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A Chebyshev Generated Block Method for Directly Solving Nonlinear and Ill-posed Fourth-Order ODEs

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Abstract

This study presents a block method induced by Chebyshev polynomials of first kind for directly solving initial value problems of fourth-order ordinary differential equations without reducing the problems to a system of first-order differential equations. The method was developed by applying interpolation and collocation procedures to a Chebyshev approximate polynomial. The unknown parameters were obtained using the Gaussian elimination method, then substituted into the approximate solution to get the continuous scheme and evaluated at the selected point to give the discrete scheme. The method was zero stable, consistent, and convergent, p-stable as shown by the region of absolute stability and has an order of seven. The accuracy and usability of the developed method were tested by applying it to solve six numerical examples. The method was found to be efficient as it gives minimal error. The numerical results of the method were compared with other works cited in the literature and found to be better as it gives minor errors.

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1 Introduction

This research provided an approximate solution to the general fourth order ordinary differential equations of the form:

$$y^{(4)}(x) = f(x, y, y', y'', y'''), \tag{1}$$

$$y(x_0) = \eta_0, \quad y'(x_0) = \eta_1, \quad y''(x_0) = \eta_2, \quad y'''(x_0) = \eta_3,$$

where f is a continuous real value function.

Many mathematical models in engineering and sciences often results in high-order linear and nonlinear IVPs. The static deflection of a uniform beam and that of a cantilever beam (with the left end embedded and the right end free giving birth to fourth-order IVPs) is one of the applications of fourth-order problems (see [1]). According to [2] [3] [4] [5] and numerous others, the reduction strategy is typically used to numerically solve equation (1). Despite the fact that this strategy has been very effective, it does have certain drawbacks. For instance, the subroutines that provide the beginning values for the techniques' implementation are usually sophisticated computer programs, which increases computational work and computing time. Although, [6] discovered that these methods do not take additional information related to a certain ordinary differential equation, such the oscillatory behavior of the solution. To address the drawback of the reduction methodology's limitations, a direct approach presented as an alternative strategy. In the literature, there are a number of direct numerical techniques, but only a few are intended specifically for the solution of fourth-order ordinary differential equations. To tackle fourth-order IVPs, for instance, [7] created the Block Hybrid Collocation Method (BHCM). The collocation makes use of three off-grid locations. In order to directly solve fourth-order IVPs, [8] created a four-step implicit block approach with three generalized off-step points. An algorithmic collocation method for achieving the approximation of fourth-order IVPs was proposed in a study by [9]. For directly solving (1), [1] presented a Runge-Kutta type approach. Single-step methods are efficient in terms of accuracy since they incorporate hybrid points, according to [9] [10] [11]. The proposed method is a fully hybrid block method that build on the success of [10] which is a motivation for this work where the approximate solution of (1) is sorted in the interval $[x_n, x_{n+\frac{1}{5}}]$ that included six intermediate points.

2 Mathematical Formulation

Let the exact solution $y(x)$ of the fourth-order initial value problem of ordinary differential equation (1) be approximated by a Chebyshev polynomial of the first kind

$$y(x) = \sum_{j=0}^{(p+q)-1} a_j T_j(x). \tag{2}$$

The fourth derivative of equation (2) is given as:

$$y^{(4)}(x) = \sum_{j=4}^{(p+q)-1} a_j T_j^{(4)}(x). \tag{3}$$

Equations (1) and (3) yields a differential system:

$$f(x, y, y', y'', y''') = \sum_{j=4}^{(p+q)-1} a_j T_j^{(4)}(x) \tag{4}$$

where, x is continuous and differentiable, parameters a_j's in (2), (3), and (4) are linear terms to be determined. To get the system of algebraic equations in equations (3) and (4), $x = x_{n+i}$, $i = \frac{1}{15}\left(\frac{1}{30}\right)\frac{1}{6}$ was applied to equation (2) and $x = x_{n+i}$, $i = 0\left(\frac{1}{30}\right)\frac{1}{5}$ applied to equation (3).

$$y(x_{n+i}) = \sum_{j=0}^{10} a_j T_j(x_{n+i}), i = \frac{1}{15}\left(\frac{1}{30}\right)\frac{1}{6}, \tag{5}$$

$$f(x_{n+i}) = \sum_{j=4}^{10} a_j T_j^{(4)}(x_{n+i}), i = 0\left(\frac{1}{30}\right)\frac{1}{5} \tag{6}$$

Using the relation $x_{n+i} = x_n + ih$, (5) and (6) were written as matrix form and solved using CAS in Wolfram Mathematica to obtain the parameters a_j's for $j = 0,\ 1,\ 2,\ \ldots 10$. The parameters a_j's were then substituted into (2) and after replacing x with $x_n + th$ yields the following continuous scheme

$$y(t) = \sum_{j=2}^{5} \alpha_j(t) y_{n+\frac{j}{30}} + h^4 \sum_{j=0}^{6} \beta_j(t) f_{n+\frac{j}{30}}, \tag{7}$$

where

$$
\begin{aligned}
\alpha_1 &= -5\left(900t^3 - 360t^2 + 47t - 2\right), \\
\alpha_2 &= 10\left(1350t^3 - 495t^2 + 57t - 2\right), \\
\alpha_3 &= -15\left(900t^3 - 300t^2 + 31t - 1\right), \\
\alpha_4 &= 2\left(2250t^3 - 675t^2 + 65t - 2\right), \\
\beta_0 &= \frac{5625h^4t^{10}}{28} - \frac{1875h^4t^9}{8} + \frac{1875h^4t^8}{16} - \frac{525h^4t^7}{16} + \frac{203h^4t^6}{36} - \frac{49h^4t^5}{80} + \frac{h^4t^4}{24} \\
&\quad - \frac{5257h^4t^3}{3110400} + \frac{19823h^4t^2}{544320000} - \frac{547h^4t}{1749600000} + \frac{h^4}{3061800000}, \\
\beta_1 &= -\frac{1}{14}16875h^4t^{10} + \frac{9375h^4t^9}{7} - \frac{34875h^4t^8}{56} + \frac{2175h^4t^7}{14} - \frac{87h^4t^6}{4} + \frac{3h^4t^5}{2} \\
&\quad - \frac{5519h^4t^3}{680400} + \frac{161141h^4t^2}{272160000} - \frac{14531h^4t}{816480000} + \frac{79h^4}{408240000}, \\
\beta_2 &= \frac{84375h^4t^{10}}{28} - \frac{178125h^4t^9}{56} + \frac{154125h^4t^8}{112} - \frac{34575h^4t^7}{112} + \frac{585h^4t^6}{16} - \frac{15h^4t^5}{8} \\
&\quad - \frac{55219h^4t^3}{21772800} + \frac{28759h^4t^2}{27216000} - \frac{41087h^4t}{544320000} + \frac{461h^4}{272160000}, \\
\beta_3 &= -\frac{1}{7}28125h^4t^{10} + \frac{28125h^4t^9}{7} - \frac{45375h^4t^8}{28} + \frac{2325h^4t^7}{7} - \frac{635h^4t^6}{18} + \frac{5h^4t^5}{3} \\
&\quad - \frac{3923h^4t^3}{544320} + \frac{3643h^4t^2}{2721600} - \frac{35563h^4t}{306180000} + \frac{131h^4}{38272500},
\end{aligned}
$$

