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Differential Sandwich Theorems for a Certain Class of Analytic Functions Defined by Differential Operator

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Abstract

In this paper, we establish some applications of first order differential subordination and superordination results involving Hadamard product for a certain class of analytic functions with differential operator defined in the open unit disk. These results are applied to obtain sandwich results.

1. Introduction and Preliminaries

Let $\mathcal{H}$ indicate the family of analytic functions in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ and let $\mathcal{H}[a, p]$ be the subclass of $\mathcal{H}$ consisting of functions of the form:

$$f(z) = a + a_p z^p + a_{p+1} z^{p+1} + \cdots, \quad (a \in \mathbb{C},\ p \in \mathbb{N} = \{1, 2, \ldots\}).$$

Also, let $\mathcal{A}$ denote the subclass of $\mathcal{H}$ consisting of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

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Let $f, g \in \mathcal{H}$. The function f is said to be subordinate to g, or g is said to be superordinate to f, if there exists a Schwarz function w analytic in U with $w(0) = 0$ and $|w(z)| < 1$ $(z \in U)$ such that $f(z) = g(w(z))$. This subordination is denoted by $f \prec g$ or $f(z) \prec g(z)$ $(z \in U)$. It is well known that, if the function g is univalent in U, we have the following equivalence (see [12]):

$$f \prec g \ (z \in U) \Leftrightarrow f(0) = g(0), \quad f(U) \subset g(U).$$

Let $k, h \in \mathcal{H}$ and $\psi(r, s, t; z) : \mathbb{C}^3 \times U \to \mathbb{C}$. If k and $\psi(k(z), zk'(z), z^2k''(z); z)$ are univalent functions in U and if k satisfies the second-order differential superordination:

$$h(z) \prec \psi(k(z), zk'(z), z^2k''(z); z), \tag{1.2}$$

then k is called a solution of the differential superordination (1.2). (If f is subordinate to g, then g is superordinate to f). An analytic function q is called a subordinant of (1.2), if $q \prec k$ for all k satisfying (1.2). A univalent subordinant $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all the subordinants q of (1.2) is called the best subordinant.

For the functions $f \in \mathcal{A}$ given by (1.1) and $g \in \mathcal{A}$ defined by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n,$$

we define the Hadamard product (or convolution) $f * g$ of the functions f and g (as usual) by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_n b_n z^n = (g * f)(z).$$

For $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\rho, \lambda \geq 0$, $\mu, \nu, \delta > 0$ and $\rho \neq \nu$, we consider the differential operator $A^m_{\mu, \nu, \lambda}(\rho, \delta) : \mathcal{A} \to \mathcal{A}$, introduced by Amourah and Darus [2], where

$$A^m_{\mu, \nu, \lambda}(\rho, \delta) f(z) = z + \sum_{n=2}^{\infty} \left[1 + \frac{(n-1)[(\nu - \rho)\delta + n\lambda]}{\mu + \nu}\right]^m a_n z^n. \tag{1.3}$$

It is readily verified from (1.3) that

$$z(A^{m}_{\mu,\,\nu,\,\lambda}(\rho,\,\delta)\,f(z))' = \frac{\mu+\nu}{(\nu-\rho)\delta+n\lambda}\,A^{m+1}_{\mu,\,\nu,\,\lambda}(\rho,\,\delta)\,f(z) - \left(1-\frac{\mu+\nu}{(\nu-\rho)\delta+n\lambda}\right)A^{m}_{\mu,\,\nu,\,\lambda}(\rho,\,\delta)\,f(z). \qquad (1.4)$$

Would like to point out here that some of the special cases of the operator defined by (1.3) can be found in [1, 4, 11, 13].

Recently several authors, Goyal et al. [5], Murugusundaramoorthy and Magesh [9, 10], Magesh et al. [7], Ibrahim and Darus [6], Wanas [14, 15], Wanas and Joudah [16] and Wanas and Majeed [17] have obtained sandwich results for certain classes of analytic functions.

