**Earthline Journal of Mathematical Sciences**
E-ISSN: 2581-8147
Volume 14, Number 5, 2024, Pages 1141-1157
https://doi.org/10.34198/ejms.14524.11411157

# A New Class of Univalent Functions Defined by Differential Operator

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## Abstract

In the present work, we submit and study a new class $\mathcal{S}_{\lambda,\alpha,b}^{z,m,t}(\beta,\gamma,\omega,\mu)$ containing analytic univalent functions defined by new differential operator $D_{\lambda,\alpha}^{z,m,t}$ in the open unit disk $E = \{s \in \mathbb{C}: |s| < 1\}$. We get some geometric properties, such as, coefficient estimate, growth and distortion theorems, convex set, radii of convexity and starlikeness, weighted mean, arithmetic mean and partial sums for functions belonging to the class $\mathcal{S}_{\lambda,\alpha,b}^{z,m,t}(\beta,\gamma,\omega,\mu)$.

## 1. Introduction

Let $\mathcal{A}$ be the class of functions $k$ that are analytic in the open unit disk $E = \{s : s \in \mathbb{C}, |s| < 1\}$ of the form

$$k(s) = s + \sum_{n=2}^{\infty} d_n s^n, \tag{1.1}$$

Let $\mathcal{M}$ be denote the function subclass of $\mathcal{A}$ consisting of functions of the form:

$$k(s) = s - \sum_{n=2}^{\infty} d_n s^n \quad (d_n \geq 0). \tag{1.2}$$

Received: July 5, 2024; Accepted: July 18, 2024; Published: July 24, 2024

2020 Mathematics Subject Classification: 30C45, 30C50.

Keywords and phrases: analytic function, univalent function, differential operator, coefficient estimate, distortion bound, radii of starlikeness and convexity, partial sums.

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If $k(s),\ g(s) \in \mathcal{M}$, then their Hadamard product is

$$k(s) * g(s) = s - \sum_{n=2}^{\infty} d_n f_n s^n = g(s) * \ k(s) \tag{1.3}$$

for

$$k(s) = s - \sum_{n=2}^{\infty} d_n s^n, \quad g(s) = s - \sum_{n=2}^{\infty} f_n s^n.$$

We say that $k(s)$ is starlike in domain $E$ if $k : E \to \mathbb{C}$ is univalent and $k(E)$ is a starlike domain with respect to origin. Then $k(s) \in \mathcal{A}$, is said to be starlike of order $\xi$ if it satisfies

$$Re\left(\frac{sk'(s)}{k(s)}\right) > \xi$$

for some $\xi$ $(0 \le \xi < 1)$ and for all $s \in E$.

Also, a univalent function $k(s) \in \mathcal{A}$ is said to be convex of order $\xi$ if and only if $sk'(s)$ is starlike of order $\xi$. In other words, if

$$Re\left(1 + \frac{sk''(s)}{k'(s)}\right) > \xi$$

for some $\xi$ $(0 \le \xi < 1)$ and for all $s \in E$.

Furthermore, a univalent function $k(s) \in \mathcal{A}$ is said to be close-to-convex of order $\xi$ if

$$Re(sk'(s)) > \xi$$

for some $\xi$ $(0 \le \xi < 1)$ and for all $s \in E$.

Symbolize by $S^\star(\xi)$ and $C^*(\xi)$ the classes of univalent starlike and univalent convex function of order $\xi$, respectively.

Now, by making use of the binomial series

$$(1-\alpha)^t = \sum_{m=0}^{t} \binom{t}{m} (-1)^m \alpha^m, \qquad (t \in \mathbb{N} = \{1,2,\cdots\},\ m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}),$$

for $k \in \mathcal{A}$, $0 < \lambda < 1$, $0 < \alpha < 1$, $m \in \mathbb{N} = \{1,2,\cdots\}$, $b \in \mathbb{C}\backslash Z_0^-$, $z \in \mathbb{C}$, $t \in \mathbb{N}_0$ and motivated by [13] and [3], Rossdy *et al.* [8] introduced the following generalized operator $D_{\lambda,\alpha}^{z,m,t}k(s)$

$$D_{\lambda,\alpha}^{z,m,t}k(s) = s + \sum_{n=2}^{\infty}\left(\frac{1+b}{n+b}\right)^{z}[1+\lambda(n-1)(1-\alpha)^{m}]^{t}d_n s^n. \tag{1.4}$$

The operator $D_{\lambda,\alpha}^{z,m,t}k(s)$ generalized other earlier established operators. For example:

1) For $m = 1$, the operator $D_{\lambda,\alpha}^{z,1,t} = \vartheta_{\alpha,\lambda}^{n,s}$ was introduced and studied by [13].