$$\begin{aligned}\beta_4 &= \frac{84375h^4t^{10}}{28} - \frac{159375h^4t^9}{56} + \frac{120375h^4t^8}{112} - \frac{23025h^4t^7}{112} + \frac{165h^4t^6}{8} - \frac{15h^4t^5}{16} \\ &\quad + \frac{10459h^4t^3}{21772800} + \frac{28183h^4t^2}{108864000} - \frac{11609h^4t}{408240000} + \frac{359h^4}{408240000}, \\ \beta_5 &= -\frac{1}{14}16875h^4t^{10} + \frac{7500h^4t^9}{7} - \frac{21375h^4t^8}{56} + \frac{975h^4t^7}{14} - \frac{27h^4t^6}{4} + \frac{3h^4t^5}{10} \\ &\quad - \frac{1201h^4t^3}{2721600} + \frac{1451h^4t^2}{272160000} + \frac{89h^4t}{151200000} - \frac{13h^4}{680400000}, \\ \beta_6 &= \frac{5625h^4t^{10}}{28} - \frac{9375h^4t^9}{56} + \frac{6375h^4t^8}{112} - \frac{1125h^4t^7}{112} + \frac{137h^4t^6}{144} - \frac{h^4t^5}{24} \\ &\quad + \frac{89h^4t^3}{1451520} - \frac{251h^4t^2}{272160000} - \frac{307h^4t}{4898880000} + \frac{h^4}{489888000}\end{aligned}$$

The coefficients α_j's and β_j's define the continuous scheme. The main linear multistep formula of the proposed hybrid Chebyshev induced block method is obtained by evaluating (7) at $t = \frac{1}{5}$. This gives

$$\begin{aligned}&y_{n+\frac{1}{5}} - 4y_{n+\frac{1}{6}} + 6y_{n+\frac{2}{15}} - 4y_{n+\frac{1}{10}} + y_{n+\frac{1}{15}} \\ &\quad = \frac{h^4}{12247200000}\Big(5f_n - 30f_{n+\frac{1}{30}} + 54f_{n+\frac{1}{15}} + 2504f_{n+\frac{1}{10}} \\ &\qquad + 10029f_{n+\frac{2}{15}} + 2574f_{n+\frac{1}{6}} - 16f_{n+\frac{1}{5}}\Big). \qquad (8)\end{aligned}$$

2.1 The additional formulas

The next task is to implement the main formula (7) in block mode. Five additional formulas are require to achieve this. These formulas are obtained by evaluating (7) at $t = 0,\ \frac{1}{6}$ and its first, second and third derivative at $t = 0$. They are given respectively as

$$\begin{aligned}&y_n - 10y_{n+\frac{1}{15}} + 20y_{n+\frac{1}{10}} - 15y_{n+\frac{2}{15}} + 4y_{n+\frac{1}{6}} \\ &\quad = \frac{h^4}{12247200000}\Big(4f_n + 2370f_{n+\frac{1}{30}} + 20745f_{n+\frac{1}{15}} \\ &\qquad 41920f_{n+\frac{1}{10}} + 10770f_{n+\frac{2}{15}} - 234f_{n+\frac{1}{6}}. + 25f_{n+\frac{1}{5}}\Big), \qquad (9)\end{aligned}$$

$$\begin{aligned} & y_{n+\frac{1}{30}} - 4y_{n+\frac{1}{15}} + 6y_{n+\frac{1}{10}} - 4y_{n+\frac{2}{15}} + y_{n+\frac{1}{6}} \\ & \quad = \frac{h^4}{12247200000}\Big(5f_n - 51f_{n+\frac{1}{30}} + 2679f_{n+\frac{1}{15}} + 9854f_{n+\frac{1}{10}} \\ & \qquad + 2679f_{n+\frac{2}{15}} - 51f_{n+\frac{1}{6}} + 5f_{n+\frac{1}{5}}\Big), \quad (10) \end{aligned}$$

$$\begin{aligned} & hy'_n + 235y_{n+\frac{1}{15}} - 570y_{n+\frac{1}{10}} + 465y_{n+\frac{2}{15}} - 130y_{n+\frac{1}{6}} \\ & \quad = \frac{-h^4}{24494400000}\Big(7658f_n + 1535f_{n+\frac{1}{5}} - 14418f_{n+\frac{1}{6}} \\ & \qquad + 2845040f_{n+\frac{1}{10}} + 1848915f_{n+\frac{1}{15}} + 696540f_{n+\frac{2}{15}} + 435930f_{n+\frac{1}{30}}\Big), \quad (11) \end{aligned}$$

$$\begin{aligned} & h^2y''_n + 2700y_{n+\frac{1}{6}} + 9900y_{n+\frac{1}{10}} - 3600y_{n+\frac{1}{15}} - 9000y_{n+\frac{2}{15}} \\ & \quad = \frac{h^4}{272160000}\Big(19823f_n - 502f_{n+\frac{1}{5}} + 2902f_{n+\frac{1}{6}} \\ & \qquad + 728600f_{n+\frac{1}{10}} + 575180f_{n+\frac{1}{15}} + 140915f_{n+\frac{2}{15}} + 322282f_{n+\frac{1}{30}}\Big), \quad (12) \end{aligned}$$

and

$$\begin{aligned} & h^3y'''n - 27000y_{n+\frac{1}{6}} - 81000y_{n+\frac{1}{10}} + 27000y_{n+\frac{1}{15}} + 81000y_{n+\frac{2}{15}} \\ & \quad = \frac{-h^4}{3628800}\Big(36799f_n - 1335f_{n+\frac{1}{5}} \\ & \qquad + 9608f_{n+\frac{1}{6}} + 156920f_{n+\frac{1}{10}}. + 55219f_{n+\frac{1}{15}} - 10459f_{n+\frac{2}{15}} + 176608f_{n+\frac{1}{30}}\Big). \quad (13) \end{aligned}$$

2.2 Block formulation of the derived formula

The Linear Multistep Method (8) can be written in block form by expressing the formulas (8)-(13) as a matrix equation

$$\hat{U}_0\hat{Y} = \hat{U}_1Y_n + h\hat{U}_2Y_n^{'} + h^2\hat{U}_3Y_n^{''} + h^3\hat{U}_4Y_n^{'''} + h^4\left(\mathrm{F}_0\ \bar{F} + \mathrm{F}_1\ \hat{F}\right), \quad (14)$$

where $\hat{U}_i$, $i = 0, \ldots 4$, F_i, $i = 0, 1$ are 6×6 matrices whose entries are coefficients of equations (8) -(13);

