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On the Convergence Region of Multi-step Chebyshev-Halley-type Schemes for Solving Equations

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Abstract

The aim of this article is to extend the convergence region of certain multi-step Chebyshev-Halley-type schemes for solving Banach space valued nonlinear equations. In particular, we find an at least as small region as the region of the operator involved containing the iterates. This way the majorant functions are tighter than the ones related to the original region, leading to a finer local as well as a semi-local convergence analysis under the same computational effort. Numerical examples complete this article.

1. Introduction

Let $F : D \subset \mathcal{B}_1 \to \mathcal{B}_2$ be a twice continuously differentiable operator in the sense of Fréchet, where $\mathcal{B}_1$, $\mathcal{B}_2$ are Banach spaces and D is a nonempty and open set. We shall denote by $\mathcal{L}(\mathcal{B}_1, \mathcal{B}_2)$ the space of bounded linear operators from $\mathcal{B}_1$ into $\mathcal{B}_2$.

Numerous problems in mathematical, scientific and engineering computing [1-19] are usually formulated like an equation of the form

$$F(x) = 0. \tag{1.1}$$

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However, a solution x_* of equation (1.1) can rarely be found in closed form. Therefore, researchers and practitioners resort to iterative schemes to produce a sequence approximating x_*. Recently, a great effort has been given to generate fast iterative schemes which converge under some Lipschitz-type criteria.

In particular, we consider the Chebyshev-Halley-type scheme defined for each $n = 0, 1, 2, \ldots,$ by

$$y_n = x_n - \left[I + \frac{1}{2}M_n(I - \alpha M_n)^{-1}\right]F'(x_n)^{-1}$$

$$z_n = y_n - F'(x_n)^{-1}[F'(x_n) + F''(u_n)(x_n - y_n)]F'(x_n)^{-1}F(y_n)$$

$$x_{n+1} = z_n - [I + M_n + \beta M_n^2]F'(x_n)^{-1}F(z_n), \tag{1.2}$$

where $u_n = x_n - \frac{2}{3}F'(x_n)^{-1}F(x_n)$, $M_n = F'(x_n)^{-1}F''(u_n)F'(x_n)^{-1}F(x_n)$, $v_n = x_n - F'(x_n)^{-1}F(x_n)$ and $\alpha, \beta \in \mathbb{R}$ (in the local convergence case) and $\alpha \in [0, 1]$, $\beta \in [-1, 1]$ (in the semi-local convergence case). Iterative schemes-type (1.2) have been considered in [19]. However, in this article, we study the local as well as the semi-local convergence of scheme (1.2) under generalized ω-conditions. Moreover, by introducing the center ω-condition, we locate a subset of D containing the iterates. This subset helps us define tighter majorant functions and parameters than before leading to larger radius of convergence (i.e., we obtain a wider choice of initial guesses); tighter error bounds on the distances $\| x_{n+1} - x_n \|$, $\| x_n - x_* \|$ (i.e., fewer iterates are needed to obtain a desired error tolerance $\varepsilon > 0$) and an at least as precise information on the location of the solution. Scheme (1.2) is especially useful, when F'' is a constant. Other favorable cases can be found in [19].

The design of the article is as follows: Section 2 and Section 3 contain the local and semi-local convergence of scheme (1.2), respectively. The numerical examples appear in the concluding Section 4.

2. Local Convergence Analysis

We rely on some parameters and scalar functions to show the local convergence analysis of scheme (1.2). Let $\gamma_0 : [0, +\infty) \to [0, +\infty)$ be a continuous and increasing function with $\gamma_0(0) = 0.$ Suppose that the equation

$$\gamma_0(t) = 1 \tag{2.1}$$

has at least one positive solution. We denote by ρ_0 the smallest such solution. Let $\gamma : [0, +\infty) \to [0, +\infty),$ $\lambda_1 : [0, \rho_0) \to [0, +\infty),$ $\lambda_2 : [0, \rho_0) \to [0, +\infty)$ be continuous and increasing functions with $\gamma(0) = 0.$ Define functions μ_0 and μ on the interval $[0, \rho_0)$ by

$$\mu_0(t) = \frac{\int_0^1 \gamma((1-\theta)t)\, d\theta}{1 - \gamma_0(t)}$$

and

$$\mu(t) = \mu_0(t) - 1.$$

We have $\mu(0) = -1$ and $\mu(t) \to +\infty$ as $t \to \rho_0^-.$ The intermediate value theorem assures that equation $\mu(t) = 0$ has at least one positive solution. We denote the smallest such solution by $r_0.$

