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The Interesting Characterizations of Some Solitons in Lorentzian Para-Kenmotsu Manifolds

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Abstract

In this article, we have investigated some special solitons in Lorentz para-Kenmotsu manifolds. We have studied in detail some important solitons such as almost η-Ricci soliton, conformal Ricci soliton and η-Ricci Bourguignon solitons in Lorentz para-Kenmotsu manifolds. While examining particularly some important symmetry conditions of Lorentz para-Kenmotsu manifolds, we have obtained characterizations based on both certain special solitons and the generalized $\mathcal{B}$-curvature tensor, which is the generalization of quasi-conformal, Weyl-conformal, concircular and conharmonic curvature tensors.

1 Introduction

Para-Kenmotsu manifolds were first defined by Sinha and Sai Prasad in 1989 [1]. Since their definition, para-Kenmotsu manifolds have undergone significant evolution. They have provided mathematically deep structures and have been

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studied under different geometric classes, such as Lorentzian manifolds. These manifolds have been considered a special type of almost paracontact metric manifolds. During their definition, the authors examined the geometric structures and characteristic properties of these manifolds. This process resulted in fundamental findings that contributed to a better understanding of the geometric structure of manifolds. The initial studies focused on the fundamental symmetry properties of the manifolds and their behavior according to curvature tensors. Following the definition of para-Kenmotsu manifolds, these structures began to attract increasing interest. Many mathematicians and researchers conducted studies aimed at discovering the deep geometric properties of these manifolds. Topics such as solitons, Ricci symmetry, and semi-symmetric curvature tensors were investigated in relation to these manifolds.

The year 2018 marked an important turning point in the development of Para-Kenmotsu manifolds. This year, Lorentzian para-Kenmotsu manifolds, or Lorentzian almost paracontact metric manifolds, were introduced [2]. This new class, associated with Lorentz geometry, was added to the literature as a different form of manifolds. The introduction of Lorentzian manifolds also leads to physical applications of these manifolds [3], [4].

On the other hand, the development of Ricci solitons holds significant importance in the fields of differential geometry and general relativity. The concept of Ricci solitons dates back to the early 1980s. During this period, Richard S. Palais, N. S. Trudinger, and others began their studies on the analysis of Ricci flow and curvature properties. Ricci solitons became an important tool for understanding the curvature behavior of manifolds. In 1982, Richard S. Hamilton made a significant advancement by introducing Ricci flow. Ricci flow is a process that examines how the curvature properties of a Riemannian manifold change over time. Hamilton demonstrated the existence and uniqueness of this flow, thus laying the foundations for the study of Ricci solitons. In the 1990s, research on Ricci solitons intensified. During this time, G. Perelman provided significant results regarding Ricci flow and the characterization of Ricci solitons. Perelman's work revealed the long-term behavior of Ricci flow and the existence of solitons, thereby uncovering the topological properties of manifolds [5, 6].

In recent years, research on Ricci solitons has worked on with the examination of various types of manifolds and geometric structures. The generalization of the concept and the definition of Ricci solitons in different geometric contexts has become important areas of research [7–11]. So, there is ongoing intensive work on the stability of Ricci solitons, the discovery of special solutions and their physical applications [12]- [20].

The fundamental concepts of Ricci-Bourguignon solitons emerged in 1990. J. M. Bourguignon conducted an in-depth examination of the relationship between Ricci flows and solitons [21]. Ricci-Bourguignon solitons have been studied across different geometric structures, with a particular focus on how the metrics and curvatures that compose the manifold interact. These solitons have played an important role in geometric analysis, contributing to the understanding of the structural properties of manifolds. Ricci-Bourguignon solitons are also significant in physical applications [22,23]. The relationship between the geometric structures of manifolds and physical systems has been explored with Ricci-Bourguignon solitons. In recent years, researches on Ricci-Bourguignon solitons have increased, leading to the development of studies on their mathematical structure and physical applications [24–26].

Motivated by all these studies, we have investigate some special solitons in Lorentz para-Kenmotsu manifolds. We have studied in detail some important solitons such as almost η-Ricci soliton, conformal Ricci soliton and η-Ricci Bourguignon solitons in Lorentz para-Kenmotsu manifolds. While examining particularly some important symmetry conditions of Lorentz para-Kenmotsu manifolds, we have obtained characterizations based on both certain special solitons and the generalized $\mathcal{B}$-curvature tensor, which is the generalization of quasi-conformal, Weyl-conformal, concircular and conharmonic curvature tensors.

2 Preliminaries

Let Λ^n be an $n-$dimensional Lorentzian metric manifold. This means that it is endowed with a structure $(\alpha, \beta, \gamma, \delta)$, where α is a $(1,1)-$type tensor field, β is a vector field, γ is a $1-$form on Λ^n and δ is a Lorentzian metric tensor satisfying;

$$\begin{cases} \alpha^2\Omega_1 = \Omega_1 + \gamma(\Omega_1)\beta, \\ \\ \delta(\alpha\Omega_1, \alpha Y) = \delta(\Omega_1, \Omega_2) + \gamma(\Omega_1)\gamma(\Omega_2), \end{cases} \tag{1}$$

and

$$\begin{cases} \gamma(\beta) = -1, \gamma(\Omega_1) = \delta(\Omega_1, \beta), \\ \\ \alpha\beta = 0, \gamma(\alpha\Omega_1) = 0, \end{cases} \tag{2}$$

for all vector fields Ω_1, Ω_2 on Λ^n. Then $\Lambda^n(\alpha, \beta, \gamma, \delta)$ is said to be a Lorentzian almost paracontact manifold [29].