$\hat{Y} = \left(y_{n+\frac{1}{30}}, y_{n+\frac{1}{15}}, y_{n+\frac{1}{10}}, y_{n+\frac{2}{15}}, y_{n+\frac{1}{6}}, y_{n+\frac{1}{5}}\right)$,

$Y_n = \left(y_{n-\frac{1}{30}}, y_{n-\frac{1}{15}}, y_{n-\frac{1}{10}}, y_{n-\frac{2}{15}}, y_{n-\frac{1}{6}}, y_n\right)$

$Y'_n = \left(y'_{n-\frac{1}{30}}, y'_{n-\frac{1}{15}}, y'_{n-\frac{1}{10}}, y'_{n-\frac{2}{15}}, y'_{n-\frac{1}{6}}, y'_n\right)$,

$Y''_n = \left(y''_{n-\frac{1}{30}}, y''_{n-\frac{1}{15}}, y''_{n-\frac{1}{10}}, y''_{n-\frac{2}{15}}, y''_{n-\frac{1}{6}}, y''_n\right)$

$Y'''_n = \left(y'''_{n-\frac{1}{30}}, y'''_{n-\frac{1}{15}}, y'''_{n-\frac{1}{10}}, y'''_{n-\frac{2}{15}}, y'''_{n-\frac{1}{6}}, y'''_n\right)$,

$\hat{F} = \left(f_{n+\frac{1}{30}}, f_{n+\frac{1}{15}}, f_{n+\frac{1}{10}}, f_{n+\frac{2}{15}}, f_{n+\frac{1}{6}}, f_{n+\frac{1}{5}}\right)$ and

$\bar{F} = \left(f_{n-\frac{1}{30}}, f_{n-\frac{1}{15}}, f_{n-\frac{1}{10}}, f_{n-\frac{2}{15}}, f_{n-\frac{1}{6}}, f_n\right)$.

Let start by combining equations (8) and (9)-(13) as a matrix equation of the type (14) and then solve using matrix inversion to obtain a version (14) with following coefficient matrices;

$$\hat{U}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{1}{162000} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{20250} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{6000} \\ 0 & 0 & 0 & 0 & 0 & \frac{4}{10125} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{1296} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{750} \end{pmatrix}, \hat{U}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{1}{1800} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{450} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{200} \\ 0 & 0 & 0 & 0 & 0 & \frac{2}{225} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{72} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{50} \end{pmatrix},$$

$$\hat{U}_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{1}{30} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{15} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{10} \\ 0 & 0 & 0 & 0 & 0 & \frac{2}{15} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{5} \end{pmatrix}, \hat{U}_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\bar{F} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{95929}{2939328000000} \\ 0 & 0 & 0 & 0 & 0 & \frac{4127}{11481750000} \\ 0 & 0 & 0 & 0 & 0 & \frac{5471}{4032000000} \\ 0 & 0 & 0 & 0 & 0 & \frac{488}{143521875} \\ 0 & 0 & 0 & 0 & 0 & \frac{6457}{940584960} \\ 0 & 0 & 0 & 0 & 0 & \frac{191}{15750000} \end{pmatrix},$$

and

$$\hat{F} = \begin{pmatrix} \frac{4001}{104976000000} & -\frac{23033}{587865600000} & \frac{811}{24494400000} & -\frac{10693}{587865600000} & \frac{4219}{734832000000} & -\frac{2323}{2939328000000} \\ \frac{4391}{5740875000} & -\frac{199}{328050000} & \frac{97}{191362500} & -\frac{127}{459270000} & \frac{499}{5740875000} & -\frac{137}{11481750000} \\ \frac{423}{112000000} & -\frac{39}{17920000} & \frac{29}{14400000} & -\frac{99}{89600000} & \frac{39}{112000000} & -\frac{193}{4032000000} \\ \frac{7808}{717609375} & -\frac{632}{143521875} & \frac{256}{47840625} & -\frac{58}{20503125} & \frac{128}{143521875} & -\frac{88}{717609375} \\ \frac{1123}{47029248} & -\frac{1265}{188116992} & \frac{95}{7838208} & -\frac{1045}{188116992} & \frac{61}{33592320} & -\frac{47}{188116992} \\ \frac{39}{875000} & -\frac{3}{350000} & \frac{19}{787500} & -\frac{3}{350000} & \frac{3}{875000} & -\frac{1}{2250000} \end{pmatrix},$$

$\hat{U}_0$ is an identity matrix of dimension six.

Remark 1. This result contains a block of formulas that can be used to obtain direct numerical solution of fourth order initial value problems where lower derivatives are absent. However, the block of formulas needs to be modified to handle general fourth order ordinary differential equation. To achieve this objective, first, second and third derivatives of (7) are evaluated at $t = \frac{i}{30}$, $i = 1, 2, \ldots, 6$ which simultaneously yields the following Chebyshev induced hybrid block method;

$$\left.\begin{aligned}
y_{n+\frac{1}{30}} &= \frac{h^3}{162000}y_n''' + \frac{h^2}{1800}y_n'' + \frac{h}{30}y_n' + y_n + \frac{h^4}{2939328000000}(95929f_n \\
&\quad +112028f_{n+\frac{1}{30}} - 115165f_{n+\frac{1}{15}} + 97320f_{n+\frac{1}{10}} - 53465f_{n+\frac{2}{15}} + 16876f_{n+\frac{1}{6}} \\
&\quad -2323f_{n+\frac{1}{5}}\Big) \\
y_{n+\frac{1}{15}} &= \frac{h^3}{20250}y_n''' + \frac{h^2}{450}y_n'' + \frac{h}{15}y_n' + y_n + \frac{h^4}{11481750000}\Big(4127f_n + 8782f_{n+\frac{1}{30}} \\
&\quad -6965f_{n+\frac{1}{15}} + 5820f_{n+\frac{1}{10}} - 3175f_{n+\frac{2}{15}} + 998f_{n+\frac{1}{6}} - 137f_{n+\frac{1}{5}}\Big) \\
y_{n+\frac{1}{10}} &= \frac{h^3}{6000}y_n''' + \frac{h^2}{200}y_n'' + \frac{h}{10}y_n' + y_n + \frac{h^4}{4032000000}\Big(5471f_n + 15228f_{n+\frac{1}{30}} \\
&\quad -8775f_{n+\frac{1}{15}} + 8120f_{n+\frac{1}{10}} - 4455f_{n+\frac{2}{15}} + 1404f_{n+\frac{1}{6}} - 193f_{n+\frac{1}{5}}\Big) \\
y_{n+\frac{2}{15}} &= \frac{4h^3}{10125}y_n''' + \frac{2h^2}{225}y_n'' + \frac{2h}{15}y_n' + y_n + \frac{h^4}{717609375}\Big(2440f_n + 7808f_{n+\frac{1}{30}} \\
&\quad -3160f_{n+\frac{1}{15}} + 3840df_{n+\frac{1}{10}} - 2030f_{n+\frac{2}{15}} + 640f_{n+\frac{1}{6}} - 88f_{n+\frac{1}{5}}\Big) \\
y_{n+\frac{1}{6}} &= +\frac{h^3}{1296}y_n''' + \frac{h^2}{72}y_n'' + \frac{h}{6}y_n' + y_n + \frac{h^4}{940584960}\Big(6457f_n + 22460f_{n+\frac{1}{30}} \\
&\quad -6325f_{n+\frac{1}{15}} + 11400f_{n+\frac{1}{10}} - 5225f_{n+\frac{2}{15}} + 1708f_{n+\frac{1}{6}} - 235f_{n+\frac{1}{5}}\Big) \\
y_{n+\frac{1}{5}} &= \frac{h^3}{750}y_n''' + \frac{h^2}{50}\frac{h}{5}y_n' + y_n + \frac{h^4}{15750000}\Big(191f_n + 702f_{n+\frac{1}{30}} - 135f_{n+\frac{1}{15}} \\
&\quad +380f_{n+\frac{1}{10}} - 135f_{n+\frac{2}{15}} + 54f_{n+\frac{1}{6}} - 7f_{n+\frac{1}{5}}\Big)
\end{aligned}\right\}, \tag{15}$$