Define functions $\mu_1,$ μ_2 on the interval $[0, \rho_0)$ by

$$\mu_1(t) = \mu_0(t) + \frac{\int_0^1 \lambda_1(\theta t)\, d\theta}{3(1 - \gamma_0(t))}$$

and

$$\mu_2(t) = \mu_1(t) - 1.$$

Suppose that

$$\lambda_1(0) < 3. \tag{2.2}$$

Then, we get $\mu_2(0) < 0$ by (2.2) and $\mu_2(t) \to +\infty$ as $t \to \rho_0^-.$ Denote by r_1 the smallest positive solution of equation $\mu_2(t) = 0.$ Notice that $r_1 \le r_0.$ Define functions p and p_1 on $[0, \rho_0)$ by

$$p(t) = \frac{\lambda_2(t)\int_0^1 \lambda_1(\theta t)\, d\theta t}{(1-\gamma_0(t))^2}$$

and

$$p_1(t) = |\alpha|\, p(t) - 1.$$

We get $p_1(0) = -1$ and $p_1(t) \to +\infty$ as $t \to \rho_0^-$. We denote by ρ_1 the smallest positive solution of equation $p_1(t) = 0$. Set

$$\rho = \min\{\rho_0, \rho_1\}.$$

Define functions μ_3, μ_4 on the interval $[0, \rho)$ by

$$\mu_3(t) = \mu_0(t) + \frac{p(t)\int_0^1 \lambda_1(\theta t)\, d\theta}{2(1-|\alpha|\, p(t))(1-\gamma_0(t))}$$

and

$$\mu_4(t) = \mu_3(t) - 1.$$

We obtain $\mu_3(0) = -1$ and $\mu_3(t) \to +\infty$ as $t \to \rho^-$. Denote by r_2 the smallest positive solution of equation $\mu_4(t) = 0$. Suppose that equation

$$\gamma_0(\mu_3(t)t) = 1 \tag{2.3}$$

has at least one positive solution. We denote by ρ_2 the smallest such solution. Define the functions μ_5, μ_6 on the interval $[0, \rho_2)$, $\rho_2 = \min\{\rho, \rho_1\}$ by

$$\mu_5(t) = \frac{\int_0^1 \gamma((1-\theta)\mu_3(t)t)\, d\theta \mu_3(t)}{1-\gamma_0(\mu_3(t)t)}$$

$$+ \frac{(\gamma_0(\mu_3(t)t) + \gamma_0(t))\int_0^1 \lambda_1(\theta\mu_3(t)t)\, d\theta\mu_3(t)}{(1-\gamma_0(\mu_3(t)t))(1-\gamma_0(t))}$$

$$+\frac{(1+\mu_0(t))\lambda_2(\mu_1(t)t)\int_0^1\lambda_1(\theta\mu_3(t)t)\,d\theta\mu_3(t)t}{(1-\gamma_0(t))^2}$$

and

$$\mu_6(t)=\mu_5(t)-1.$$

We have $\mu_6(0)=-1$ and $\mu_6(t)\to+\infty$ as $t\to\rho_3^-$. Denote by r_3 the smallest positive solution of equation $\mu_6(t)=0$. Suppose that equation

$$\gamma_0(\mu_5(t)t)=1 \tag{2.4}$$

has at least one positive solution. We denote by ρ_3 the smallest such solution and set $\rho_4=\min\{\rho_2,\rho_3\}$. Define functions μ_7, μ_8 on the interval $[0,\rho_4)$, by

$$\mu_7(t)=\mu_5(t)+\frac{(\gamma_0(\mu_5(t)t)+\gamma_0(t))\int_0^1\lambda_1(\theta\mu_5(t)t)d\theta\mu_5(t)}{(1-\gamma_0(\mu_5(t)t))(1-\gamma_0(t))}$$

$$+\frac{p(t)\int_0^1\lambda_1(\theta\mu_5(t)t)\,d\theta\mu_5(t)}{1-\gamma_0(t)}$$

$$+\frac{|\beta|p^2(t)\int_0^1\lambda_1(\theta\mu_5(t)t)\,d\theta\mu_5(t)t}{1-\gamma_0(t)}$$

and

$$\mu_8(t)=\mu_7(t)-1.$$

We get $\mu_8(0)=-1$ and $\mu_8(t)\to+\infty$ as $t\to\rho_5^-$. Denote by r_4 the smallest positive solution on equation $\mu_8(t)=0$. Define radius of convergence r by

$$r=\min\{r_1\},\quad i=1,2,3,4. \tag{2.5}$$

Let $U(x,a)=\{y\in\mathcal{B}_1:\|x-y\|<a\}$ and $\overline{U}(x,a)$ be its closure.

The local convergence analysis of method (1.2) is based on the hypotheses (H):

(h$_1$) $F : D \subset \mathcal{B}_1 \to \mathcal{B}_2$ is a continuously differentiable operator in the sense of Fréchet and there exists $x_* \in D$ such that $F(x_*) = 0$ and $F'(x_*)^{-1} \in \mathcal{L}(\mathcal{B}_2, \mathcal{B}_1)$.

(h$_2$) There exists a function $\gamma_0 : [0, +\infty) \to [0, +\infty)$ continuous and increasing with $\gamma_0(0) = 0$ such that for each $x \in D$

$$\| F'(x_*)^{-1}(F'(x) - F'(x_*)) \| \le \gamma_0(\| x - x_* \|).$$

Set $D_0 = D \cap U(x_*, \rho_0)$, where ρ_0 is given in (2.1).