A Lorentzian almost paracontact manifold $\Lambda^n(\alpha, \beta, \gamma, \delta)$ is called Lorentzian para-Kenmotsu manifold if

$$(\nabla_{\Omega_1}\alpha)\Omega_2 = -\delta(\alpha\Omega_1, \Omega_2)\beta - \gamma(\Omega_2)\alpha\Omega_1, \tag{3}$$

for all $\Omega_1, \Omega_2 \in \Gamma(T\Lambda)$, where ∇ and $\Gamma(T\Lambda)$ denote the Levi-Civita connection and differentiable vector fields set on Λ^n, respectively.

From now on, Lorentz para-Kenmotsu manifolds will be referred to as $\mathcal{LPK}$-manifolds for the sake of brevity.

Lemma 1. *Let $\Lambda^n(\alpha, \beta, \gamma, \delta)$ be an $n-$dimensional $\mathcal{LPK}$-manifold. The following relations are provided for $\Lambda^n(\alpha, \beta, \gamma, \delta)$.*

$$\tilde{\nabla}_{\Omega_1}\beta = -\alpha^2\Omega_1, \tag{4}$$

$$\left(\tilde{\nabla}_{\Omega_1}\gamma\right)\Omega_2 = -\delta(\Omega_1, \Omega_2) - \gamma(\Omega_1)\gamma(\Omega_2), \tag{5}$$

$$R(\Omega_1, \Omega_2)\beta = \gamma(\Omega_2)\Omega_1 - \gamma(\Omega_1)\Omega_2, \tag{6}$$

$$\gamma\left(R\left(\Omega_1,\Omega_2\right)\Omega_3\right)=\delta\left(\gamma\left(\Omega_1\right)\Omega_2-\gamma\left(\Omega_2\right)\Omega_1,\Omega_3\right), \tag{7}$$

$$S\left(\Omega_1,\beta\right)=(n-1)\,\gamma\left(\Omega_1\right), \tag{8}$$

where R and S are the Riemann curvature tensor and Ricci curvature tensor of $\Lambda^n(\alpha,\beta,\gamma,\delta)$, respectively.

In 2014, Shaik and Kundu defined and studied a type of tensor field, called generalized $\mathcal{B}-$curvature tensor on a Riemannian manifold [28]. It is a generalizations of quasi-conformal, Weyl-conformal, conharmonic and concircular curvature tensors and is given by

$$\begin{aligned}\mathcal{B}\left(\Omega_1,\Omega_2\right)\Omega_3 &= p_0R\left(\Omega_1,\Omega_2\right)\Omega_3+p_1\left[S\left(\Omega_2,\Omega_3\right)\Omega_1\right.\\ &\quad -S\left(\Omega_1,\Omega_3\right)\Omega_2+\delta\left(\Omega_2,\Omega_3\right)Q\Omega_1\\ &\quad \left.-\delta\left(\Omega_1,\Omega_3\right)Q\Omega_2\right]+2p_2r\left[\delta\left(\Omega_2,\Omega_3\right)\Omega_1\right.\\ &\quad \left.-\delta\left(\Omega_1,\Omega_3\right)\Omega_2\right].\end{aligned} \tag{9}$$

From (9) for n-dimensional $\mathcal{LPK}$-manifold, we easily to see that

$$\begin{aligned}\mathcal{B}\left(\Omega_1,\Omega_2\right)\beta &= \left[p_0+(n-1)\,p_1+2p_2r\right]\left[\gamma\left(\Omega_2\right)\Omega_1-\gamma\left(\Omega_1\right)\Omega_2\right]\\ &\quad +p_1\left[\gamma\left(\Omega_2\right)Q\Omega_1-\gamma\left(\Omega_1\right)Q\Omega_2\right],\end{aligned} \tag{10}$$

and

$$\begin{aligned}\gamma\left(\mathcal{B}\left(\Omega_1,\Omega_2\right)\Omega_3\right) &= \left[p_0+(n-1)\,p_1+2p_2r\right]\delta\left(\gamma\left(\Omega_1\right)\Omega_2-\gamma\left(\Omega_2\right)\Omega_1,\Omega_3\right)\\ &\quad +p_1S\left(\gamma\left(\Omega_1\right)\Omega_2-\gamma\left(\Omega_2\right)\Omega_1,\Omega_3\right).\end{aligned} \tag{11}$$