$$\left.\begin{aligned}
y'_{n+\frac{1}{30}} &= \frac{h^2}{1800}y'''_n + \frac{h}{30}y''_n + y'_n + \frac{h^3}{97977600000}\Big(343801f_n + 506604f_{n+\frac{1}{30}} \\
&\quad -494715f_{n+\frac{1}{15}} + 414160f_{n+\frac{1}{10}} - 226605f_{n+\frac{2}{15}} + 71364f_{n+\frac{1}{6}} - 9809f_{n+\frac{1}{5}}\Big) \\
y'_{n+\frac{1}{15}} &= \frac{h^2}{450}y'''_n + \frac{h}{15}y''_n + y'_n + \frac{h^3}{765450000}\Big(13774f_n + 35976f_{n+\frac{1}{30}} - 24465f_{n+\frac{1}{15}} \\
&\quad +20800f_{n+\frac{1}{10}} - 11370f_{n+\frac{2}{15}} + 3576f_{n+\frac{1}{6}} - 491f_{n+\frac{1}{5}}\Big) \\
y'_{n+\frac{1}{10}} &= \frac{h^2}{200}y'''_n + \frac{h}{10}y''_n + y'_n + \frac{h^3}{134400000}\Big(5877f_n + 19188f_{n+\frac{1}{30}} - 8055f_{n+\frac{1}{15}} \\
&\quad +8960f_{n+\frac{1}{10}} - 4905f_{n+\frac{2}{15}} + 1548f_{n+\frac{1}{6}} - 213f_{n+\frac{1}{5}}\Big) \\
y'_{n+\frac{2}{15}} &= \frac{2h^2}{225}y'''_n + \frac{h^3}{47840625}\Big(3863f_n + 13992f_{n+\frac{1}{30}} - 3390f_{n+\frac{1}{15}} + 6800f_{n+\frac{1}{10}} \\
&\quad -3255f_{n+\frac{2}{15}} + 1032f_{n+\frac{1}{6}} - 142f_{n+\frac{1}{5}}\Big) \\
y'_{n+\frac{1}{6}} &= \frac{h^2}{72}y'''_n + \frac{h}{6}y''_n + y'_n + \frac{h^3}{31352832}\Big(4045f_n + 15564f_{n+\frac{1}{30}} - 2055f_{n+\frac{1}{15}} \\
&\quad +8560f_{n+\frac{1}{10}} - 2865f_{n+\frac{2}{15}} + 1092f_{n+\frac{1}{6}} - 149f_{n+\frac{1}{5}}\Big) \\
y'_{n+\frac{1}{5}} &= \frac{h^2}{50}y'''_n + \frac{h}{5}y''_n + y'_n + \frac{h^3}{1050000}\Big(198f_n + 792f_{n+\frac{1}{30}} - 45f_{n+\frac{1}{15}} + 480f_{n+\frac{1}{10}} \\
&\quad -90f_{n+\frac{2}{15}} + 72f_{n+\frac{1}{6}} - 7f_{n+\frac{1}{5}}\Big)
\end{aligned}\right\}, \tag{16}$$

$$
\left.
\begin{aligned}
y''_{n+\frac{1}{30}} &= \frac{h}{30} + y''_n + \frac{h^2}{108864000}\Big(28549 f_n + 57750 f_{n+\frac{1}{30}} - 51453 f_{n+\frac{1}{15}} + 42484 f_{n+\frac{1}{10}} \\
&\quad -23109 f_{n+\frac{2}{15}} + 7254 f_{n+\frac{1}{6}} - 995 f_{n+\frac{1}{5}}\Big) \\
y''_{n+\frac{1}{15}} &= \frac{h}{15} y'''_n + y''_n + \frac{h^2}{1701000}\Big(1027 f_n + 3492 f_{n+\frac{1}{30}} - 1680 f_{n+\frac{1}{15}} + 1576 f_{n+\frac{1}{10}} \\
&\quad -873 f_{n+\frac{2}{15}} + 276 f_{n+\frac{1}{6}} - 38 f_{n+\frac{1}{5}}\Big) \\
y''_{n+\frac{1}{10}} &= \frac{h}{10} y'''_n + y''_n + \frac{h^2}{1344000}\Big(1265 f_n + 4950 f_{n+\frac{1}{30}} - 801 f_{n+\frac{1}{15}} + 2100 d f_{n+\frac{1}{10}} \\
&\quad -1089 f_{n+\frac{2}{15}} + 342 f_{n+\frac{1}{6}} - 47 f_{n+\frac{1}{5}}\Big) \\
y''_{n+\frac{2}{15}} &= \frac{2h}{15} y'''_n + y''_n + \frac{2h^2}{212625}\Big(272 f_n + 1128 f_{n+\frac{1}{30}} - 18 f_{n+\frac{1}{15}} + 656 f_{n+\frac{1}{10}} \\
&\quad -210 f_{n+\frac{2}{15}} + 72 f_{n+\frac{1}{6}} - 10 f_{n+\frac{1}{5}}\Big) \\
y''_{n+\frac{1}{6}} &= \frac{h}{6} y'''_n + y''_n + \frac{h^2}{870912}\Big(1409 f_n + 6030 f_{n+\frac{1}{30}} + 375 f_{n+\frac{1}{15}} + 4100 f_{n+\frac{1}{10}} \\
&\quad -225 f_{n+\frac{2}{15}} + 462 f_{n+\frac{1}{6}} - 55 f_{n+\frac{1}{5}}\Big) \\
y''_{n+\frac{1}{5}} &= \frac{h}{5} y'''_n + y''_n \frac{h^2}{21000}\Big(41 f_n + 180 f_{n+\frac{1}{30}} + 18 f_{n+\frac{1}{15}} + 136 f_{n+\frac{1}{10}} \\
&\quad +9 f_{n+\frac{2}{15}} + 36 f_{n+\frac{1}{6}}\Big)
\end{aligned}
\right\}, \tag{17}
$$