2) For $t = 0$, the operator $D_{\lambda,\alpha}^{z,m,0} = J_{s,b}$ was introduced and studied by [11].

3) For $z = 0$, $\alpha = 0$ and $z = 0$, $m = 0$, the operator $D_{\lambda,0}^{0,m,t} = D_{\lambda,\alpha}^{0,0,t} = D_{\lambda}^{n}$ was introduced and studied by [1].

4) For $z = 0$, $\alpha = 0$, $\lambda = 1$, and $z = 0$, $m = 0$, $\lambda = 1$ the operator $D_{1,0}^{0,m,t} = D_{1,\alpha}^{0,0,t} = D^n$ was introduced and studied by [9].

The generalized operator is easily reduced to the following relations:

$$D_{\lambda,\alpha}^{0,m,0}k(s) = k(s),$$

$$D_{\lambda,\alpha}^{0,m,1}k(s) = (1-\lambda(1-\alpha)^{m})\,k(s) + \lambda(1-\alpha)^{m}sk'(s) = D_{\lambda,\alpha}^{m}k(s), \tag{1.5}$$

$$\vdots$$

$$D_{\lambda,\alpha}^{z,m,t}k(s) = D_{\lambda,\alpha}^{m}\left(D_{\lambda,\alpha}^{z,m,t-1}k(s)\right). \tag{1.6}$$

To show (1.6), let $k(s)$ be given by (1.1). Then, from (1.5), we have

$$D_{\lambda,\alpha}^{m}k(s) = (1-\lambda(1-\alpha)^{m})\,k(s) + \lambda(1-\alpha)^{m}sk'(s), \tag{1.7}$$

Then, we obtain

$$\begin{aligned} D_{\lambda,\alpha}^{z,m,t}k(s) &= (1-\lambda(1-\alpha)^{m})\left(D_{\lambda,\alpha}^{z,m,t-1}k(s)\right) + \lambda(1-\alpha)^{m}s\left(D_{\lambda,\alpha}^{z,m,t-1}k(s)\right)' \\ &= D_{\lambda,\alpha}^{m}\left(D_{\lambda,\alpha}^{z,m,t-1}k(s)\right). \end{aligned}$$

**Definition 1.1.** A function $k(s)$ in $\mathcal{M}$ is in the class $S_{\lambda,\alpha,b}^{z,m,t}(\beta,\gamma,\omega,\mu)$ if and only if it satisfies the condition:

$$\left|\frac{\beta s^2\left(D_{\lambda,\alpha}^{z,m,t}k(s)\right)'' + \gamma\left(s\left(D_{\lambda,\alpha}^{z,m,t}k(s)\right)' - D_{\lambda,\alpha}^{z,m,t}k(s)\right)}{\omega s\left(D_{\lambda,\alpha}^{z,m,t}k(s)\right)' + (1-\gamma)D_{\lambda,\alpha}^{z,m,t}k(s)}\right| < \mu,$$

where $s \in E, 0 \leq \gamma < 1, 0 \leq \omega < 1, 0 < \mu < 1$ and $0 \leq \beta \leq 1$.

**Remark 1.1.** If $z = 0, t = 0$ and $\beta = 0$, the class $\mathcal{S}_{\lambda,\alpha,b}^{z,m,t}(\beta,\gamma,\omega,\mu)$ shortens to the class $S(\gamma,\alpha,\mu)$ which is introduced by Atshan and Ghawi [2].

Some of the following properties studied for other classes in [2,4,5,6,7,12].

## 2. Coefficient Estimate

In the following theorem, we obtain a necessary and sufficient condition for function to be in the class $\mathcal{S}_{\lambda,\alpha,b}^{z,m,t}(\beta,\gamma,\omega,\mu)$.