The main object of the present investigation is to find sufficient condition for certain normalized analytic functions f in U such that $(f*\Psi)(z)\neq 0$ and f to satisfy

$$q_1(z) \prec \left(\frac{A^{m+1}_{\mu,\,\nu,\,\lambda}(\rho,\,\delta)(f*\Phi)(z)}{A^{m}_{\mu,\,\nu,\,\lambda}(\rho,\,\delta)(f*\Psi)(z)}\right)^{\gamma} \prec q_2(z),$$

where q_1 and q_2 are given univalent functions in U with $q_1(0)=q_2(0)=1$ and $\Phi(z)=z+\sum_{n=2}^{\infty} r_n z^n$, $\Psi(z)=z+\sum_{n=2}^{\infty} e_n z^n$ are analytic functions in U with $r_n\geq 0$, $e_n\geq 0$.

To establish our main results, we need the following definition and lemmas.

Definition 1.1 [8]**.** Denote by Q the set of all functions f that are analytic and injective on $\overline{U}\setminus E(f)$, where

$$E(f)=\left\{\zeta\in\partial U : \lim_{z\to\zeta} f(z)=\infty\right\}$$

and are such that $f'(\zeta)\neq 0$ for $\zeta\in\partial U\setminus E(f)$.

Lemma 1.1 [8]**.** *Let q be univalent in the unit disk U and let θ and ϕ be analytic in a domain D containing $q(U)$ with $\phi(w)\neq 0$ when $w\in q(U)$. Set $Q(z)=zq'(z)\phi(q(z))$ and $h(z)=\theta(q(z))+Q(z)$. Suppose that*

(1) $Q(z)$ *is starlike univalent in U*,

(2) $\operatorname{Re}\left\{\frac{zh'(z)}{Q(z)}\right\} > 0$ *for* $z \in U$.

If k is analytic in U, with $k(0) = q(0)$, $k(U) \subset D$ *and*

$$\theta(k(z)) + zk'(z)\phi(k(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)), \tag{1.5}$$

then $k \prec q$ *and q is the best dominant of* (1.5).

Lemma 1.2 [3]**.** *Let q be convex univalent in the unit disk U and let* θ *and* ϕ *be analytic in a domain D containing* $q(U)$. *Suppose that*

(1) $\operatorname{Re}\left\{\frac{\theta'(q(z))}{\phi(q(z))}\right\} > 0$ *for* $z \in U$,

(2) $Q(z) = zq'(z)\phi(q(z))$ *is starlike univalent in U.*

If $k \in \mathcal{H}[q(0), 1] \cap Q$, *with* $k(U) \subset D$, $\theta(k(z)) + zk'(z)\phi(k(z))$ *is univalent in U and*

$$\theta(q(z)) + zq'(z)\phi(q(z)) \prec \theta(k(z)) + zk'(z)\phi(k(z)), \tag{1.6}$$

then $q \prec k$ *and q is the best subordinant of* (1.6).

2. Main Results

Theorem 2.1. *Let* $\Phi, \Psi \in \mathcal{A}$, $\alpha, \beta, \tau \in \mathbb{C}$, $\eta, \gamma \in \mathbb{C} \setminus \{0\}$ *and let q be convex univalent in U with* $q(0) = 1$ *and assume that q satisfies*:

$$\operatorname{Re}\left\{1 + \frac{\alpha\tau}{\eta} + \frac{\beta(\tau+1)}{\eta}q(z) + (\tau - 1)\frac{zq'(z)}{q(z)} + \frac{zq''(z)}{q'(z)}\right\} > 0. \tag{2.1}$$