(h$_3$) There exist functions $\gamma : [0, \rho_0) \to [0, +\infty)$, $\lambda_1 : [0, \rho_0) \to [0, +\infty)$, $\lambda_2 : [0, \rho_0) \to [0, +\infty)$ with $\gamma(0) = 0$, continuous and increasing such that for each $x, y \in D_0$

$$\| F'(x_*)^{-1}(F'(y) - F'(x)) \| \le \gamma_0(\| y - x \|)$$

$$\| F'(x_*)^{-1} F'(x) \| \le \lambda_1(\| x - x_* \|)$$

and

$$\| F'(x_*)^{-1} F''(x) \| \le \lambda_2(\| x - x_* \|).$$

(h$_4$) $\overline{U}(x_*, r) \subset D$, ρ_0, ρ_1, ρ_3 given in (2.1), (2.3), (2.4), respectively exist and (2.2) holds.

(h$_5$) There exists $r_* \ge r$ such that

$$\int_0^1 \gamma_0(\theta r_*)\, d\theta < 1.$$

Set $D_1 = D \cap \overline{U}(x_*, r_*)$.

The aforementioned hypotheses (H) and notation lead to the local convergence result for method (1.2).

Theorem 2.1. *Under the hypotheses* (*H*), *sequence* $\{x_n\}$ *generated by scheme* (1.2) *for* $x_0 \in U(x_*, r) - \{x_*\}$ *converges to* x_* *so that*

$$\| y_n - x_* \| \le \mu_3(\| x_n - x_* \|) \| x_n - x_* \| \le \| x_n - x_* \| < r \tag{2.6}$$

$$\| z_n - x_* \| \le \mu_5(\| x_n - x_* \|) \| x_n - x_* \| \le \| x_n - x_* \| \tag{2.7}$$

and

$$\| x_{n+1} - x_* \| \le \mu_7(\| x_n - x_* \|) \| x_n - x_* \| \le \| x_n - x_* \|, \tag{2.8}$$

where functions μ_3, μ_5, μ_7 *are given previously and r is defined in* (2.5). *Moreover,* x_* *is the unique solution of equation* $F(x) = 0$ *in* D_1.

Proof. The definition of the convergence radius r guarantees that for each $t \in [0, r)$

$$0 \le \gamma_0(t) < 1, \tag{2.9}$$

$$0 \le \mu_0(t) < 1, \tag{2.10}$$

$$0 \le \mu_1(t) < 1, \tag{2.11}$$

$$0 \le |\alpha| p(t) < 1, \tag{2.12}$$

$$0 \le \mu_3(t) < 1, \tag{2.13}$$

$$0 \le \gamma_0(\mu_3(t)t) < 1, \tag{2.14}$$

$$0 \le \mu_5(t) < 1, \tag{2.15}$$

$$0 \le \gamma_0(\mu_5(t)t) < 1, \tag{2.16}$$

and

$$0 \le \mu_7(t) < 1. \tag{2.17}$$

The proof is based on the estimates (2.9)-(2.17) and mathematical induction. Let $x \in U(x_*, r) - \{x_*\}$. Using ($h_1$), ($h_2$), (2.5) and (2.9), we have:

$$\| F'(x_*)^{-1}(F'(x) - F'(x_*)) \| \le \gamma_0(\| x - x_* \|) \le \gamma_0(r) < 1,$$

which together with the Banach Perturbation Lemma [3-5], imply that $F'(x)^{-1} \in \mathcal{L}(\mathcal{B}_2, \mathcal{B}_2)$ and

$$\| F'(x)^{-1} F'(x_*) \| \le \frac{1}{1 - \gamma_0(\| x - x_* \|)}. \tag{2.18}$$

We must also, show $(I - \alpha M_0)^{-1} \in \mathcal{L}(\mathcal{B}_2, \mathcal{B}_1)$. By (2.5), (2.12), (2.18) and (h$_3$), we get

$$\| \alpha M_0 \| \leq \frac{|\alpha| \lambda_2(\| x_0 - x_* \|) \int_0^1 \lambda_1(\theta \| x_0 - x_* \|) d\theta \| x_0 - x_* \|}{(1 - \gamma_0(\| x_0 - x_* \|))^2}$$

$$= |\alpha| p(\| x_0 - x_* \|) \leq |\alpha| p(r) < 1,$$

so

$$\| (I - \alpha M_0)^{-1} \| \leq \frac{1}{1 - |\alpha| p(\| x_0 - x_* \|)}. \tag{2.19}$$

We can write by (h$_1$),

$$F(x) = F(x) - F(x_*) = \int_0^1 F'(x_* + \theta(x - x_*)) d\theta (x - x_*). \tag{2.20}$$

Then, by (h$_3$) and (2.20)

$$\| F'(x_*)^{-1} F(x) \| \leq \int_0^1 \lambda_1(\theta \| x - x_* \|) d\theta \| x - x_* \|. \tag{2.21}$$