In particular, the $\mathcal{B}-$curvature tensor is reduced to

i. the quasi-conformal curvature tensor if taking

$$p_0=a, p_1=b \text{ and } p_2=-\frac{1}{2n}\left(\frac{a}{n-1}+2b\right).$$

ii. the Weyl-conformal curvature tensor if taking

$$p_0 = 1, p_1 = -\frac{1}{n-1} \text{ and } p_2 = -\frac{1}{2(n-1)(n-2)}.$$

iii. the concircular curvature tensor if taking

$$p_0 = 1, p_1 = 0 \text{ and } p_2 = -\frac{1}{n(n-1)}.$$

iv. the conharmonic curvature tensor if taking

$$p_0 = 1, p_1 = -\frac{1}{n-1} \text{ and } p_2 = 0.$$

Definition 1. *Let $\Lambda^n(\alpha,\beta,\gamma,\delta)$ be an n-dimensional $\mathcal{LPK}$-manifold. If $R\cdot S$ and $Q(\delta,S)$ are linearly dependent, then the $\Lambda^n(\alpha,\beta,\gamma,\delta)$ is said to be* ***Ricci pseudosymmetric****.*

In this case, there exists a function L on $\Lambda^n(\alpha,\beta,\gamma,\delta)$ such that

$$R\cdot S = LQ(\delta,S).$$

In particular, if $L = 0$, the manifold $\Lambda^n(\alpha,\beta,\gamma,\delta)$ is said to be **Ricci semisymmetric**.

Let us consider Ricci pseudosymmetric and Ricci semisymmetric manifolds according to generalized $\mathcal{B}$-curvature tensor on $\Lambda^n(\alpha,\beta,\gamma,\delta)$ by using different soliton equations. First, let us calculate the derivative $(L_\beta\delta)(\Omega_1,\Omega_2)$, which we will frequently use in soliton equations. Then we can write

$$(L_\beta\delta)(\Omega_1,\Omega_2) = L_\beta\delta(\Omega_1,\Omega_2) - \delta(L_\beta\Omega_1,\Omega_2) - \delta(\Omega_1, L_\beta\Omega_2)$$

$$= \beta[\delta(\Omega_1,\Omega_2)] - \delta([\beta,\Omega_1],\Omega_2) - \delta(\Omega_1,[\beta,\Omega_2])$$

$$= \delta(\nabla_\beta\Omega_1,\Omega_2) + \delta(\Omega_1,\nabla_\beta\Omega_2) - \delta(\nabla_\beta\Omega_1,\Omega_2)$$

$$+\delta(\nabla_{\Omega_1}\beta,\Omega_2) - \delta(\nabla_\beta\Omega_2,\Omega_1) + \delta(\Omega_1,\nabla_{\Omega_2}\beta),$$

and if we use (4) in here, then we get

$$(L_\beta\delta)(\Omega_1,\Omega_2) = -2[\delta(\Omega_1,\Omega_2) + \gamma(\Omega_1)\gamma(\Omega_2)] \tag{12}$$

for all $\Omega_1,\Omega_2 \in \Gamma(T\Lambda)$.

3 Almost η-Ricci Solitons by via of Generalized $\mathcal{B}$-Curvature Tensor

A η-Ricci soliton on a $\mathcal{LPK}$-manifold $\Lambda^n(\alpha,\beta,\gamma,\delta)$ is defined as a quadruple $(\delta,\beta,\lambda,\mu)$ on $\Lambda^n(\alpha,\beta,\gamma,\delta)$ satisfying

$$\frac{1}{2}L_\beta\delta + S + \lambda\delta + \mu\gamma\otimes\gamma = 0, \tag{13}$$

where λ and μ are real constants, γ is the dual of the vector field β and S is the Ricci curvature tensor of $\Lambda^n(\alpha,\beta,\gamma,\delta)$. if λ and μ are smooth functions on $\Lambda^n(\alpha,\beta,\gamma,\delta)$, then it called almost η-Ricci soliton on $\Lambda^n(\alpha,\beta,\gamma,\delta)$. An almost η-Ricci soliton $(\delta,\beta,\lambda,\mu)$ is called steady if $\lambda = 0$; if shrinking $\lambda < 0$ and if expanding $\lambda > 0$.

Now let $(\delta,\beta,\lambda,\mu)$ be almost η-Ricci soliton on $\mathcal{LPK}$-manifold $\Lambda^n(\alpha,\beta,\gamma,\delta)$. Then we have from (12) and (13)

$$S(\Omega_1,\Omega_2) = (1-\lambda)\delta(\Omega_1,\Omega_2) + (1-\mu)\gamma(\Omega_1)\gamma(\Omega_2). \tag{14}$$

Thus, we can give the following theorem from (14).

Theorem 1. *The n-dimensional $\mathcal{LPK}$-manifold $\Lambda^n(\alpha,\beta,\gamma,\delta)$ admitting the almost η-Ricci soliton $(\delta,\beta,\lambda,\mu)$ is an η-Einstein manifold.*