and

$$\left.\begin{array}{l} y'''_{n+\frac{1}{30}} = y'''_n + \frac{h}{1814400}\Big(19087f_n + 65112f_{n+\frac{1}{30}} - 46461f_{n+\frac{1}{15}} + 37504f_{n+\frac{1}{10}} \\ \qquad -20211f_{n+\frac{2}{15}} + 6312f_{n+\frac{1}{6}} - 863f_{n+\frac{1}{5}}\Big) \\ y'''_{n+\frac{1}{15}} = y'''_n + \frac{h}{113400}\Big(1139f_n + 5640f_{n+\frac{1}{30}} + 33f_{n+\frac{1}{15}} + 1328f_{n+\frac{1}{10}} - 807f_{n+\frac{2}{15}} \\ \qquad +264f_{n+\frac{1}{6}} - 37f_{n+\frac{1}{5}}\Big) \\ y'''_{n+\frac{1}{10}} = y'''_n + \frac{h}{67200}\Big(685f_n + 3240f_{n+\frac{1}{30}} + 1161f_{n+\frac{1}{15}} + 2176f_{n+\frac{1}{10}} - 729f_{n+\frac{2}{15}} \\ \qquad +216f_{n+\frac{1}{6}} - 29f_{n+\frac{1}{5}}\Big) \\ y'''_{n+\frac{2}{15}} = y'''_n + \frac{h}{14175}\Big(143f_n + 696f_{n+\frac{1}{30}} + 192f_{n+\frac{1}{15}} + 752f_{n+\frac{1}{10}} + 87f_{n+\frac{2}{15}} \\ \qquad +24f_{n+\frac{1}{6}} - 4f_{n+\frac{1}{5}}\Big) \\ y'''_{n+\frac{1}{6}} = y'''_n + \frac{h}{72576}\Big(743f_n + 3480f_{n+\frac{1}{30}} + 3200f_{n+\frac{1}{10}} + 1275f_{n+\frac{1}{15}} + 2325f_{n+\frac{2}{15}} \\ \qquad +1128f_{n+\frac{1}{6}} - 55f_{n+\frac{1}{5}}\Big) \\ y'''_{n+\frac{1}{5}} = y'''_n + \frac{h}{4200}\Big(41f_n + 216f_{n+\frac{1}{30}} + 27f_{n+\frac{1}{15}} + 272f_{n+\frac{1}{10}} + 27f_{n+\frac{2}{15}} + 216f_{n+\frac{1}{6}} \\ \qquad +41f_{n+\frac{1}{5}}\Big) \end{array}\right\}. \tag{18}$$

3 Analysis of the Properties of the Derived Method

In this section, the analysis of the basic properties of the derived Chebyshev induced hybrid block method is presented.

3.1 Local truncation and order of the proposed method

Proposition 1. *The theoretical order of each of the six primary formulas for the proposed methods is seven while the local truncation error is* $\Upsilon_{\frac{r}{30}}\{y(x) : h\} = c_{n+11}y^{(11)}(t_n)h^{11} + 0(h^{12})$.

Proof. Since $y(x)$ is continuous and differentiable, and following [2], the linear difference operator equivalent to formulas in (15)-(18) is defined by

$$\Upsilon_{\frac{r}{30}}\{y(x):h\} = y\left(x_n + \frac{r}{30}h\right) - \left\{\sum_{b=0}^{3}\alpha_b y^{(b)}(x)h^b - h^4\sum_{r=0}^{6}\beta_r y^{(4)}\left(x+\frac{r}{30}h\right)\right\}. \tag{19}$$

The Taylor series expansion about the point x to (19) provides a formula for the local truncation errors of the proposed scheme

$$\begin{aligned}\Upsilon_{\frac{r}{30}}\{y(x):h\} =& c_0 y(x) + c_1 h y^{'}(x) + c_2 h^2 y^{''}(x) + \cdots + c_{p+3}h^{p+3}y^{(p+3)}(x) \\ &+ c_{p+4}h^{p+4}y^{(p+4)}(x)\end{aligned} \tag{20}$$

The term c_{p+4} is called the error constant and it implies that the local truncation error is defined as:

$$\Upsilon_{\frac{r}{30}}\{y(x):h\} = c_{p+4}h^{p+4}y^{(p+4)}(x_n) + Oh^{p+5} \tag{21}$$

where

$$\left.\begin{aligned}
\Upsilon_{\frac{1}{30}}\{y(x):h\} &= \frac{15739h^{11}y^{(11)}(x_n)}{530335602720000000000000} + O\left(h^{12}\right)\\
\Upsilon_{\frac{2}{30}}\{y(x):h\} &= \frac{733h^{11}y^{(11)}(x_n)}{1657298758500000000000} + O\left(h^{12}\right)\\
\Upsilon_{\frac{3}{30}}\{y(x):h\} &= \frac{h^{11}y^{(11)}(x_n)}{561330000000000000} + O\left(h^{12}\right)\\
\Upsilon_{\frac{4}{30}}\{y(x):h\} &= \frac{37h^{11}y^{(11)}(x_n)}{80922790942382812 50} + O\left(h^{12}\right)\\
\Upsilon_{\frac{5}{30}}\{y(x):h\} &= \frac{317h^{11}y^{(11)}(x_n)}{339414785740800000000} + O\left(h^{12}\right)\\
\Upsilon_{\frac{6}{30}}\{y(x):h\} &= \frac{h^{11}y^{(11)}(x_n)}{601425000000000000} + O\left(h^{12}\right)
\end{aligned}\right\}. \tag{22}$$

for each formula in (15). This procedure was also repeated for the formulas in (16)-(18). Since $c_0 = c_1 = ... = c_{p+3} = 0$, $c_{p+4} \neq 0$, refer to [14]; then formulas in (15)-(18) have uniform order $p = 7$. □

3.2 Consistency of the Method

Definition 1. (see [2]) The linear multistep method is said to be consistent if it has order $p \geq 1$. It is obvious that the present method is consistent.

3.3 Zero Stability of the Block Method

Authors such [3], [14], and [22] described the zero stability of a numerical method as the one that tells its behaviors as $h \to 0$. That is, if we set h to zero in the primary formulas of the proposed in (15), it will reduce to a set of equations that be written as in matrix form

$$\hat{U}_0 \hat{Y} = \hat{U}_1 Y_n \tag{23}$$

where, $\hat{U}_0$ and $\hat{U}_1$ remain as earlier defined.