In particular, for $x = x_0$, since $x_0 \in U(x_*, r) - \{x_*\}$, y_0 is well defined, if $n = 0$ by the first substep of scheme (1.2). By (2.5), (2.13), (h$_1$)-(h$_3$), (2.18), scheme (1.2) for $n = 0$, we get in turn that

$$\| y_0 - x_* \| = \left\| x_0 - x_* - F'(x_0)^{-1} F(x_0) + \frac{1}{2} M_0 (I - \alpha M_0)^{-1} F'(x_0)^{-1} F(x_0) \right\|$$

$$\leq \| F'(x_0)^{-1} F(x_*) \| \int_0^1 \| F'(x_*)^{-1} (F'(x_0 + \theta(x_0 - x_*)) - F'(x_0)) \| d\theta$$

$$\| x_0 - x_* \|$$

$$+ \frac{1}{2} \| M_0 \| \| (I - \alpha M_0)^{-1} \| \| F'(x_0)^{-1} F'(x_*) \| \| F'(x_*)^{-1} F(x_0) \|$$

$$\leq \left[\frac{\int_0^1 \gamma((1 - \theta) \| x_0 - x_* \|) d\theta}{1 - \gamma_0(\| x_0 - x_* \|)} \right.$$

$$\left.\frac{p(\| x_0 - x_* \|)\int_0^1 \lambda_1(\theta\| x_0 - x_* \|)\,d\theta}{2(1-|\alpha| p(\| x_0 - x_* \|))(1-\gamma_0(\| x_0 - x_* \|))}\right]\| x_0 - x_* \|$$

$$= \mu_3(\| x_0 - x_* \|)\| x_0 - x_* \| \le \| x_0 - x_* \| < r, \tag{2.22}$$

so (2.6) holds for $n = 0$ and $y_0 \in U(x_*, r)$. By the definition of r, v_0 and (2.22), $v_0 \in U(x_*, r)$. Concerning, u_0, we have in turn as in (2.22)

$$\| u_0 - x_* \| = \left\| (x_0 - x_* - F'(x_0)^{-1}F(x_0)) + \frac{1}{3}F'(x_0)^{-1}F(x_0)\right\|$$

$$\le \frac{\int_0^1 \gamma((1-\theta)\| x_0 - x_* \|)\,d\theta\| x_0 - x_* \|}{1-\gamma_0(\| x_0 - x_* \|)}$$

$$+\frac{1}{3}\frac{\int_0^1 \lambda_1(\theta\| x_0 - x_* \|)\,d\theta\| x_0 - x_* \|}{1-\gamma_0(\| x_0 - x_* \|)}$$

$$= \mu_1(\| x_0 - x_* \|)\| x_0 - x_* \| \le \| x_0 - x_* \| < r,$$

so $v_0 \in U(x_*, r)$. Then, by the last condition in (h_3), we get that

$$\| F'(x_*)^{-1}F''(u_0) \| \le \lambda_3(\| u_0 - x_* \|)$$

$$\le \lambda_3(\mu_1(\| x_0 - x_* \|)\| x_0 - x_* \|). \tag{2.23}$$

Hence, z_0 is well defined. Using (2.5), (2.18), (2.14), (2.21), (2.22), (2.23) and second substep of scheme (1.2) for $n = 0$ we have in turn that

$$\| z_0 - x_* \| = \| (y_0 - x_* - F'(y_0)^{-1}F(y_0)) + (F'(y_0)^{-1} - F'(x_0)^{-1})F(y_0)$$

$$- F'(x_0)^{-1}F''(u_0)[(x_0 - x_*) + (x_* - v_0)]F'(x_0)^{-1}F(y_0) \|$$

$$\le \frac{\int_0^1 \gamma((1-\theta)\| y_0 - x_* \|)\,d\theta\| y_0 - x_* \|}{1-\gamma_0(\| y_0 - x_* \|)}$$

$$\leq \frac{\Gamma_1}{(1-\gamma_0(\| y_0 - x_* \|))(1-\gamma_0(\| x_0 - x_* \|))} + \frac{\Gamma_2}{(1-\gamma_0(\| x_0 - x_* \|))^2}$$

$$\leq \mu_5(\| x_0 - x_* \|)\| x_0 - x_* \| \leq \| x_0 - x_* \| < r, \tag{2.24}$$

where

$$\Gamma_1 = (\gamma_0(\| x_0 - x_* \|) + \gamma_0(\| y_0 - x_* \|))\int_0^1 \lambda_1(\theta\| y_0 - x_* \|)\, d\theta \| y_0 - x_* \|$$

and

$$\Gamma_2 = (1 + \mu_0(\| x_0 - x_* \|))\int_0^1 \lambda_2(\mu_1(\| x_0 - x_* \|)\| x_0 - x_* \|)$$

$$\int_0^1 \lambda_1(\theta\| y_0 - x_* \|)\, d\theta\| y_0 - x_* \|\| x_0 - x_* \|,$$

so (2.7) holds for $n = 0$ and $z_0 \in U(x_*, r)$. We also have that x_1 is well defined by the third substep of scheme (1.2) for $n = 0$. Next, using (2.5), (2.16), (2.17), (2.18), (2.21), (2.22) and (2.24), we get in turn that

$$\| x_1 - x_* \| = \| z_0 - x_* - F'(z_0)^{-1} F(z_0) + (F'(z_0)^{-1} - F'(x_0)^{-1}) F(z_0)$$

$$- M_0 F'(x_0)^{-1} F(z_0) - \beta M_0^2 F'(x_0)^{-1} F(z_0) \|$$

$$\leq \mu_5(\| x_0 - x_* \|)\| x_0 - x_* \|$$

$$+ \frac{(\gamma_0(\| z_0 - x_* \|) + \gamma_0(\| x_0 - x_* \|))\int_0^1 \lambda_1(\theta\| z_0 - x_* \|)\, d\theta\| z_0 - x_* \|}{(1-\gamma_0(\| z_0 - x_* \|))(1-\gamma_0(\| x_0 - x_* \|))}$$