If we choose $\Omega_2=\beta$ in (14), we get

$$S(\Omega_1,\beta)=(\mu-\lambda)\gamma(\Omega_1). \tag{15}$$

It is also clear from (8) and (15), one can easily to see that

$$\mu-\lambda=n-1. \tag{16}$$

Theorem 2. *Let $\Lambda^n(\alpha,\beta,\gamma,\delta)$ be an n-dimensional $\mathcal{LPK}$-manifold and $(\delta,\beta,\lambda,\mu)$ be an almost η-Ricci soliton on $\Lambda^n(\alpha,\beta,\gamma,\delta)$. If $\Lambda^n(\alpha,\beta,\gamma,\delta)$ is the generalized $\mathcal{B}$-Ricci pseudosymmetric manifold, then we have*

$$L_1=p_0+(n-\lambda)p_1+2rp_2.$$

Proof. Let us assume that n-dimensional $\mathcal{LPK}$-manifold $\Lambda^n(\alpha,\beta,\gamma,\delta)$ be generalized $\mathcal{B}$-Ricci pseudosymmetric and $(\delta,\beta,\lambda,\mu)$ be almost η-Ricci soliton on $\Lambda^n(\alpha,\beta,\gamma,\delta)$. That means

$$(\mathcal{B}(\Omega_1,\Omega_2)\cdot S)(\Omega_4,\Omega_5)=L_1Q(\delta,S)(\Omega_4,\Omega_5;\Omega_1,\Omega_2),$$

for all $\Omega_1,\Omega_2,\Omega_4,\Omega_5\in\Gamma(T\Lambda)$. From the last equation, we can easily write

$$\begin{aligned}&S(\mathcal{B}(\Omega_1,\Omega_2)\Omega_4,\Omega_5)+S(\Omega_4,\mathcal{B}(\Omega_1,\Omega_2)\Omega_5)\\&=L_1\{S((\Omega_1\wedge_\delta\Omega_2)\Omega_4,\Omega_5)+S(\Omega_4,(\Omega_1\wedge_\delta\Omega_2)\Omega_5)\}.\end{aligned} \tag{17}$$

If we choose $\Omega_5=\beta$ in (17), we get

$$\begin{aligned}&S(\mathcal{B}(\Omega_1,\Omega_2)\Omega_4,\beta)+S(\Omega_4,\mathcal{B}(\Omega_1,\Omega_2)\beta)\\&=L_1\{S(\delta(\Omega_2,\Omega_4)\Omega_1-\delta(\Omega_1,\Omega_4)\Omega_2,\beta)\\&+S(\Omega_4,\gamma(\Omega_2)\Omega_1-\gamma(\Omega_1)\Omega_2)\}.\end{aligned}$$

If we make use of $(2), (8), (10)$ and (11) in the last equality, we have

$$\begin{aligned}&(n-1)\left[p_0+(n-1)p_1+2rp_2\right]\delta\left(\gamma\left(\Omega_1\right)\Omega_2-\gamma\left(\Omega_2\right)\Omega_1,\Omega_4\right)\\&-\left[p_0+2rp_2\right]S\left(\gamma\left(\Omega_1\right)\Omega_2-\gamma\left(\Omega_2\right)\Omega_1,\Omega_4\right)\\&+p_1S\left(\gamma\left(\Omega_2\right)Q\Omega_1-\gamma\left(\Omega_1\right)Q\Omega_2,\Omega_4\right)\\&=L_1\left\{(n-1)\delta\left(\gamma\left(\Omega_1\right)\Omega_2-\gamma\left(\Omega_2\right)\Omega_1,\Omega_4\right)\right.\\&\left.+S\left(\gamma\left(\Omega_2\right)\Omega_1-\gamma\left(\Omega_1\right)\Omega_2,\Omega_4\right)\right\}.\end{aligned} \tag{18}$$

If we use (18) in the (14), we get

$$\begin{aligned}&\Big[(n-2)p_0+(n-1)^2p_1+2(n-2)rp_2\\&+\left(p_0+2rp_2\right)\lambda\Big]\delta\left(\gamma\left(\Omega_1\right)\Omega_2-\gamma\left(\Omega_2\right)\Omega_1,\Omega_4\right)\\&+(1-\lambda)p_1S\left(\gamma\left(\Omega_2\right)\Omega_1-\gamma\left(\Omega_1\right)\Omega_2,\Omega_4\right)\\&=L_1(n+\lambda-2)\delta\left(\gamma\left(\Omega_1\right)\Omega_2-\gamma\left(\Omega_2\right)\Omega_1,\Omega_4\right).\end{aligned} \tag{19}$$

If we use (14) in (19), we can infer

$$\begin{aligned}&\left[(n+\lambda-2)p_0+(n+\lambda-2)(n-\lambda)p_1+2r(n+\lambda-2)p_2\right.\\&\left.-L_1(n+\lambda-2)\right]\delta\left(\gamma\left(\Omega_2\right)\Omega_1-\gamma\left(\Omega_1\right)\Omega_2,\Omega_4\right)=0.\end{aligned}$$

This completes the proof. □

We can give the following corollaries as follows.

Corollary 1. *Let $\Lambda^n(\alpha,\beta,\gamma,\delta)$ be an n-dimensional $\mathcal{LPK}$-manifold and $(\delta,\beta,\lambda,\mu)$ be an almost η-Ricci soliton on $\Lambda^n(\alpha,\beta,\gamma,\delta)$. If $\Lambda^n(\alpha,\beta,\gamma,\delta)$ is* *the generalized $\mathcal{B}$-Ricci pseudosymmetric manifold, the roots of characteristic polynomial according to λ given*

$$p_1\lambda^2 - (p_0 - 2p_{12} - 2rp_2)\,\lambda + (n-2)\,(p_0 + np_1 + 2rp_2) = 0$$

characterize the almost η-Ricci soliton.