Suppose that $z(q(z))^{\tau-1}q'(z)$ *is starlike univalent in U. If* $f \in \mathcal{A}_p$ *satisfies the differential subordination*:

$$\varphi_1(f, \Phi, \Psi, \alpha, \beta, \tau, \eta, \gamma, \rho, \delta, \mu, \nu, \lambda, m; z)$$

$$\prec (\alpha + \beta q(z))(q(z))^{\tau} + \eta z(q(z))^{\tau-1}q'(z), \tag{2.2}$$

where

$$\varphi_1(f, \Phi, \Psi, \alpha, \beta, \tau, \eta, \gamma, \rho, \delta, \mu, \nu, \lambda, m; z)$$

$$= \alpha\left(\frac{A^{m+1}_{\mu,\nu,\lambda}(\rho, \delta)(f * \Phi)(z)}{A^{m}_{\mu,\nu,\lambda}(\rho, \delta)(f * \Psi)(z)}\right)^{\gamma\tau} + \beta\left(\frac{A^{m+1}_{\mu,\nu,\lambda}(\rho, \delta)(f * \Phi)(z)}{A^{m}_{\mu,\nu,\lambda}(\rho, \delta)(f * \Psi)(z)}\right)^{\gamma(\tau+1)}$$

$$+ \frac{\gamma\eta(\mu+\nu)}{(\nu-\rho)\delta + n\lambda}\left(\frac{A^{m+1}_{\mu,\nu,\lambda}(\rho, \delta)(f * \Phi)(z)}{A^{m}_{\mu,\nu,\lambda}(\rho, \delta)(f * \Psi)(z)}\right)^{\gamma\tau}\left(\frac{A^{m+2}_{\mu,\nu,\lambda}(\rho, \delta)(f * \Phi)(z)}{A^{m+1}_{\mu,\nu,\lambda}(\rho, \delta)(f * \Phi)(z)}\right.$$

$$\left. - \frac{A^{m+1}_{\mu,\nu,\lambda}(\rho, \delta)(f * \Psi)(z)}{A^{m}_{\mu,\nu,\lambda}(\rho, \delta)(f * \Psi)(z)}\right), \tag{2.3}$$

then

$$\left(\frac{A^{m+1}_{\mu,\nu,\lambda}(\rho, \delta)(f * \Phi)(z)}{A^{m}_{\mu,\nu,\lambda}(\rho, \delta)(f * \Psi)(z)}\right)^{\gamma} \prec q(z)$$

and q is the best dominant of (2.2).

Proof. Let the function k be defined by

$$k(z) = \left(\frac{A^{m+1}_{\mu,\nu,\lambda}(\rho, \delta)(f * \Phi)(z)}{A^{m}_{\mu,\nu,\lambda}(\rho, \delta)(f * \Psi)(z)}\right)^{\gamma}, \quad (z \in U). \tag{2.4}$$

Then the function k is analytic in U and $k(0) = 1$.

A simple computation using (2.4) gives

$$\frac{zk'(z)}{k(z)} = \gamma\left(\frac{z(A^{m+1}_{\mu,\nu,\lambda}(\rho, \delta)(f * \Phi)(z))'}{A^{m+1}_{\mu,\nu,\lambda}(\rho, \delta)(f * \Phi)(z)} - \frac{z(A^{m}_{\mu,\nu,\lambda}(\rho, \delta)(f * \Psi)(z))'}{A^{m}_{\mu,\nu,\lambda}(\rho, \delta)(f * \Psi)(z)}\right).$$

In view of (1.4), we obtain

$$\frac{zk'(z)}{k(z)} = \frac{\gamma(\mu+\nu)}{(\nu-\rho)\delta + n\lambda}\left(\frac{A^{m+2}_{\mu,\nu,\lambda}(\rho, \delta)(f * \Phi)(z)}{A^{m+1}_{\mu,\nu,\lambda}(\rho, \delta)(f * \Phi)(z)} - \frac{A^{m+1}_{\mu,\nu,\lambda}(\rho, \delta)(f * \Psi)(z)}{A^{m}_{\mu,\nu,\lambda}(\rho, \delta)(f * \Psi)(z)}\right).$$

Also, we find that

$$(\alpha + \beta k(z))(k(z))^{\tau} + \eta z(k(z))^{\tau-1} k'(z)$$

$$= \varphi_1(f, \Phi, \Psi, \alpha, \beta, \tau, \eta, \gamma, \rho, \delta, \mu, \nu, \lambda, m; z), \tag{2.5}$$

where $\varphi_1(f, \Phi, \Psi, \alpha, \beta, \tau, \eta, \gamma, \rho, \delta, \mu, \nu, \lambda, m; z)$ is given by (2.3).