$$+ \frac{p(\| x_0 - x_* \|)\int_0^1 \lambda_1(\theta\| z_0 - x_* \|)\, d\theta\| z_0 - x_* \|}{1-\gamma_0(\| x_0 - x_* \|)}$$

$$+ \frac{|\beta| p^2(\| x_0 - x_* \|)\int_0^1 \lambda_1(\theta\| z_0 - x_* \|)\, d\theta\| z_0 - x_* \|}{1-\gamma_0(\| x_0 - x_* \|)}$$

$$\leq \mu_7(\| x_0 - x_* \|) \| x_0 - x_* \| \leq \| x_0 - x_* \| < r, \tag{2.25}$$

so (2.8) holds for $n = 0$ and $x_1 \in U(x_*, r)$.

To finish the induction for estimates (2.6)-(2.8), substitute $x_0, y_0, z_0, v_0, u_0, x_1$ by $x_k, y_k, z_k, v_k, u_k, x_{k+1}$ in the preceding estimates. Then, from the estimate

$$\| x_{k+1} - x_* \| \leq c\| x_k - x_* \| < r, \tag{2.26}$$

where $c = \mu_7(\| x_0 - x_* \|) \in [0, 1)$, we conclude that $\lim_{n\to+\infty} x_k = x_*$ and $x_{k+1} \in U(x_*, r)$. The uniqueness of the solution part, is shown by letting $G = \int_0^1 F'(x_* + \theta(y_* - x_*))\,d\theta$ for some $y_* \in D_1$ with $F(y_*) = 0$. Then, by (h_1), (h_2) and (h_5) we obtain that

$$\| F'(x_*)^{-1}(G - F'(x_*)) \| \leq \int_0^1 \gamma_0(\theta\| x_* - y_* \|)\,d\theta$$

$$\leq \int_0^1 \gamma_0(\theta r_*)\,d\theta < 1,$$

so $G^{-1} \in \mathcal{L}(\mathcal{B}_2, \mathcal{B}_1)$. Finally, in view of the identity

$$0 = F(y_*) - F(x_*) = G(y_* - x_*),$$

we deduce that $x_* = y_*$. □

Remark 2.2. (a) Let $\gamma_0(t) = L_0 t$, $\gamma(t) = Lt$. The radius $\tilde{\rho}_1 = \frac{2}{2L_0 + L}$ was obtained by Argyros as the convergence radius for Newton's method under condition (h_1)–(h_3). Notice that the convergence radius for Newton's method given independently by Rheinboldt [17] and Traub [18] is given by

$$\tilde{\rho} = \frac{2}{3L} < \tilde{\rho}_1.$$

Let $f(x) = e^x - 1$. Then $x^* = 0$. Set $\Omega = U(0, 1)$. Then, we have that $L_0 = e - 1 < L = e^{1/L_0}$, so $\tilde{\rho} = 0.24252961 < \tilde{\rho}_1 = 0.3827$.

Moreover, the new error bounds [3-5] are:

$$\| x_{n+1} - x^* \| \leq \frac{L}{1 - L_0 \| x_n - x^* \|} \| x_n - x^* \|^2,$$

whereas the old ones [17, 18]

$$\| x_{n+1} - x^* \| \leq \frac{L}{1 - L \| x_n - x^* \|} \| x_n - x^* \|^2.$$

Clearly, the new error bounds are more precise, if $L_0 < L$. Clearly, the radius of convergence of method (1.2) given by ρ^* is smaller than $\tilde{\rho}_1$.

(b) Method (1.2) stays the same if we use the new instead of the old conditions [19]. We can use the computational order of convergence (COC) [3-5]

$$\xi = \frac{\ln \frac{\| x_{n+2} - x_{n+1} \|}{\| x_{n+1} - x_n \|}}{\ln \frac{\| x_{n+1} - x_n \|}{\| x_n - x_{n-1} \|}}, \quad \text{for each } n = 1, 2, \ldots$$

or the approximate computational order of convergence (ACOC) [3-5]

$$\xi^* = \frac{\ln \frac{\| x_{n+2} - x^* \|}{\| x_{n+1} - x^* \|}}{\ln \frac{\| x_{n+1} - x^* \|}{\| x_n - x^* \|}}, \quad \text{for each } n = 0, 1, 2, \ldots$$

(c) Using (2.6) and

$$\begin{aligned} \| F'(x^*)^{-1} F'(x) \| &= \| F'(x^*)^{-1}(F'(x) - F'(x^*)) + I \| \\ &\leq 1 + \| F'(x^*)^{-1}(F'(x) - F'(x^*)) \| \\ &\leq 1 + q_0(\| x - x^* \|) \end{aligned}$$

condition (2.9) can be replaced by

$$\lambda_1(t) = 1 + \gamma_0(t)$$

or

$$\lambda_1 = 1 + \gamma_0(\rho_0).$$

3. Semi-local convergence analysis

We study the semi-local convergence analysis of scheme (1.2) in an analogous way to the local convergence analysis appearing in Section 2. That is why we omit the proofs for which you can also see [19]. The hypotheses on which we base our analysis are (A):