Corollary 2. *Let $\Lambda^n(\alpha,\beta,\gamma,\delta)$ be an n-dimensional $\mathcal{LPK}$-manifold and $(\delta,\beta,\lambda,\mu)$ be an almost η-Ricci soliton on $\Lambda^n(\alpha,\beta,\gamma,\delta)$. If $\Lambda^n(\alpha,\beta,\gamma,\delta)$ is the quasi–conformal Ricci pseudosymmetric manifold, then we have*

$$L_1 = \left(1 - \frac{r}{n-1}\right)a + (n - \lambda - 2r)\,b.$$

Corollary 3. *Let $\Lambda^n(\alpha,\beta,\gamma,\delta)$ be an n-dimensional $\mathcal{LPK}$-manifold and $(\delta,\beta,\lambda,\mu)$ be an almost γ-Ricci soliton on $\Lambda^n(\alpha,\beta,\gamma,\delta)$. If $\Lambda^n(\alpha,\beta,\gamma,\delta)$ is the Weyl-conformal Ricci pseudosymmetric manifold, then we have*

$$L_1 = \frac{(n-2)\,(\lambda-1) - r}{(n-1)\,(n-2)}.$$

Corollary 4. *Let $\Lambda^n(\alpha,\beta,\gamma,\delta)$ be an n-dimensional $\mathcal{LPK}$-manifold and $(\delta,\beta,\lambda,\mu)$ be an almost γ-Ricci soliton on $\Lambda^n(\alpha,\beta,\gamma,\delta)$. If $\Lambda^n(\alpha,\beta,\gamma,\delta)$ is the concircular Ricci pseudosymmetric manifold, then we have*

$$L_1 = \frac{n\,(n-1) - 2r}{n\,(n-1)}.$$

Corollary 5. *Let $\Lambda^n(\alpha,\beta,\gamma,\delta)$ be an n-dimensional $\mathcal{LPK}$-manifold and $(\delta,\beta,\lambda,\mu)$ be an almost γ-Ricci soliton on $\Lambda^n(\alpha,\beta,\gamma,\delta)$. If $\Lambda^n(\alpha,\beta,\gamma,\delta)$ is the conharmonic Ricci pseudosymmetric manifold, then we have*

$$L_1 = \frac{\lambda-1}{n-1}.$$

4 Conformal Ricci Solitons on the Generalized $\mathcal{B}$-Curvature Tensor

A conformal Ricci soliton on an $\mathcal{LPK}$-manifold $\Lambda^n(\alpha,\beta,\gamma,\delta)$ is defined as a triple (δ,β,λ) satisfying

$$L_\beta\delta + 2S = \left[2\lambda - \left(\Theta + \frac{2}{n}\right)\right]\delta, \tag{20}$$

where λ is real constant and Θ is the conformal pressure. An conformal Ricci soliton (δ,β,λ) is called steady if $\lambda = 0$; if shrinking $\lambda < 0$ and if expanding $\lambda > 0$.

Now let (δ,β,λ) be conformal Ricci soliton on $\mathcal{LPK}$-manifold $\Lambda^n(\alpha,\beta,\gamma,\delta)$. From (12) and (20), then we have

$$S(\Omega_1,\Omega_2) = \left[\lambda - \left(\frac{\Theta}{2} + \frac{1}{n}\right) + 1\right]\delta(\Omega_1,\Omega_2) + \gamma(\Omega_1)\gamma(\Omega_2). \tag{21}$$

From (21), we have the following theorem.

Theorem 3. *The n-dimensional $\mathcal{LPK}$-manifold $\Lambda^n(\alpha,\beta,\gamma,\delta)$ admitting the conformal Ricci soliton (δ,β,λ) is an γ-Einstein manifold.*

If we choose $\Omega_2 = \beta$ in (21), we get

$$S(\Omega_1,\beta) = \lambda - \left(\frac{\Theta}{2} + \frac{1}{n}\right). \tag{22}$$

From (8) and (22), we conclude that

$$\lambda = \frac{1}{2}\Theta + \frac{n(n-1)+1}{n}. \tag{23}$$

Theorem 4. *Let $\Lambda^n(\alpha,\beta,\gamma,\delta)$ be an n-dimensional $\mathcal{LPK}$-manifold and (δ,β,λ) be an conformal Ricci soliton on $\Lambda^n(\alpha,\beta,\gamma,\delta)$. If $\Lambda^n(\alpha,\beta,\gamma,\delta)$ is the generalized $\mathcal{B}$-Ricci pseudosymmetric manifold, then we have*

$$L_2 = p_0 + \left[(n+\lambda) - \left(\frac{\Theta}{2} + \frac{1}{n}\right)\right]p_1 + 2rp_2.$$

