delta, \lambda, m, n; x)$. Then there exists two holomorphic functions $\phi, \psi : \mathbb{D} \longrightarrow \mathbb{D}$ given by

$$\phi(z) = r_1 z + r_2 z^2 + r_3 z^3 + \cdots \quad (z \in \mathbb{D}) \tag{2.2}$$

and

$$\psi(w) = s_1 w + s_2 w^2 + s_3 w^3 + \cdots \quad (w \in \mathbb{D}), \tag{2.3}$$

with $\phi(0) = \psi(0) = 0$, $|\phi(z)| < 1$, $|\psi(w)| < 1$, $z, w \in \mathbb{D}$ such that

$$(1-\delta)\frac{(m-n)z\,(f'(z))^{\lambda}}{f(mz) - f(nz)} + \delta \frac{(m-n)\left((zf'(z))'\right)^{\lambda}}{(f(mz) - f(nz))'}$$
$$= -1 + L_{M,N,0}(x) + L_{M,N,1}(x)\phi(z) + L_{M,N,2}(x)\phi^2(z) + \cdots \tag{2.4}$$

and

$$(1-\delta)\frac{(m-n)w\,(g'(w))^{\lambda}}{g(mw) - g(nw)} + \delta \frac{(m-n)\left((wg'(w))'\right)^{\lambda}}{(g(mw) - g(nw))'}$$
$$= -1 + L_{M,N,0}(x) + L_{M,N,1}(x)\psi(w) + L_{M,N,2}(x)\psi^2(w) + \cdots. \tag{2.5}$$

Combining (2.2), (2.3), (2.4) and (2.5), yield

$$(1-\delta)\frac{(m-n)z\,(f'(z))^{\lambda}}{f(mz) - f(nz)} + \delta \frac{(m-n)\left((zf'(z))'\right)^{\lambda}}{(f(mz) - f(nz))'}$$
$$= 1 + L_{M,N,1}(x) r_1 z + \left[L_{M,N,1}(x) r_2 + L_{M,N,2}(x) r_1^2 \right] z^2 + \cdots \tag{2.6}$$

and

$$(1-\delta)\frac{(m-n)w\,(g'(w))^{\lambda}}{g(mw) - g(nw)} + \delta \frac{(m-n)\left((wg'(w))'\right)^{\lambda}}{(g(mw) - g(nw))'}$$
$$= 1 + L_{M,N,1}(x) s_1 w + \left[L_{M,N,1}(x) s_2 + L_{M,N,2}(x) s_1^2 \right] w^2 + \cdots. \tag{2.7}$$

It is quite well-known that if $|\phi(z)| < 1$ and $|\psi(w)| < 1$, $z, w \in \mathbb{D}$, we get

$$|r_j| \leq 1 \quad \text{and} \quad |s_j| \leq 1 \; (j \in \mathbb{N}). \tag{2.8}$$

In the light of (2.6) and (2.7), after simplifying, we find that

$$(\delta+1)(2\lambda - m - n)a_2 = L_{M,N,1}(x)r_1, \tag{2.9}$$

$$\begin{aligned}&(2\delta+1)(3\lambda - m^2 - n^2 - mn)a_3 + (3\delta+1)\left[(m+n)^2 - 2\lambda(m+n-\lambda+1)\right]a_2^2 \\ &= L_{M,N,1}(x)r_2 + L_{M,N,2}(x)r_1^2,\end{aligned} \tag{2.10}$$

$$-(\delta+1)(2\lambda - m - n)a_2 = L_{M,N,1}(x)s_1 \tag{2.11}$$

and

$$\begin{aligned}&\left[(6\lambda - m^2 - n^2) - 2\lambda(m+n-\lambda+1) - \delta\left(6\lambda(m+n-\lambda-1) + (m-n)^2\right)\right]a_2^2 \\ &-(2\delta+1)(3\lambda - m^2 - n^2 - mn)a_3 = L_{M,N,1}(x)s_2 + L_{M,N,2}(x)s_1^2.\end{aligned} \tag{2.12}$$

It follows from (2.9) and (2.11) that

$$r_1 = -s_1 \tag{2.13}$$

and

$$2\left(\delta+1\right)^2\left(2\lambda - m - n\right)^2 a_2^2 = L_{M,N,1}^2(x)(r_1^2 + s_1^2). \tag{2.14}$$

If we add (2.10) to (2.12), we obtain

$$\begin{aligned}&2\left\{\delta\left[(m^2+n^2+4mn) - 6\lambda(m+n-\lambda)\right] + \lambda\left(1 - 2(m+n-\lambda)\right) - mn\right\}a_2^2 \\ &= L_{M,N,1}(x)(r_2+s_2) + L_{M,N,2}(x)(r_1^2+s_1^2).\end{aligned} \tag{2.15}$$

Substituting the value of $r_1^2 + s_1^2$ from (2.14) in the right hand side of (2.15), we deduce that

$$2\left[\Omega(\delta,\lambda,m,n) - mn - \frac{L_{M,N,2}(x)}{L_{M,N,1}^2(x)}\left(\delta+1\right)^2\left(2\lambda - m - n\right)^2\right]a_2^2 = L_{M,N,1}(x)(r_2+s_2), \tag{2.16}$$

where $\Omega(\delta,\lambda,m,n)$ is given by (2.2).