Definition 2. (see [2]) The linear multistep method (15) is said to be zero-stable if no root of the first characteristic polynomial has modulus greater than one, and if every root with modulus one is simple.

From (23), the characteristics polynomial of the primary formulas of the proposed method can be written as:

$$Det\left(\lambda \hat{U}_0 - \hat{U}_1\right) = 0 \tag{24}$$

$$Det\left(\lambda \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}\right) \tag{25}$$

This gives $\lambda^5(\lambda - 1) = 0$, that is $\lambda = (0,\ 0,\ 0,\ 0,\ 0,\ 1)$. Similar results were obtained when this procedure was repeated for the formulas in (16)-(18) indicating that the Chebyshev induced block method is zero stable in line with Definition 2.

3.4 Convergence

It is important to state the fundamental Dahlquist theorem on linear multistep methods.

Theorem 1. *The necessary and sufficient conditions for a linear multi- step method to be convergent are that it be consistent and zero-stable.*

The proposed method is convergent since it consistent and zero stable.

3.5 Region of Absolute Stability of the Method

Definition 3. [2] The block method MHBM2 is P-stable if the periodicity interval of the method $(0, +\infty)$.

Proposition 2. *The proposed method is P-stable.*

This proposition is ascertained by first considering the characteristics polynomial of the primary formulas of (15) given as

$$p(r) = \begin{bmatrix} r^{1/30} & 0 & 0 & 0 & 0 & -1 \\ 0 & r^{1/15} & 0 & 0 & 0 & -1 \\ 0 & 0 & r^{1/10} & 0 & 0 & -1 \\ 0 & 0 & 0 & r^{2/15} & 0 & -1 \\ 0 & 0 & 0 & 0 & \sqrt[6]{r} & -1 \\ 0 & 0 & 0 & 0 & 0 & \sqrt[5]{r} - 1 \end{bmatrix} \tag{26}$$

and

$$q(r) = \begin{bmatrix} \frac{39\,r^{1/30}}{875000} & \frac{-3\,r^{1/15}}{350000} & \frac{19\,r^{1/10}}{787500} & \frac{-3\,r^{2/15}}{350000} & \frac{3\,\sqrt[6]{r}}{875000} & \frac{-\sqrt[5]{r}}{2250000} + \frac{95929}{2939328000000} \\ \frac{1123\,r^{1/30}}{47029248} & \frac{-1265\,r^{1/15}}{188116992} & \frac{95\,r^{1/10}}{7838208} & \frac{-1045\,r^{2/15}}{188116992} & \frac{61\,\sqrt[6]{r}}{33592320} & \frac{-47\,\sqrt[5]{r}}{188116992} + \frac{6457}{940584960} \\ \frac{7808\,r^{1/30}}{717609375} & \frac{-632\,r^{1/15}}{143521875} & \frac{256\,r^{1/10}}{47840625} & \frac{-58\,r^{2/15}}{20503125} & \frac{128\,\sqrt[6]{r}}{143521875} & \frac{-88\,\sqrt[5]{r}}{717609375} + \frac{5471}{4032000000} \\ \frac{4391\,r^{1/30}}{5740875000} & \frac{-199\,r^{1/15}}{328050000} & \frac{97\,r^{1/10}}{191362500} & \frac{-127\,r^{2/15}}{459270000} & \frac{499\,\sqrt[6]{r}}{5740875000} & \frac{-137\,\sqrt[5]{r}}{11481750000} + \frac{4127}{11481750000} \\ \frac{423\,r^{1/30}}{112000000} & \frac{-39\,r^{1/15}}{17920000} & \frac{29\,r^{1/10}}{14400000} & \frac{-99\,r^{2/15}}{89600000} & \frac{39\,\sqrt[6]{r}}{112000000} & \frac{-193\,\sqrt[5]{r}}{4032000000} + \frac{488}{143521875} \\ \frac{4001\,r^{1/30}}{104976000000} & \frac{-23033\,r^{1/15}}{587865600000} & \frac{811\,r^{1/10}}{24494400000} & \frac{-10693\,r^{2/15}}{587865600000} & \frac{4219\,\sqrt[6]{r}}{734832000000} & \frac{-2323\,\sqrt[5]{r}}{2939328000000} + \frac{191}{15750000} \end{bmatrix}. \tag{27}$$

The stability polynomial of (15) is given by

$$\Pi(r, z) = p(r) + zq(r) \tag{28}$$

where, $z = \lambda^4 h^4$ and $\lambda = \frac{\partial f}{\partial y}$. Note that the stability polynomial (28) is obtained by applying the formulas in (15) to the scalar test function

$$y^{(4)} = -\lambda^4 y. \tag{29}$$

Replacing r with $e^{I\theta}$ and solving the stability polynomial (28) for $\Pi(r, z) = 0$ yields after some simplifications

$$z = \frac{18223519635182674907136000000000000000000000000000(e^{\frac{I}{5}\theta} - 1)}{18662400e^{\frac{I}{5}\theta} - 15611903689201}. \tag{30}$$

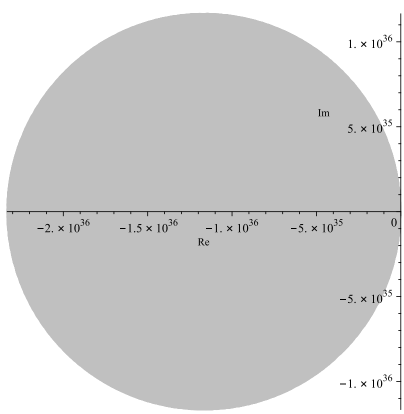

Figure 1: Region of absolute stability of the method.

This is plotted in Maplesoft environment using the following commands.
with(plots);
complexplot(-z, theta = 0 .. 360, filled = true, labels = ["Re", "Im"], color = grey);
Figure 1 shown the region of absolute stability of the proposed method, the shaded region is where the method is unstable. This confirmed that the left-hand side of the complex plane $(0, +\infty)$ is included in the stability region of the proposed method. Hence, the derived method is p-stable.

4 Numerical Experiments

To test how well the proposed method works, six numerical examples are considered for the numerical experiment. The accuracy was measured by calculating the absolute error using the relation $Error = |y_n - y(x)|$.

Problem 1. The general fourth-order IVP of ordinary differential equation

$$y^{(4)} = y''' + y'' + y' + 2y,\ y(0) = 0,\ y'(0) = 0,\ y''(0) = 0,\ y'''(0) = 30$$

whose exact solution is reported as $y(x) = 2e^{2x} - 5e^{-x} + 3\cos x - 9\sin x$ is considered as the first test problem (Source: see [6] and [7]. The solutions to problem (1) were obtained within $[0, 2]$ over 10 iterations and are compared with the exact solution, as presented in Figure 2. It is clear from Table 1 and Figure 2 that methods shows good performance over the methods in [6] and [7].