Proof. Let us assume n-dimensional $\mathcal{LPK}$-manifold $\Lambda^n(\alpha,\beta,\gamma,\delta)$ be generalized $\mathcal{B}$-Ricci pseudosymmetric and (δ,β,λ) be conformal Ricci soliton on $\Lambda^n(\alpha,\beta,\gamma,\delta)$. That means

$$(\mathcal{B}(\Omega_1,\Omega_2)\cdot S)(\Omega_4,\Omega_5)=L_2Q(\delta,S)(\Omega_4,\Omega_5;\Omega_1,\Omega_2),$$

for all $\Omega_1,\Omega_2,\Omega_4,\Omega_5\in\Gamma(T\Lambda)$. The last equation is of

$$\begin{aligned}&S(\mathcal{B}(\Omega_1,\Omega_2)\Omega_4,\Omega_5)+S(\Omega_4,\mathcal{B}(\Omega_1,\Omega_2)\Omega_5)\\&=L_2\{S((\Omega_1\wedge_\delta\Omega_2)\Omega_4,\Omega_5)+S(\Omega_4,(\Omega_1\wedge_\delta\Omega_2)\Omega_5)\}.\end{aligned}\tag{24}$$

If we choose $\Omega_5=\beta$ in (24), we get

$$\begin{aligned}&S(\mathcal{B}(\Omega_1,\Omega_2)\Omega_4,\beta)+S(\Omega_4,\mathcal{B}(\Omega_1,\Omega_2)\beta)\\&=L_2\{S(\delta(\Omega_2,\Omega_4)\Omega_1-\delta(\Omega_1,\Omega_4)\Omega_2,\beta)\\&+S(\Omega_4,\gamma(\Omega_2)\Omega_1-\gamma(\Omega_1)\Omega_2)\}.\end{aligned}$$

If we make use of $(2),(8),(10)$ and (11) in the last equality, we have

$$\begin{aligned}&(n-1)[p_0+(n-1)p_1+2rp_2]\delta(\gamma(\Omega_1)\Omega_2-\gamma(\Omega_2)\Omega_1,\Omega_4)\\&-[p_0+2rp_2]S(\gamma(\Omega_1)\Omega_2-\gamma(\Omega_2)\Omega_1,\Omega_4)\\&+p_1S(\gamma(\Omega_2)Q\Omega_1-\gamma(\Omega_1)Q\Omega_2,\Omega_4)\\&=L_2\{(n-1)\delta(\gamma(\Omega_1)\Omega_2-\gamma(\Omega_2)\Omega_1,\Omega_4)\\&+S(\gamma(\Omega_2)\Omega_1-\gamma(\Omega_1)\Omega_2,\Omega_4)\}.\end{aligned}\tag{25}$$

If we use (21) in the (25), we get

$$\begin{aligned}
&\left\{\left[(n-\lambda-2)+\tfrac{\Theta}{2}+\tfrac{1}{n}\right]p_0+\left[2(n-\lambda-2)+\Theta+\tfrac{2}{n}\right]rp_2\right.\\
&\left.+(n-1)^2p_1\right\}\delta\left(\gamma(\Omega_1)\Omega_2-\gamma(\Omega_2)\Omega_1,\Omega_4\right)\\
&+p_1\left[\lambda-\left(\tfrac{\Theta}{2}+\tfrac{1}{n}\right)+1\right]S\left(\gamma(\Omega_2)\Omega_1-\gamma(\Omega_1)\Omega_2,\Omega_4\right)\\
&=L_2\left[(n-\lambda-2)+\tfrac{\Theta}{2}+\tfrac{1}{n}\right]\delta\left(\gamma(\Omega_1)\Omega_2-\gamma(\Omega_2)\Omega_1,\Omega_4\right).
\end{aligned}\tag{26}$$

By using (21) in (26), we can drive

$$\begin{aligned}
&\left\{\left[(n-\lambda-2)+\tfrac{\Theta}{2}+\tfrac{1}{n}\right]p_0+2r\left[(n-\lambda-2)+\tfrac{\Theta}{2}+\tfrac{1}{n}\right]p_2\right.\\
&+\left[(n-\lambda-2)+\tfrac{\Theta}{2}+\tfrac{1}{n}\right]\left[(n+\lambda)-\left(\tfrac{\Theta}{2}+\tfrac{1}{n}\right)\right]p_1\\
&\left.-L_2\left[(n-\lambda-2)+\tfrac{\Theta}{2}+\tfrac{1}{n}\right]\right\}\delta\left(\gamma(\Omega_1)\Omega_2-\gamma(\Omega_2)\Omega_1,\Omega_4\right)=0.
\end{aligned}$$

This completes the proof. □

Thus we have the following corollaries.

Corollary 6. *Let $\Lambda^n(\alpha,\beta,\gamma,\delta)$ be an n-dimensional $\mathcal{LPK}$-manifold and (δ,β,λ) be an conformal Ricci soliton on $\Lambda^n(\alpha,\beta,\gamma,\delta)$. If $\Lambda^n(\alpha,\beta,\gamma,\delta)$ is the quasi–conformal Ricci pseudosymmetric manifold, then we have*

$$L_2=\left[\frac{n(n-1)-r}{n(n-1)}\right]a+\left[(n+\lambda)-\frac{1}{n}(1+2r)-\frac{\Theta}{2}\right]b.$$