**Theorem 2.1.** *Let the function $k$ be defined by* (1.2). *Then* $k(s) \in \mathcal{S}_{\lambda,\alpha,b}^{z,m,t}(\beta,\gamma,\omega,\mu)$ *if and only if*

$$\sum_{n=2}^{\infty}[(n-1)(\beta n+\gamma)+\mu(n\omega+1-\gamma)]\varphi_n^{z,m,t}(\lambda,\alpha,b)d_n \leq \mu(\omega+1-\gamma), \tag{2.1}$$

*where* $\varphi_n^{z,m,t}(\lambda,\alpha,b) = \left(\frac{1+b}{n+b}\right)^z[1+\lambda(n-1)(1-\alpha)^m]^t,\ s \in E,\ 0 \leq \gamma < 1,\ 0 \leq \omega < 1,\ 0 < \mu < 1$ *and* $0 \leq \beta \leq 1$. *The result* (2.1) *is sharp for the function*

$$k(s) = s - \frac{\mu(\omega+1-\gamma)}{[(n-1)(\beta n+\gamma)+\mu(n\omega+1-\gamma)]\varphi_n^{z,m,t}(\lambda,\alpha,b)}s^n. \tag{2.2}$$

**Proof.** Suppose that the inequality (2.1) holds true and $|s| = 1$. Then we obtain

$$\left|\beta s^2\left(D_{\lambda,\alpha}^{z,m,t}k(s)\right)'' + \gamma\left(s\left(D_{\lambda,\alpha}^{z,m,t}k(s)\right)' - D_{\lambda,\alpha}^{z,m,t}k(s)\right)\right|$$
$$- \mu\left|\omega s\left(D_{\lambda,\alpha}^{z,m,t}k(s)\right)' + (1-\gamma)D_{\lambda,\alpha}^{z,m,t}k(s)\right|$$

$$= \left|-\sum_{n=2}^{\infty}(n-1)(\beta n+\gamma)\,\varphi_n^{z,m,t}(\lambda,\alpha,b)d_n s^n\right|$$

$$-\mu\left|(\omega+(1-\gamma))s-\sum_{n=2}^{\infty}(n\omega+1-\gamma)\,\varphi_n^{z,m,t}(\lambda,\alpha,b)d_n s^n\right|$$

$$\leq \sum_{n=2}^{\infty}[(n-1)(\beta n+\gamma)+\mu(n\omega+1-\gamma)]\varphi_n^{z,m,t}(\lambda,\alpha,b)d_n-\mu(\omega+1-\gamma)\leq 0.$$

Hence, by maximum modules principle, $k\in\mathcal{S}_{\lambda,\alpha,b}^{z,m,t}(\beta,\gamma,\omega,\mu)$. Now, assume that $k\in\mathcal{S}_{\lambda,\alpha,b}^{z,m,t}(\beta,\gamma,\omega,\mu)$ so that

$$\left|\frac{\beta s^2\left(D_{\lambda,\alpha}^{z,m,t}k(s)\right)''+\gamma\left(s\left(D_{\lambda,\alpha}^{z,m,t}k(s)\right)'-D_{\lambda,\alpha}^{z,m,t}k(s)\right)}{\omega s\left(D_{\lambda,\alpha}^{z,m,t}k(s)\right)'+(1-\gamma)D_{\lambda,\alpha}^{z,m,t}k(s)}\right|<\mu,\qquad s\in E.$$

Hence

$$\left|\beta s^2\left(D_{\lambda,\alpha}^{z,m,t}k(s)\right)''+\gamma\left(s\left(D_{\lambda,\alpha}^{z,m,t}k(s)\right)'-D_{\lambda,\alpha}^{z,m,t}k(s)\right)\right|$$

$$<\mu\left|\omega s\left(D_{\lambda,\alpha}^{z,m,t}k(s)\right)'+(1-\gamma)D_{\lambda,\alpha}^{z,m,t}k(s)\right|.$$

Therefore, we get

$$\left|\sum_{n=2}^{\infty}-(\beta n+\gamma)(n-1)\,\varphi_n^{z,m,t}(\lambda,\alpha,b)d_n s^n\right|$$

$$<\mu\left|(\omega+(1-\gamma))s-\sum_{n=2}^{\infty}(n\omega+1-\gamma)\,\varphi_n^{z,m,t}(\lambda,\alpha,b)d_n s^n\right|.$$

Thus

$$\sum_{n=2}^{\infty}[(n-1)(\beta n+\gamma)+\mu(n\omega+1-\gamma)]\varphi_n^{z,m,t}(\lambda,\alpha,b)d_n\leq\mu(\omega+1-\gamma)$$

and this proof is complete.