Moreover computations using (1.3), (2.8) and (2.16), we find that

$$|a_2| \leq \frac{|M(x)|\sqrt{|M(x)|}}{\sqrt{\left|(\Omega(\delta,\lambda,m,n) - mn)M^2(x) - \left(\delta+1\right)^2\left(2\lambda - m - n\right)^2\left(M^2(x) + 2N(x)\right)\right|}}.$$

Next, if we subtract (2.12) from (2.10), we can easily see that

$$2(2\delta+1)(3\lambda - m^2 - n^2 - mn)(a_3 - a_2^2) = L_{M,N,1}(x)(r_2 - s_2) + L_{M,N,2}(x)(r_1^2 - s_1^2). \tag{2.17}$$

In view of (2.13) and (2.14), we get from (2.17)

$$a_3 = \frac{L^2_{M,N,1}(x)}{2\left(\delta+1\right)^2\left(2\lambda-m-n\right)^2}(r_1^2+s_1^2) + \frac{L_{M,N,1}(x)}{2(2\delta+1)(3\lambda-m^2-n^2-mn)}(r_2-s_2).$$

Thus applying (1.3), we conclude that

$$|a_3| \le \frac{M^2(x)}{\left(\delta+1\right)^2\left(2\lambda-m-n\right)^2} + \frac{|M(x)|}{(2\delta+1)(3\lambda-m^2-n^2-mn)}.$$

□

Putting $\delta=0$, $m=1$ and $n=-1$ in Theorem 2.1, we conclude the following result:

Corollary 2.1. *If f belongs to the family $\mathcal{L}S^*_\Sigma(\lambda;x)$, then*

$$|a_2| \le \frac{|M(x)|\sqrt{|M(x)|}}{\sqrt{|(\lambda-1)M^2(x)-2\lambda^2\left(M^2(x)+4N(x)\right)|}}$$

and

$$|a_3| \le \frac{M^2(x)}{4\lambda^2} + \frac{|M(x)|}{(3\lambda-1)}.$$

Putting $\delta=1$, $m=1$ and $n=-1$ in Theorem 2.1, we conclude the following result:

Corollary 2.2. *If f belongs to the family $\mathcal{L}C_\Sigma(\lambda;x)$, then*

$$|a_2| \le \frac{|M(x)|\sqrt{|M(x)|}}{\sqrt{|(\lambda-3)M^2(x)-8\lambda^2\left(M^2(x)+4N(x)\right)|}}$$

and

$$|a_3| \le \frac{M^2(x)}{16\lambda^2} + \frac{|M(x)|}{3(3\lambda-1)}.$$

In the next theorem, we discuss the Fekete-Szegö problem for the family $\mathcal{G}_\Sigma(\delta,\lambda,m,n;x)$.

Theorem 2.2. *For $0\le\delta\le1$, $\lambda\ge1$, $m,n\in\mathbb{C}$, $m\ne n$, $|n|\le1$ and $\eta\in\mathbb{R}$, let $f\in\mathcal{A}$ belong to the family $\mathcal{G}_\Sigma(\delta,\lambda,m,n;x)$. Then*

$$\left|a_3-\eta a_2^2\right| \le \begin{cases} \frac{|M(x)|}{(2\delta+1)(3\lambda-m^2-n^2-mn)}; \\ \qquad for\ |\eta-1| \le \frac{\left|(\Omega(\delta,\lambda,m,n)-mn)-(\delta+1)^2(2\lambda-m-n)^2\left(1+\frac{2N(x)}{M^2(x)}\right)\right|}{(2\delta+1)(3\lambda-m^2-n^2-mn)}, \\ \\ \frac{|M(x)|^3|\eta-1|}{\left|(\Omega(\delta,\lambda,m,n)-mn)M^2(x)-(\delta+1)^2(2\lambda-m-n)^2(M^2(x)+2N(x))\right|}; \\ \qquad for\ |\eta-1| \ge \frac{\left|(\Omega(\delta,\lambda,m,n)-mn)-(\delta+1)^2(2\lambda-m-n)^2\left(1+\frac{2N(x)}{M^2(x)}\right)\right|}{(2\delta+1)(3\lambda-m^2-n^2-mn)}. \end{cases}$$