(a_1) $F : D \subset \mathcal{B}_1 \to \mathcal{B}_2$ is twice continuously differentiable operator in the sense of Fréchet and there exists $x_0 \in D,\ b > 0,\ \eta \geq 0$ such that $F'(x_0)^{-1} \in \mathcal{L}(\mathcal{B}_2, \mathcal{B}_1)$,

$$\| F'(x_0)^{-1} \| \leq b$$

and

$$\| F'(x_0)^{-1} F(x_0) \| \leq \eta.$$

(a_2) There exists a function $w_0 : [0, +\infty) \to [0, +\infty)$ continuous and non-decreasing such that for each $x \in D$

$$\| F'(x) - F'(x_0) \| \leq w_0(\| x - x_0 \|),$$

and equation

$$w_0(t) = 1$$

has at least one positive solution. Denote by ρ_0 the smallest such solution and set $D_0 = D \cap U(x_0, \rho_0)$. Moreover, suppose that $w_0(t) \geq 0$, for each $t > 0$, $w_0(t\xi) \leq t^q w_0(\xi)$ for each $t \in [0, 1]$, $\xi \in (0, +\infty)$ and $q \in [0, 1]$.

(a_3) There exists function $w : [0, \rho_0) \to [0, +\infty)$ continuous and non-decreasing such that for each $x, y \in D_0$

$$\| F'(y) - F'(x) \| \leq w(\| y - x \|),$$

where $w(t) \geq 0$, for each $t > 0$, $w(t\xi) \leq t^q w(\xi)$ for each $t \in [0, 1]$, $\xi \in (0, +\infty)$ and $q \in [0, 1]$.

(a_4) There exists $K \geq 0$ such that for each $x \in D_0$

$$F''(x) \leq K.$$

In the literature the following conditions are used instead of (a_2), (a_3) and (a_4).

(a_3)' There exists function $w_1 : [0, +\infty) \to [0, +\infty)$ continuous and non-decreasing such that for each $x, y \in D$

$$\| F'(y) - F'(x) \| \leq w_1(\| y - x \|),$$

where $w_1(t) \geq 0,$ for each $t > 0,$ $w_1(t\xi) \leq t^q w_1(\xi)$ for each $t \in [0, 1],$ $\xi \in (0, +\infty)$ and $q \in [0, 1].$

(a_4)' There exists $K_1 \geq 0$ such that for each $x \in D$

$$F''(x) \leq K_1.$$

Clearly, we have

$$D_0 \subseteq D, \tag{3.1}$$

so for each $t \in [0, \rho_0)$

$$w_0(t) \leq w_1(t), \tag{3.2}$$

$$w(t) \leq w_1(t) \tag{3.3}$$

and

$$K \leq K_1. \tag{3.4}$$

Hence, w, K can replace $w_1,$ K_1 in the semi-local convergence of scheme (1.2) and other schemes using the same function w_1 and parameter K. This way, we expand the convergence region, provide tighter error bounds on the distances $\| x_{n+1} - x_n \|,$ $\| x_n - x_* \|$ and give a more precise information on the location of the solution x_*. These benefits are obtained under the same computational effort as before, since the computation of $w_0,$ w (or K) as special cases. Notice also that condition (a_2) helps us define D_0 and then w (i.e., $w = w(w_0, D)$). The set D_0 contains the iterates x_n and by (3.2) is at least as precise as D.

Define

$$h_0(t) = f_1(t) + (1+t)\,f_2(t) + (1 + t + |\beta|\,t^2)\,f_3(t), \tag{3.5}$$

$$h(t) = \frac{1}{1 - h_0(t)\,t}, \tag{3.6}$$

$$\begin{aligned} f(t, u) = {} & \left(\frac{2}{3}\right)^q g_2((t, u)u + t^2(1 + |\beta| + |\beta|\,t))\,g_2(t, u) \\ & + \frac{1}{q+1}(1 + t + |\beta|\,t^2)\,g_2(t, u)u + \frac{t^2(1 + t + |\beta|\,t^2)}{2(1 - \alpha t)}\,g_2(t, u) \\ & t(1+t)\,g_1(t, u)(1 + t + |\beta|\,t^2)\,g_2(t, u) \\ & \frac{t}{2}(1 + t + |\beta|\,t^2)^2\,g_2(t, u)^2, \end{aligned} \tag{3.7}$$

where

$$f_1(t) = 1 + \frac{1}{2}\frac{t}{1 - \alpha t},$$

$$f_2(t) = \frac{t}{2} + \frac{t}{2(1 - \alpha t)} + \frac{t^2}{2(1 - \alpha t)} + \frac{t^3}{2(1 - \alpha t)^2},$$

$$f_3(t) = tf_2(t) + t(1+t)\,f_2(t) + \frac{t^2(1+t)}{2(1 - \alpha t)}\,f_2(t) + \frac{t}{2}(1+t)^2\,f_2(t)^2,$$

$$g_1(t, u) = \frac{2^{q-1}}{3^q}u + \frac{u}{(q+1)(q+2)} + \frac{(1+\alpha)\,t^2}{2(1 - \alpha t)} + \frac{t^3}{8(1 - \alpha t)^2},$$

$$g_2(t, u) = \left[\left(\frac{2}{3}\right)^q u + t^2 + \frac{(1+t)\,u}{q+1} + \frac{t^2(1+t)}{2(1 - \alpha t)}\right] g_1(t, u) + \frac{t}{2}(1+t)^2\,g_1(t, u)^2.$$

Define $g(t) = h_0(t)\,t - 1,$ notice that $h_0(0) = -1 < 0,$ $h_0\left(\frac{1}{2}\right) > 0,$ so $h_0(t) = 0$ has at least a root in $\left(0, \frac{1}{2}\right)$. Let s^* be the smallest positive root of $h_0(t)\,t - 1 = 0,$ then $s^* < \frac{1}{2}.$

We need some auxiliary results.