By using (2.5) in (2.2), we have

$$(\alpha + \beta k(z))(k(z))^{\tau} + \eta z(k(z))^{\tau-1} k'(z) \prec (\alpha + \beta q(z))(q(z))^{\tau} + \eta z(q(z))^{\tau-1} q'(z).$$

By setting

$$\theta(w) = (\alpha + \beta w) w^{\tau} \text{ and } \phi(w) = \eta w^{\tau-1}, \quad w \neq 0,$$

it can be easily observed that $\theta(w)$ is analytic in $\mathbb{C}$, $\phi(w)$ is analytic in $\mathbb{C} \setminus \{0\}$ and that $\phi(w) \neq 0$, $w \in \mathbb{C} \setminus \{0\}$. Also, we get

$$Q(z) = zq'(z)\phi(q(z)) = \eta z(q(z))^{\tau-1} q'(z)$$

and

$$h(z) = \theta(q(z)) + Q(z) = (\alpha + \beta q(z))(q(z))^{\tau} + \eta z(q(z))^{\tau-1} q'(z).$$

In light of the hypothesis of Theorem 2.1, we see that $Q(z)$ is starlike univalent in U and

$$\operatorname{Re}\left\{\frac{zh'(z)}{Q(z)}\right\} = \operatorname{Re}\left\{1 + \frac{\alpha\tau}{\eta} + \frac{\beta(\tau+1)}{\eta} q(z) + (\tau - 1)\frac{zq'(z)}{q(z)} + \frac{zq''(z)}{q'(z)}\right\} > 0.$$

Hence the result now follows by an application of Lemma 1.1.

By taking $q(z) = \dfrac{1 + Az}{1 + Bz}$ $(-1 \leq B < A \leq 1)$ in Theorem 2.1, we obtain the following corollary:

Corollary 2.1. *Let* $\Phi, \Psi \in \mathcal{A}$, $\alpha, \beta, \tau \in \mathbb{C}$, $\eta, \gamma \in \mathbb{C} \setminus \{0\}$, $-1 \leq B < A \leq 1$ *and assume that*

$$\operatorname{Re}\left\{\frac{\alpha\tau}{\eta} + \frac{\beta(\tau+1)(1 + Az)}{\eta(1 + Bz)} + \frac{1 + \tau(A - B)z - ABz^2}{(1 + Az)(1 + Bz)}\right\} > 0,$$

If $f \in \mathcal{A}$ *satisfies the differential subordination*:

$$\varphi_1(f, \Phi, \Psi, \alpha, \beta, \tau, \eta, \gamma, \rho, \delta, \mu, \nu, \lambda, m; z)$$

$$\prec \left(\alpha + \beta\left(\frac{1+Az}{1+Bz}\right)\right)\left(\frac{1+Az}{1+Bz}\right)^{\tau} + \frac{\eta(A-B)(1+Az)^{\tau-1}z}{(1+Bz)^{\tau+1}}, \qquad (2.6)$$

where $\varphi_1(f, \Phi, \Psi, \alpha, \beta, \tau, \eta, \gamma, \rho, \delta, \mu, \nu, \lambda, m; z)$ *is given by* (2.3), *then*

$$\left(\frac{A^{m+1}_{\mu,\nu,\lambda}(\rho, \delta)(f * \Phi)(z)}{A^{m}_{\mu,\nu,\lambda}(\rho, \delta)(f * \Psi)(z)}\right)^{\gamma} \prec \frac{1+Az}{1+Bz}$$

and $q(z) = \dfrac{1+Az}{1+Bz}$ *is the best dominant of* (2.6).