Corollary 7. *Let $\Lambda^n(\alpha,\beta,\gamma,\delta)$ be an n-dimensional $\mathcal{LPK}$-manifold and (δ,β,λ) be an conformal Ricci soliton on $\Lambda^n(\alpha,\beta,\gamma,\delta)$. If $\Lambda^n(\alpha,\beta,\gamma,\delta)$ is the Weyl-conformal Ricci pseudosymmetric manifold, then we have*

$$L_2=\frac{(n-1)(n-2)-(n-2)\left[2n(n+\lambda)-(n\Theta+2)\right]-r}{(n-1)(n-2)}.$$

Corollary 8. *Let $\Lambda^n(\alpha,\beta,\gamma,\delta)$ be an n-dimensional $\mathcal{LPK}$-manifold and (δ,β,λ) be an conformal Ricci soliton on $\Lambda^n(\alpha,\beta,\gamma,\delta)$. If $\Lambda^n(\alpha,\beta,\gamma,\delta)$ is the concircular Ricci pseudosymmetric manifold, then we have*

$$L_2 = \frac{n(n-1) - 2r}{n(n-1)}.$$

Corollary 9. *Let $\Lambda^n(\alpha,\beta,\gamma,\delta)$ be an n-dimensional $\mathcal{LPK}$-manifold and (δ,β,λ) be an conformal Ricci soliton on $\Lambda^n(\alpha,\beta,\gamma,\delta)$. If $\Lambda^n(\alpha,\beta,\gamma,\delta)$ is the conharmonic Ricci pseudosymmetric manifold, then we have*

$$L_2 = \frac{2n(n-1) - 2n(n+\lambda) + (\Theta n + 2)}{2n(n-1)}.$$

5 η-Ricci-Bourguignon Solitons by means of Generalized $\mathcal{B}$-Curvature Tensor

An η-Ricci-Bourguignon soliton on a $\mathcal{LPK}$-manifold $\Lambda^n(\alpha,\beta,\gamma,\delta)$ is defined as a quadruple $(\delta,\beta,\lambda,\mu)$ on $\Lambda^n(\alpha,\beta,\gamma,\delta)$ satisfying

$$L_\beta\delta + 2S + 2(\lambda - \sigma r)\delta + 2\mu\gamma\otimes\gamma = 0, \tag{27}$$

where λ and μ are real constants, σ is a non-zero constant, r is the scalar curvature of manifold and S is the Ricci curvature tensor of $\Lambda^n(\alpha,\beta,\gamma,\delta)$. An η-Ricci-Bourguignon soliton $(\delta,\beta,\lambda,\mu)$ is called steady if $\lambda = 0$, if shrinking $\lambda < 0$, and if expanding $\lambda > 0$.

Now let $(\delta,\beta,\lambda,\mu)$ be η-Ricci-Bourguignon soliton on $\mathcal{LPK}$-manifold $\Lambda^n(\alpha,\beta,\gamma,\delta)$. From (12) and (27), then we have

$$S(\Omega_1,\Omega_2) = (1 + \sigma r - \lambda)\delta(\Omega_1,\Omega_2) + (1-\mu)\gamma(\Omega_1)\gamma(\Omega_2). \tag{28}$$

Thus, we can easily obtain the following theorem from (28).

Theorem 5. *The n-dimensional $\mathcal{LPK}$-manifold $\Lambda^n(\alpha,\beta,\gamma,\delta)$ admitting the η-Ricci-Bourguignon soliton $(\delta,\beta,\lambda,\mu)$ is an η-Einstein manifold.*

If we choose $\Omega_2 = \beta$ in (28), we get

$$S(\Omega_1, \beta) = (\sigma r + \mu - \lambda)\, \gamma(\Omega_1). \tag{29}$$

From (8) and (29), then we give us

$$\sigma r + \mu - \lambda = n - 1. \tag{30}$$

Theorem 6. *Let $\Lambda^n(\alpha, \beta, \gamma, \delta)$ be an n-dimensional $\mathcal{LPK}$-manifold and $(\delta, \beta, \lambda, \mu)$ be an η-Ricci-Bourguignon soliton on $\Lambda^n(\alpha, \beta, \gamma, \delta)$. If $\Lambda^n(\alpha, \beta, \gamma, \delta)$ is the generalized $\mathcal{B}$-Ricci pseudosymmetric manifold, then we have*

$$L_3 = p_0 + (n - \lambda + \sigma r)\, p_1 + 2rp_2.$$

Proof. Let us assume n-dimensional $\mathcal{LPK}$-manifold $\Lambda^n(\alpha, \beta, \gamma, \delta)$ be generalized $\mathcal{B}$-Ricci pseudosymmetric and $(\delta, \beta, \lambda, \mu)$ be η-Ricci-Bourguignon soliton on $\Lambda^n(\alpha, \beta, \gamma, \delta)$. That means

$$(\mathcal{B}(\Omega_1, \Omega_2) \cdot S)(\Omega_4, \Omega_5) = L_3 Q(\delta, S)(\Omega_4, \Omega_5; \Omega_1, \Omega_2),$$

for all $\Omega_1, \Omega_2, \Omega_4, \Omega_5 \in \Gamma(T\Lambda)$. From the last equation, we can easily write

$$\begin{aligned} & S(\mathcal{B}(\Omega_1, \Omega_2)\Omega_4, \Omega_5) + S(\Omega_4, \mathcal{B}(\Omega_1, \Omega_2)\Omega_5) \\ &= L_3 \{S((\Omega_1 \wedge_\delta \Omega_2)\Omega_4, \Omega_5) + S(\Omega_4, (\Omega_1 \wedge_\delta \Omega_2)\Omega_5)\}. \end{aligned} \tag{31}$$