Lemma 3.1. *Let the functions* h_0, h, f *be defined in* (3.5), (3.7). *Then*

(a) $h_0(t)$ *and* $h(t)$ *are increasing,* $h_0(t) > 1$ *and* $h(t) > 1$ *for* $0 < t < s^*$;

(b) *For* $t \in (0, s^*)$ *and a fixed* $u > 0$, $f(t, u)$ *is increasing function of t, for* $u > 0$ *and a fixed* $t \in (0, s^*)$, $f(t, u)$ *is increasing function of u.*

(c) *For* $0 < \alpha < 1, t \in (0, s^*)$ *and* $u > 0$, $f(\alpha t, \alpha^{1+q}u) < \alpha^{3+3q} f(t, u)$.

Let $\beta_0 = b, \eta_0 = \eta, \alpha_0 = Kb\eta, b_0 = b\eta w(\eta)$ and $c_0 = h(a_0) f(a_0, b_0)$. Moreover, we define the following sequences

$$\beta_{m+1} = h(a_m)\beta_m, \ \eta_{m+1} = c_m\eta_m,$$

$$a_{m+1} = K\beta_{m+1}\eta_{m+1}, \ b_{m+1} = \beta_{m+1}\eta_{m+1}w(\eta_{m+1}), \ c_{m+1} = h(a_{m+1}) f(a_{m+1}, b_{m+1})$$

where $m \geq 0$. Then it follows that

$$a_{m+1} = h(a_m)c_m a_m, \ b_{m+1} \leq h(a_m)c_m^{1+q} b_m.$$

Lemma 3.2. *Let* s^* *be the smallest positive root of* $h_0(t)t - 1 = 0$. *If*

$$a_0 < s^* \text{ and } h(a_0)c_0 < 1,$$

then (a) *for* $m \geq 0$, *it holds that* $h(a_m) > 0$ *and* $c_m < 1$, (b) *the sequence* $\{a_m\}, \{b_m\}, \{c_m\}$ *and* $\{\eta_m\}$ *are decreasing,* (c) $h_0(a_m)a_m < 1$ *and* $h(a_m)c_m < 1$ *for* $m \geq 0$.

We have the following estimates [19].

(i) $\Gamma_{m+1} = [F'(x_{m+1})]^{-1}$ exists and $\| \Gamma_{m+1} \| \leq h(a_m)\| \Gamma_m \| \leq h(a_m)\beta_m = \beta_{m+1}$,

(ii) $\| \Gamma_{m+1}F(x_{m+1}) \| \leq h(a_m) f(a_m, b_m)\| \Gamma_m F(x_m) \| \leq c_m\eta_m = \eta_{m+1}$,

(iii) $K\| \Gamma_m \| \| \Gamma_m F(x_m) \| \leq K\beta_m\eta_m = a_m$,

(iv) $\| \Gamma_m \| \| \Gamma_m F(x_m) \| w(\| \Gamma_m F(x_m) \|) \leq b_m$,

(v) $\| u_m - x_m \| = \left\| \frac{2}{3}\Gamma_m F(x_m) \right\| \leq \frac{2}{3}\eta_m$,

(vi) $\| v_m - x_m \| = \| \Gamma_m F(x_m) \| \leq \eta_m$,

(vii) $\| y_m - x_m \| \leq f_1(a_m) \| \Gamma_m F(x_m) \|$,

(viii) $\| z_m - y_m \| \leq [1 + a_m] g_2(a_m) \| \Gamma_m F(x_m) \|$,

(ix) $\| x_{m+1} - x_m \| \leq h_0(a_m) \| \Gamma_m F(x_m) \| \leq h_0(a_m) \eta_m$, where $m \geq 0$.

Lemma 3.3. *Let the assumptions of Lemma* 3.2 *and the conditions* (A) *hold. Then* $u_m, v_m, y_m, z_m, x_{m+1}$ *belong to* $U(c_0, \rho\eta)$, *where* $\rho = \dfrac{h_0(a_0)}{1 - c_0}$.

Next, we present the main semi-local convergence result for method (1.2) under the hypotheses (A).