By fixing $\Phi(z) = \Psi(z) = \dfrac{z}{1-z}$ in Theorem 2.1, we obtain the following corollary:

Corollary 2.2. *Let* $\alpha, \beta, \tau \in \mathbb{C}, \ \eta, \gamma \in \mathbb{C} \setminus \{0\}$ *and let q be convex univalent in U with* $q(0) = 1$ *and assume that* (2.1) *holds true*. *Suppose that* $z(q(z))^{\tau-1}q'(z)$ *is starlike univalent in U*. *If* $f \in \mathcal{A}$ *satisfies the differential subordination*:

$$\varphi_2(f, \alpha, \beta, \tau, \eta, \gamma, \rho, \delta, \mu, \nu, \lambda, m; z) \prec (\alpha + \beta q(z))(q(z))^{\tau} + \eta z(q(z))^{\tau-1}q'(z), \qquad (2.7)$$

where

$$\varphi_2(f, \alpha, \beta, \tau, \eta, \gamma, \rho, \delta, \mu, \nu, \lambda, m; z)$$

$$= \alpha\left(\frac{A^{m+1}_{\mu,\nu,\lambda}(\rho, \delta) f(z)}{A^{m}_{\mu,\nu,\lambda}(\rho, \delta) f(z)}\right)^{\gamma\tau} + \beta\left(\frac{A^{m+1}_{\mu,\nu,\lambda}(\rho, \delta) f(z)}{A^{m}_{\mu,\nu,\lambda}(\rho, \delta) f(z)}\right)^{\gamma(\tau+1)}$$

$$+ \frac{\gamma\eta(\mu+\nu)}{(\nu-\rho)\delta + n\lambda}\left(\frac{A^{m+1}_{\mu,\nu,\lambda}(\rho, \delta) f(z)}{A^{m}_{\mu,\nu,\lambda}(\rho, \delta) f(z)}\right)^{\gamma\tau}\left(\frac{A^{m+2}_{\mu,\nu,\lambda}(\rho, \delta) f(z)}{A^{m+1}_{\mu,\nu,\lambda}(\rho, \delta) f(z)} - \frac{A^{m+1}_{\mu,\nu,\lambda}(\rho, \delta) f(z)}{A^{m}_{\mu,\nu,\lambda}(\rho, \delta) f(z)}\right), \qquad (2.8)$$

then

$$\left(\frac{A^{m+1}_{\mu,\nu,\lambda}(\rho, \delta) f(z)}{A^{m}_{\mu,\nu,\lambda}(\rho, \delta) f(z)}\right)^{\gamma} \prec q(z)$$

and q is the best dominant of (2.7).

Theorem 2.2. *Let* $\Phi, \Psi \in \mathcal{A}$, $\alpha, \beta, \tau \in \mathbb{C}$, $\eta, \gamma \in \mathbb{C} \setminus \{0\}$ *and let* q *be convex univalent in* U *with* $q(0) = 1$ *and assume that* q *satisfies*:

$$\operatorname{Re}\left\{\frac{\alpha\tau}{\eta} q'(z) + \frac{\beta(\tau+1)}{\eta} q(z) q'(z)\right\} > 0. \tag{2.9}$$