Table 1: Solution of problem 1 obtained using the proposed method.

x	y-computed	y -exact	Error in Method	Error in [7]	Error in [6]
0.2	0.04217	0.04217	8.70415 E-14	3.5129 E-13	2.319 E-13
0.4	0.3579	0.3579	8.04246 E-13	4.1833 E-12	2.2603 E-12
1.8	62.9237	62.9237	2.93019 E-10	5.4334 E-10	9.1180 E-09
2.0	99.0875	99.0875	5.1155 E-10	8.0796 E-10	1.7409 E-08

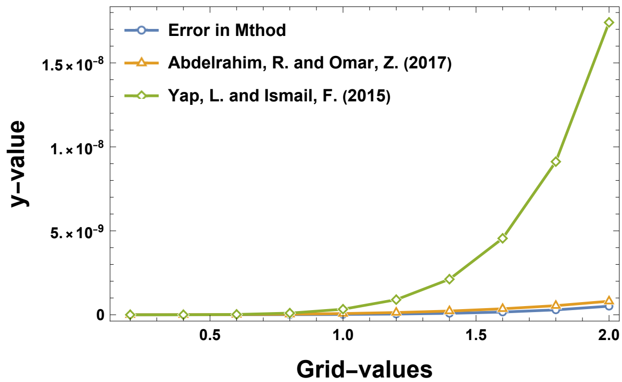

Figure 2: Comparison curves of problem 1.

Problem 2. The second sample equation considered in this work is

$$y^{(4)} = -4x^2 + e^x\left(x^2 - 4t + 1\right) - yy' + (y')^2,\ y(0) = 0,\ y'(0) = 1, y''(0) = 0,\ y'''(0) = 0$$

whose exact solution is reported as $y(x) = x^2 + e^x$ (Source: see [9]). The solutions to problem 2 were obtained within $[0, 1]$ over 10 iterations and are compared with the results of NDSolve and exact solutions, as presented in Figure 3. Table 2 presents the comparison of the maximum absolute errors of the proposed method with those of methods in [9] and [20].

Table 2: Comparison of the maximum absolute errorsobtained for Problem 2.

h	Methods	Error
0.2	Proposed Method	5.28 E-21
	BHI In [20]	1.21 E-17
	AM In [20]	5.59 E-10
	BHCM4 In [20]	2.38 E-17
0.1	Proposed Method	2.06 E-23
	BHI In [20]	5.20 E-21
	AM In [20]	2.84 E-14
	BHCM4 In [20]	1.95 E-17

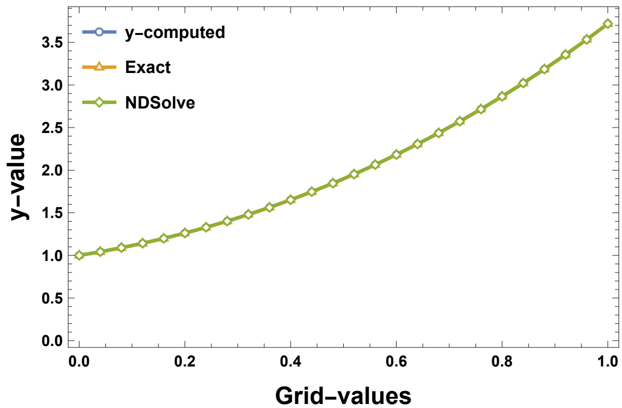

Figure 3: Comparison curves obtained for problem 2.

Problem 3. The third sample equation considered is

$$y^{(4)} = -y'', y(0) = 0, y'(0) = -\frac{1.1}{72 - 50\pi}, y''(0) = \frac{1}{144 - 100\pi}, y'''(0) = \frac{1.2}{144 - 100\pi}$$

whose exact solution is reported as $y(t) = \frac{-t-1.2\sin(t)-\cos(t)+1}{144-100\pi}$ (Source: see [7]). Problem 3 was approximated using the derived method within $[0, 2]$ over 10 iterations and the results are compared with [21] in Table 3. It is obvious from Table 3 that y-approx agreed with exact solution up to eighteen decimal places which confirmed the good performance of the present method over the method in [21].

Table 3: Comparison results of Problem 3.

x	Error in Proposed Method	Error in [21]
0.0	0.0000	0.0000
0.2	2.60208E-18	3.40060E - 15
0.4	4.33680E-18	7.40519E - 14
1.6	5.37764E-17	5.09100E - 11
1.8	6.76542E-17	9.85949E - 11
2.0	7.97972E-17	1.83206E - 10

Problem 4. The fourth sample equation considered is a nonlinear fourth-order ordinary differential equation

$$y^{(4)} = \frac{3Sin(y)(3 + 2Sin^2(y))}{Cos^7(y)}, y(0) = 0, y'(0) = 1, y''(0) = 0, y'''(0) = 1$$

with the exact $y(x) = arcsin(x)$ (Source: see [20, 21]. The problem is integrated in the interval $[0, \frac{\pi}{4}]$. Table 4 and Figure 4 showed the y-approx generated by the derived methods at the grid points $\left(\frac{\pi}{200}, \frac{\pi}{100}, \frac{3\pi}{200}, \frac{\pi}{50}, \frac{\pi}{40}, \frac{3\pi}{100}, \frac{7\pi}{200}, \frac{\pi}{25}, \frac{9\pi}{200}\right)$ as compared those generated NDSolve in Wolfram Mathematica. This confirmed that the present method is a good alternative for solving nonlinear problems.

Table 4: Computed results of Problem 4.

x	y-approx	y-Exact	NDSolve	y-Error	NDSolveError
0	0.	0.	0.	0.	0.
$\frac{\pi}{200}$	0.0157086	0.0157086	0.015708	1.0447E-23	6.4594E-7
$\frac{\pi}{100}$	0.0314211	0.0314211	0.0314159	8.6841E-23	5.1676E-6
$\frac{3\pi}{200}$	0.0471413	0.0471413	0.0471239	2.9686E-22	1.74409E-5
$\frac{\pi}{50}$	0.0628733	0.0628733	0.0628319	7.1173E-22	4.1345E-5
$\frac{\pi}{40}$	0.0786208	0.0786208	0.07854	1.4082E-21	8.07453E-5
$\frac{3\pi}{100}$	0.0943879	0.0943879	0.0942483	2.4706E-21	1.39528E-4
$\frac{7\pi}{200}$	0.110179	0.110179	0.109957	3.9938E-21	2.21566E-4
$\frac{\pi}{25}$	0.125997	0.125997	0.125666	6.0856E-21	3.30734E-4
$\frac{9\pi}{200}$	0.141847	0.141847	0.141376	8.8707E-21	4.7091E-4