Theorem 3.4. *Let* $F : D \subseteq \mathcal{B}_1 \to \mathcal{B}_2$ *be twice Fréchet differentiable*, *where* $\mathcal{B}_1$ *and* $\mathcal{B}_2$ *are Banach spaces*, *D is a non-empty open convex subset*. *Suppose that* $x_0 \in D$ *and hypotheses* (A) *hold. Let* $\overline{U}(x_0, \rho\eta) \subseteq D$, $a_0 = Kb\eta$, $b_0 = b\eta w(\eta)$, $c_0 = h(a_0) h_0(a_0, b_0)$ *satisfy* $a_0 < s^*$ *and* $h(a_0) c_0 < 1$ *where* $\rho = \dfrac{h_0(a_0)}{1 - c_0}$, s^* *is the smallest positive root of* $h_0(t)t - 1 = 0$ *and* h_0, h, f *are defined previously*. *Then*, *starting from* x_0, *the sequence* $\{x_n\}$ *generated by method* (1.2) *converge to* x_* *with* $x_n, x_* \in U(x_0, \rho\eta)$ *and the solution* x_* *of* $F(x) = 0$ *is unique in* $U(x_0, \tilde{\rho}\eta) \cap D$, *where* $\tilde{\rho} = \dfrac{2}{a_0} - \rho$. *Furthermore*,

$$\| x_n - x_* \| \leq h_0(a_0) r_0^n r^{\frac{(4+3q)^n - 1}{3+q}} \frac{1}{1 - r_0 r^{(4+3q)^n}},$$

where $r_0 = \dfrac{1}{h(a_0)}$, $r = h(a_0) c_0$.

4. Numerical Examples

The numerical examples are presented in this section.

Example 4.1. Let $\mathcal{B}_1 = \mathcal{B}_2 = \mathbb{R}^3$, $\Omega = \overline{U}(0, 1)$, $x_* = (0, 0, 0)^T$. Define function F on Ω for $u = (x, y, z)^T$ by

$$F(u) = \left(e^x - 1, \frac{e-1}{2} y^2 + y, z\right)^T.$$

Then, the Fréchet-derivative is given by

$$F'(v) = \begin{bmatrix} e^x & 0 & 0 \\ 0 & (e-1)y+1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Notice that using the (2.8)-(2.12), conditions, we get $\gamma_0(t) = (e-1)t$, $\gamma(t) = e^{\frac{1}{e-1}}t$, $\lambda_1(t) = \lambda_2(t) = e^{\frac{1}{e-1}}$.

Then using the definition of r_0 and r, we have that

$$r_0 = 0.14444885915244823348935199192056,$$

$$r = 0.041513536254307446815570159515119.$$

Example 4.2. Let $\mathcal{B}_1 = \mathcal{B}_2 = C[0, 1]$, the space of continuous functions defined on $[0, 1]$ and be equipped with the max norm. Let $\Omega = \overline{U}(0, 1)$. Define function F on Ω by

$$F(\varphi)(x) = \varphi(x) - 5\int_0^1 x\theta\varphi(\theta)^3 d\theta. \tag{4.1}$$

We have that

$$F'(\varphi(\xi))(x) = \xi(x) - 15\int_0^1 x\theta\varphi(\theta)^2 \xi(\theta) d\theta, \text{ for each } \xi \in \Omega.$$

Then, we get that $x_* = 0$, $\gamma_0(t) = 7.5t$, $\gamma(t) = 15t$, $\lambda_1(t) = \lambda_2(t) = 15$. This way, we have that

$$r_0 = 0.0029787165027481215216764720565834,$$

$$r = 0.00025772162389070053020029282819792.$$

Example 4.3. Let us return back to the motivational example. Then, we get that $w_0(t) = w(t) = 147t,\ v(t) = 147.$ So, we obtain

$$r_0 = 0.000002280599520303840650292497016504,$$

$$r = 0.0000001563215609454908465374514160.$$

Example 4.4. Let $\mathcal{B}_1 = \mathcal{B}_2 = C[0, 1],\ \ \Omega = \overline{U}(x^*, 1)$ and consider the non-linear integral equation of the mixed Hammerstein-type [1, 2, 6-9, 12] defined by

$$x(s) = \int_0^1 G(s, t)\left(x(t)^{3/2} + \frac{x(t)^2}{2}\right)dt,$$

where the kernel G is the Green's function defined on the interval $[0, 1]\times[0, 1]$ by

$$G(s, t) = \begin{cases}(1-s)t, & t \le s\\ s(1-t), & s \le t.\end{cases}$$

The solution $x_*(s) = 0$ is the same as the solution of equation (1.1), where $F : C[0, 1] \to C[0, 1]$ is defined by

$$F(x)(s) = x(s) - \int_0^1 G(s, t)\left(x(t)^{3/2} + \frac{x(t)^2}{2}\right)dt,$$

Notice that

$$\left\| \int_0^1 G(s, t)\, dt \right\| \le \frac{1}{8}.$$

Then, we have that

$$F'(x)\, y(s) = y(s) - \int_0^1 G(s, t)\left(\frac{2}{3}x(t)^{1/2} + x(t)\right)dt,$$

so since $F'(x^*(s)) = I,$

$$\| F'(x^*)^{-1}(F'(x) - F'(y)) \| \le \frac{1}{8}\left(\frac{3}{2}\| x - y \|^{1/2} + \| x - y \|\right).$$

Then, we get that $\gamma_0(t) = \gamma(t) = \frac{1}{8}\left(\frac{3}{2}t^{1/2} + t\right),\ \ \lambda_1(t) = \lambda_2(t) = 1 + \gamma_0(t).$ So, we obtain

$$r_0 = 0.74068507094596702788891207092092,$$

$$r = 0.57895531889724227703197811933933.$$

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