Suppose that $z(q(z))^{\tau-1} q'(z)$ *is starlike univalent in* U. *Let* $f \in \mathcal{A}$ *satisfy*

$$\left(\frac{A^{m+1}_{\mu,\nu,\lambda}(\rho, \delta)(f * \Phi)(z)}{A^{m}_{\mu,\nu,\lambda}(\rho, \delta)(f * \Psi)(z)}\right)^{\gamma} \in \mathcal{H}[q(0), 1] \cap Q$$

and $\varphi_1(f, \Phi, \Psi, \alpha, \beta, \tau, \eta, \gamma, \rho, \delta, \mu, \nu, \lambda, m; z)$ *as defined by* (2.3) *be univalent in* U. *If*

$$(\alpha + \beta q(z))(q(z))^{\tau} + \eta z(q(z))^{\tau-1} q'(z)$$
$$\prec \varphi_1(f, \Phi, \Psi, \alpha, \beta, \tau, \eta, \gamma, \rho, \delta, \mu, \nu, \lambda, m; z), \tag{2.10}$$

then

$$q(z) \prec \left(\frac{A^{m+1}_{\mu,\nu,\lambda}(\rho, \delta)(f * \Phi)(z)}{A^{m}_{\mu,\nu,\lambda}(\rho, \delta)(f * \Psi)(z)}\right)^{\gamma}$$

and q *is the best subordinant of* (2.10).

Proof. Let the function k be defined by (2.4).

In view of (1.4), the superordination (2.10) becomes

$$(\alpha + \beta q(z))(q(z))^{\tau} + \eta z(q(z))^{\tau-1} q'(z) \prec (\alpha + \beta k(z))(k(z))^{\tau} + \eta z(k(z))^{\tau-1} k'(z).$$

By setting $\theta(w) = (\alpha + \beta w) w^{\tau}$ and $\phi(w) = \eta w^{\tau-1}$, $w \neq 0$, it is easily observed that $\theta(w)$ is analytic in $\mathbb{C}$, $\phi(w)$ is analytic in $\mathbb{C} \setminus \{0\}$ and that $\phi(w) \neq 0$, $w \in \mathbb{C} \setminus \{0\}$. Also, we get

$$Q(z) = zq'(z)\phi(q(z)) = \eta z(q(z))^{\tau-1} q'(z).$$

It is clear that $Q(z)$ is starlike univalent in U and

$$\operatorname{Re}\left\{\frac{\theta'(q(z))}{\phi(q(z))}\right\} = \operatorname{Re}\left\{\frac{\alpha\tau}{\eta} q'(z) + \frac{\beta(\tau+1)}{\eta} q(z) q'(z)\right\} > 0.$$

Now Theorem 2.2 follows by applying Lemma 1.2.

By fixing $\Phi(z) = \Psi(z) = \dfrac{z}{1-z}$ in Theorem 2.2, we obtain the following corollary:

Corollary 2.3. *Let* $\alpha, \beta, \tau \in \mathbb{C}$, $\eta, \gamma \in \mathbb{C} \setminus \{0\}$ *and let q be convex univalent in U with* $q(0) = 1$ *and assume that* (2.9) *holds true. Suppose that* $z(q(z))^{\tau-1} q'(z)$ *is starlike univalent in U. Let* $f \in \mathcal{A}$ *satisfy*

$$\left(\frac{A^{m+1}_{\mu,\nu,\lambda}(\rho, \delta) f(z)}{A^{m}_{\mu,\nu,\lambda}(\rho, \delta) f(z)}\right)^{\gamma} \in \mathcal{H}[q(0), 1] \cap Q$$

and $\varphi_2(f, \alpha, \beta, \tau, \eta, \gamma, \rho, \delta, \mu, \nu, \lambda, m; z)$ *as defined by* (2.8) *be univalent in U. If*

$$(\alpha + \beta q(z))(q(z))^{\tau} + \eta z(q(z))^{\tau-1} q'(z) \prec \varphi_2(f, \alpha, \beta, \tau, \eta, \gamma, \rho, \delta, \mu, \nu, \lambda, m; z), \tag{2.11}$$

then

$$q(z) \prec \left(\frac{A^{m+1}_{\mu,\nu,\lambda}(\rho, \delta) f(z)}{A^{m}_{\mu,\nu,\lambda}(\rho, \delta) f(z)}\right)^{\gamma}$$

and q is the best subordinant of (2.11).

Concluding the results of differential subordination and superordination, we arrive at the following "sandwich results".