Proof. By making use of (2.16) and (2.17), we conclude that

$$a_3 - \eta a_2^2 = (1-\eta)\frac{L^3_{M,N,1}(x)(r_2+s_2)}{2\left[\left(\Omega(\delta,\lambda,m,n)-mn\right)L^2_{M,N,1}(x)-(\delta+1)^2\left(2\lambda-m-n\right)^2 L_{M,N,2}(x)\right]}$$
$$+\frac{L_{M,N,1}(x)(r_2-s_2)}{2(2\delta+1)(3\lambda-m^2-n^2-mn)}$$
$$= L_{M,N,1}(x)\left[\left(\varphi(\eta;x)+\frac{1}{2(2\delta+1)(3\lambda-m^2-n^2-mn)}\right)r_2\right.$$
$$\left.+\left(\varphi(\eta;x)-\frac{1}{2(2\delta+1)(3\lambda-m^2-n^2-mn)}\right)s_2\right],$$

where

$$\varphi(\eta;x)=\frac{L^2_{M,N,1}(x)(1-\eta)}{2\left[\left(\Omega(\delta,\lambda,m,n)-mn\right)L^2_{M,N,1}(x)-(\delta+1)^2\left(2\lambda-m-n\right)^2 L_{M,N,2}(x)\right]}.$$

According to (1.3), we find that

$$\left|a_3-\eta a_2^2\right| \le \begin{cases} \frac{|M(x)|}{(2\delta+1)(3\lambda-m^2-n^2-mn)}, & 0\le|\varphi(\eta;x)|\le\frac{1}{2(2\delta+1)(3\lambda-m^2-n^2-mn)}, \\ 2\left|M(x)\right|\left|\varphi(\eta;x)\right|, & |\varphi(\eta;x)|\ge\frac{1}{2(2\delta+1)(3\lambda-m^2-n^2-mn)}. \end{cases}$$

After some computations, we have

$$\left|a_3-\eta a_2^2\right| \le \begin{cases} \frac{|M(x)|}{(2\delta+1)(3\lambda-m^2-n^2-mn)}; \\ \qquad for\ \ |\eta-1|\le\frac{\left|(\Omega(\delta,\lambda,m,n)-mn)-(\delta+1)^2(2\lambda-m-n)^2\left(1+\frac{2N(x)}{M^2(x)}\right)\right|}{(2\delta+1)(3\lambda-m^2-n^2-mn)}, \\ \frac{|M(x)|^3|\eta-1|}{\left|(\Omega(\delta,\lambda,m,n)-mn)M^2(x)-(\delta+1)^2(2\lambda-m-n)^2(M^2(x)+2N(x))\right|}; \\ \qquad for\ \ |\eta-1|\ge\frac{\left|(\Omega(\delta,\lambda,m,n)-mn)-(\delta+1)^2(2\lambda-m-n)^2\left(1+\frac{2N(x)}{M^2(x)}\right)\right|}{(2\delta+1)(3\lambda-m^2-n^2-mn)}. \end{cases}$$

□

Putting $\delta=1$, $m=1$ and $n=-1$ in Theorem 2.2, we conclude the following result:

Corollary 2.3. *If f belongs to the family $\mathcal{LS}^*_\Sigma(\lambda;x)$, then*

$$\left|a_3-\delta a_2^2\right| \le \begin{cases} \frac{|M(x)|}{3\lambda-1}; \\ \qquad for\ \ |\delta-1|\le\frac{\left|(\lambda-1)-2\lambda^2\left(1+\frac{4N(x)}{M^2(x)}\right)\right|}{3\lambda-1}, \\ \frac{|M(x)|^3|\delta-1|}{\left|(\lambda-1)M^2(x)-2\lambda^2(M^2(x)+4N(x))\right|}; \\ \qquad for\ \ |\delta-1|\ge\frac{\left|(\lambda-1)-2\lambda^2\left(1+\frac{4N(x)}{M^2(x)}\right)\right|}{3\lambda-1}. \end{cases}$$

Putting $\delta = 0$, $m = 1$ and $n = -1$ in Theorem 2.2, we conclude the following result:

Corollary 2.4. *If f belongs to the family $\mathcal{LC}_\Sigma(\lambda; x)$, then*

$$\left|a_3 - \delta a_2^2\right| \leq \begin{cases} \frac{|M(x)|}{3(3\lambda-1)}; \\ \qquad for\ |\delta - 1| \leq \frac{\left|(\lambda-3)-8\lambda^2\left(1+\frac{4N(x)}{M^2(x)}\right)\right|}{3(3\lambda-1)}, \\ \\ \frac{|M(x)|^3|\delta-1|}{|(\lambda-3)M^2(x)-8\lambda^2(M^2(x)+4N(x))|}; \\ \qquad for\ |\delta - 1| \geq \frac{\left|(\lambda-3)-8\lambda^2\left(1+\frac{4N(x)}{M^2(x)}\right)\right|}{3(3\lambda-1)}. \end{cases}$$

Putting $\eta = 1$ in Theorem 2.2, we conclude the following result:

Corollary 2.5. *If f belongs to the family $\mathcal{G}_\Sigma(\delta, \lambda, m, n; x)$, then*

$$\left|a_3 - a_2^2\right| \leq \frac{|M(x)|}{(2\delta + 1)(3\lambda - m^2 - n^2 - mn)}.$$

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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