If we choose $\Omega_5 = \beta$ in (31), we get

$$\begin{aligned} & S(\mathcal{B}(\Omega_1, \Omega_2)\Omega_4, \beta) + S(\Omega_4, \mathcal{B}(\Omega_1, \Omega_2)\beta) \\ &= L_3 \{S(\delta(\Omega_2, \Omega_4)\Omega_1 - \delta(\Omega_1, \Omega_4)\Omega_2, \beta) \\ &+ S(\Omega_4, \gamma(\Omega_2)\Omega_1 - \gamma(\Omega_1)\Omega_2)\}. \end{aligned}$$

Making use of (8), (10) and (11) in the last equality, we have

$$\begin{aligned}
&(n-1)\left[p_0+(n-1)\,p_1+2rp_2\right]\delta\left(\gamma\left(\Omega_1\right)\Omega_2-\gamma\left(\Omega_2\right)\Omega_1,\Omega_4\right)\\
&-\left(p_0+2rp_2\right)S\left(\gamma\left(\Omega_1\right)\Omega_2-\gamma\left(\Omega_2\right)\Omega_1,\Omega_4\right)\\
&+p_1S\left(\gamma\left(\Omega_2\right)Q\Omega_1-\gamma\left(\Omega_1\right)Q\Omega_2,\Omega_4\right) \qquad (32)\\
&=L_3\left\{(n-1)\,\delta\left(\gamma\left(\Omega_1\right)\Omega_2-\gamma\left(\Omega_2\right)\Omega_1,\Omega_4\right)\right.\\
&\left.+\,S\left(\gamma\left(\Omega_2\right)\Omega_1-\gamma\left(\Omega_1\right)\Omega_2,\Omega_4\right)\right\}.
\end{aligned}$$

If we use (28) in (32), we get

$$\begin{aligned}
&\left[(n+\lambda-\sigma r-2)\,p_0+(n+\lambda-\sigma r-2)\,(n-\lambda+\sigma r)\,p_1+2r\,(n+\lambda-\sigma r-2)\,p_2\right.\\
&\left.-L_3\,(n+\lambda-\sigma r-2)\right]\delta\left(\gamma\left(\Omega_2\right)\Omega_1-\gamma\left(\Omega_1\right)\Omega_2,\Omega_4\right)=0.
\end{aligned}$$

This completes the proof. □

Thus we have the following corollaries.

Corollary 10. *Let $\Lambda^n(\alpha,\beta,\gamma,\delta)$ be an n-dimensional $\mathcal{LPK}$-manifold and $(\delta,\beta,\lambda,\mu)$ be an η-Ricci-Bourguignon soliton on $\Lambda^n(\alpha,\beta,\gamma,\delta)$. If $\Lambda^n(\alpha,\beta,\gamma,\delta)$ is the quasi-conformal Ricci pseudosymmetric manifold, then we have*

$$L_3=a+(n-\lambda+\sigma r)\,b-\frac{r}{n}\left(\frac{a}{n-1}+2b\right).$$

Corollary 11. *Let $\Lambda^n(\alpha,\beta,\gamma,\delta)$ be an n-dimensional $\mathcal{LPK}$-manifold and $(\delta,\beta,\lambda,\mu)$ be an η-Ricci-Bourguignon soliton on $\Lambda^n(\alpha,\beta,\gamma,\delta)$. If $\Lambda^n(\alpha,\beta,\gamma,\delta)$ is the Weyl-conformal Ricci pseudosymmetric manifold, then we have*

$$L_3=1-\frac{1}{n-1}\,(n-\lambda+\sigma r)-\frac{r}{(n-1)\,(n-2)}.$$

Corollary 12. *Let $\Lambda^n(\alpha,\beta,\gamma,\delta)$ be an n-dimensional $\mathcal{LPK}$-manifold and $(\delta,\beta,\lambda,\mu)$ be an η-Ricci-Bourguignon soliton on $\Lambda^n(\alpha,\beta,\gamma,\delta)$. If $\Lambda^n(\alpha,\beta,\gamma,\delta)$ is the concircular Ricci pseudosymmetric manifold, then we have*

$$L_3 = -\frac{n(n-1)+2r}{n(n-1)}.$$

Corollary 13. *Let $\Lambda^n(\alpha,\beta,\gamma,\delta)$ be an n-dimensional $\mathcal{LPK}$-manifold and $(\delta,\beta,\lambda,\mu)$ be an η-Ricci-Bourguignon soliton on $\Lambda^n(\alpha,\beta,\gamma,\delta)$. If $\Lambda^n(\alpha,\beta,\gamma,\delta)$ is the conharmonic Ricci pseudosymmetric manifold, then we have*

$$L_3 = 1 - \frac{1}{n-1}(n-\lambda+\sigma r).$$

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