Theorem 2.3. *Let* q_1 *and* q_2 *be convex univalent in U with* $q_1(0) = q_2(0) = 1$, $\alpha, \beta, \tau \in \mathbb{C}$, $\eta, \gamma \in \mathbb{C} \setminus \{0\}$. *Suppose* q_2 *satisfies* (2.1) *and* q_1 *satisfies* (2.9) *such that* $z(q(z))^{\tau-1} q'(z)$ *is starlike univalent in U. For* $f, \Phi, \Psi \in \mathcal{A}$, *let*

$$\left(\frac{A^{m+1}_{\mu,\nu,\lambda}(\rho, \delta)(f * \Phi)(z)}{A^{m}_{\mu,\nu,\lambda}(\rho, \delta)(f * \Psi)(z)}\right)^{\gamma} \in \mathcal{H}[1, 1] \cap Q$$

and $\varphi_1(f, \Phi, \Psi, \alpha, \beta, \tau, \eta, \gamma, \rho, \delta, \mu, \nu, \lambda, m; z)$ *as defined by* (2.3) *be univalent in U.*

If

$$(\alpha + \beta q_1(z))(q_1(z))^{\tau} + \eta z(q_1(z))^{\tau-1} q_1'(z)$$

$$\prec \varphi_1(f, \Phi, \Psi, \alpha, \beta, \tau, \eta, \gamma, \rho, \delta, \mu, \nu, \lambda, m; z)$$

$$\prec (\alpha + \beta q_2(z))(q_2(z))^{\tau} + \eta z(q_2(z))^{\tau-1} q_2'(z),$$

then

$$q_1(z) \prec \left(\frac{A_{\mu,\nu,\lambda}^{m+1}(\rho, \delta)(f * \Phi)(z)}{A_{\mu,\nu,\lambda}^{m}(\rho, \delta)(f * \Psi)(z)} \right)^{\gamma} \prec q_2(z)$$

and q_1, q_2 *are respectively the best subordinant and the best dominant.*

By making use of Corollaries 2.2 and 2.3, we obtain the following corollary:

Corollary 2.4. *Let* q_1 *and* q_2 *be convex univalent in U with* $q_1(0) = q_2(0) = 1$, $\alpha, \beta, \tau \in \mathbb{C}$, $\eta, \gamma \in \mathbb{C} \setminus \{0\}$. *Suppose* q_2 *satisfies* (2.1) *and* q_1 *satisfies* (2.9) *such that* $z(q(z))^{\tau-1} q'(z)$ *is starlike univalent in U. For* $f \in \mathcal{A}$, *let*

$$\left(\frac{A_{\mu,\nu,\lambda}^{m+1}(\rho, \delta) f(z)}{A_{\mu,\nu,\lambda}^{m}(\rho, \delta) f(z)} \right)^{\gamma} \in \mathcal{H}[1, 1] \cap Q$$

and $\varphi_2(f, \alpha, \beta, \tau, \eta, \gamma, \rho, \delta, \mu, \nu, \lambda, m; z)$ *as defined by* (2.8) *be univalent in U. If*

$$(\alpha + \beta q_1(z))(q_1(z))^{\tau} + \eta z(q_1(z))^{\tau-1} q_1'(z)$$

$$\prec \varphi_2(f, \alpha, \beta, \tau, \eta, \gamma, \rho, \delta, \mu, \nu, \lambda, m; z)$$

$$\prec (\alpha + \beta q_2(z))(q_2(z))^{\tau} + \eta z(q_2(z))^{\tau-1} q_2'(z),$$

then

$$q_1(z) \prec \left(\frac{A_{\mu,\nu,\lambda}^{m+1}(\rho, \delta) f(z)}{A_{\mu,\nu,\lambda}^{m}(\rho, \delta) f(z)} \right)^{\gamma} \prec q_2(z)$$

and q_1, q_2 *are respectively the best subordinant and the best dominant.*

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