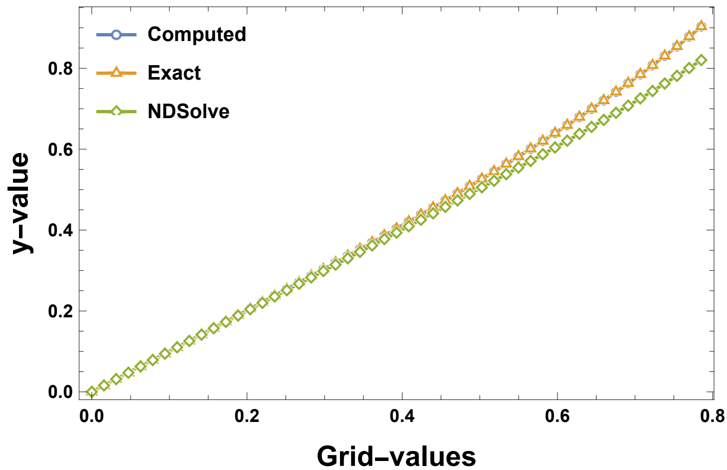

Figure 4: Solution curves of problem 4.

Problem 5. The fifth sample equation considered in this work is also a nonlinear, non-homogeneous fourth-order ordinary differential equation

$$y^{(4)} = y^2 + cos^2(x) + sin(x) - 1, y(0) = 0, y'(0) = 1, y''(0) = 0, y'''(0) = -1$$

whose exact solution is $y(x) = sin(x)$ (Source: see [21]). The problem is integrated in the interval [0, 1]. The results are presented in Table 5 and Figure 5.

Table 5: Computed results of Problem 5.

x	y-approx	y-Exact	NDSolve	y-Error	NDSolveError
0.	0.	0.	0.	0.	0.
0.02	0.0199987	0.0199987	0.0199987	2.4887E-16	2.5364E-8
0.04	0.0399893	0.0399893	0.0399897	3.1200E-14	3.2640E-7
0.1	0.0998334	0.0998334	0.0998458	1.7832E-11	1.24265E-5
0.16	0.159318	0.159318	0.159399	4.4589E-10	8.11434E-5
0.18	0.17903	0.17903	0.179159	9.9195E-10	1.29839E-4
0.2	0.198669	0.198669	0.198867	2.0216E-9	1.97695E-4

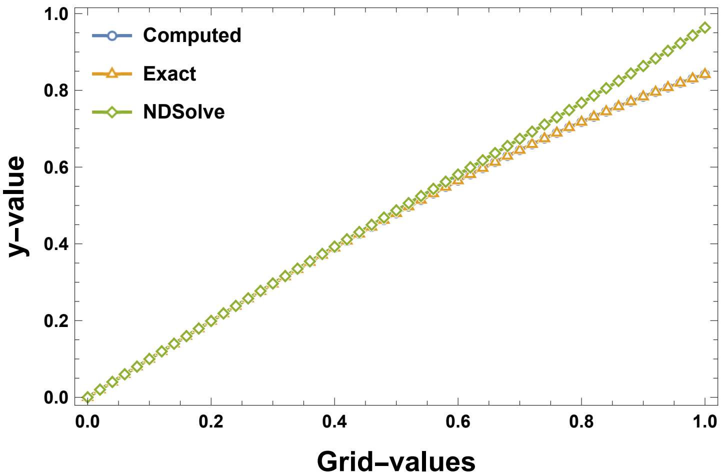

Figure 5: Solution curves of problem 5.

Problem 6. The last sample equation considered in this work is the ill-posed Problem of a Beam on Elastic Foundation. The equation is given as

$$y^{(4)} = 1 - y.$$

The ill-posed problem has an important engineering application in a beam on elastic foundation according to [23], where y is the normalized vertical displacement (deviation), and 1 is the normalized distributed load, with the following initial conditions $y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 0, \quad y'''(0) = 0$. The exact solution is given by

$$y(x) = -\frac{1}{2}e^{-\frac{t}{\sqrt{2}}}\left(-2e^{\frac{t}{\sqrt{2}}} + e^{\sqrt{2}t}\cos\left(\frac{t}{\sqrt{2}}\right) + \cos\left(\frac{t}{\sqrt{2}}\right)\right).$$

(Source: see [20, 21]). The problem is integrated in the interval $[0, 1]$. Table 6 demonstrates the good performance of the derived method which was further analysed in Figure 6.

Table 6: Computed results of Problem 6.

x	y-approx	y-Exact	NDSolve	y-Error	NDSolveError
0	0.	0.	0.	0.	0.
$\frac{1}{50}$	6.6666E-9	6.6666E-9	7.5574E-9	1.4746E-30	8.9079E-10
$\frac{1}{25}$	1.0666E-7	1.0666E-7	1.09687E-7	1.5043E-29	3.0205E-9
$\frac{1}{10}$	4.1666E-6	4.1666E-6	4.1721E-6	4.9411E-28	5.4695E-9
$\frac{3}{25}$	8.6399E-6	8.6399E-6	8.6409E-6	1.0133E-27	9.8926E-10
$\frac{9}{50}$	0.00004374	0.00004374	0.0000437401	5.0577E-27	8.1364E-11
$\frac{1}{5}$	0.0000666666	0.0000666666	0.0000666664	7.6920E-27	1.7100E-10

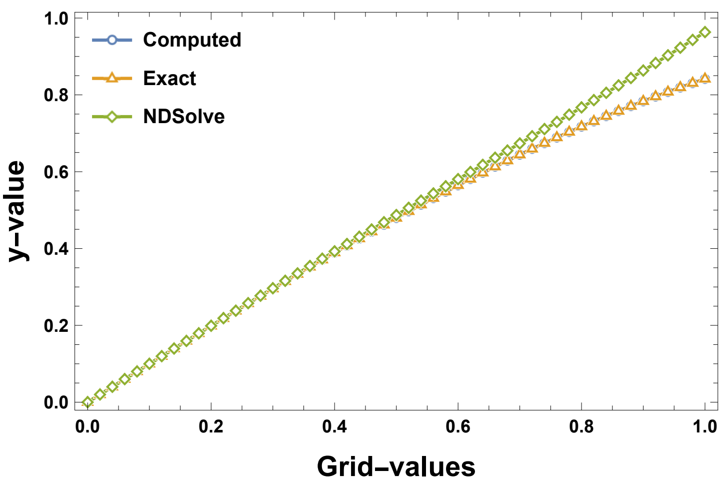

Figure 6: Solution curves of problem 6.

5 Conclusion

This work concentrated on deriving a Chebyshev-induced hybrid block method of theoretical order seven to solve initial value problems of fourth-order ordinary differential equations. The method satisfied the basic properties of linear multistep methods. The accuracy and usability of the developed method were tested by applying it to solve six numerical examples. They were efficient as they give minimal error and have higher accuracy for handling the direct solution of fourth-order initial value problems of ordinary differential equations including nonlinear and ill-posed ones.

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