**Corollary 2.1.** *If* $k(s)\in\mathcal{S}_{\lambda,\alpha,b}^{z,m,t}(\beta,\gamma,\omega,\mu)$, *then we have*

$$d_n\leq\frac{\mu(\omega+1-\gamma)}{[(n-1)(\beta n+\gamma)+\mu(n\omega+1-\gamma)]\varphi_n^{z,m,t}(\lambda,\alpha,b)}.\tag{2.3}$$

## 3. Growth and Distortion Theorems

Next, we prove the growth and distortion bounds for the class $\mathcal{S}_{\lambda,\alpha,b}^{z,m,t}(\beta,\gamma,\omega,\mu)$.

**Theorem 3.1.** *If $k$ an analytic function given by (1.2) is in the class $\mathcal{S}_{\lambda,\alpha,b}^{z,m,t}(\beta,\gamma,\omega,\mu)$, then for $0<|s|=r<1$*

$$r-\frac{\mu(\omega+1-\gamma)}{\big(2\beta+\gamma+\mu(2\omega+1-\gamma)\big)\varphi_2^{z,m,t}(\lambda,\alpha,b)}r^2\le|k(s)|$$

$$\le r+\frac{\mu(\omega+1-\gamma)}{\big(2\beta+\gamma+\mu(2\omega+1-\gamma)\big)\varphi_2^{z,m,t}(\lambda,\alpha,b)}r^2.$$

*The bounds are sharp, since the equality are attained by the function*

$$k(s)=s-\frac{\mu(\omega+1-\gamma)}{\big(2\beta+\gamma+\mu(2\omega+1-\gamma)\big)\varphi_2^{z,m,t}(\lambda,\alpha,b)}s^2. \tag{3.1}$$

**Proof.** In view of Theorem 2.1, we have

$$\sum_{n=2}^{\infty}[(n-1)(\beta n+\gamma)+\mu(n\omega+1-\gamma)]\varphi_n^{z,m,t}(\lambda,\alpha,b)d_n\le\mu(\omega+1-\gamma),$$

and

$$\big(2\beta+\gamma+\mu(2\omega+1-\gamma)\big)\varphi_2^{z,m,t}(\lambda,\alpha,b)\sum_{n=2}^{\infty}d_n$$

$$\le\sum_{n=2}^{\infty}[(n-1)(\beta n+\gamma)+\mu(n\omega+1-\gamma)]\varphi_n^{z,m,t}(\lambda,\alpha,b)d_n$$

$$\le\mu(\omega+1-\gamma).$$

Therefore, we have

$$\sum_{n=2}^{\infty}d_n\le\frac{\mu(\omega+1-\gamma)}{\big(2\beta+\gamma+\mu(2\omega+1-\gamma)\big)\varphi_2^{z,m,t}(\lambda,\alpha,b)}$$

Thus, for $k\in\mathcal{S}_{\lambda,\alpha,b}^{z,m,t}(\beta,\gamma,\omega,\mu)$, we obtain

$$|k(s)| = \left| s - \sum_{n=2}^{\infty} d_n\, s^n \right| \leq |s| + |s|^2 \sum_{n=2}^{\infty} d_n$$

$$\leq r + \frac{\mu(\omega + 1 - \gamma)}{\big(2\beta + \gamma + \mu(2\omega + 1 - \gamma)\big)\varphi_2^{z,m,t}(\lambda, \alpha, b)} r^2.$$

The other assertion can be proved as follows

$$|k(s)| = \left| s - \sum_{n=2}^{\infty} d_n\, s^n \right| \geq |s| - |s|^2 \sum_{n=2}^{\infty} d_n$$

$$\geq r - \frac{\mu(\omega + 1 - \gamma)}{\big(2\beta + \gamma + \mu(2\omega + 1 - \gamma)\big)\varphi_2^{z,m,t}(\lambda, \alpha, b)} r^2.$$

This completes the proof.

Similarly, following the same method in Theorem 3.1, we can prove the following

**Theorem 3.2.** *If* $k$ *an analytic function given by* (1.2) *is in the class* $\mathcal{S}_{\lambda,\alpha,b}^{z,m,t}(\beta, \gamma, \omega, \mu)$, *then for* $0 < |s| = r < 1$

$$1 - \frac{2\mu(\omega + 1 - \gamma)}{\big(2\beta + \gamma + \mu(2\omega + 1 - \gamma)\big)\varphi_2^{z,m,t}(\lambda, \alpha, b)} r \leq |k(s)'|$$

$$\leq 1 + \frac{2\mu(\omega + 1 - \gamma)}{\big(2\beta + \gamma + \mu(2\omega + 1 - \gamma)\big)\varphi_2^{z,m,t}(\lambda, \alpha, b)} r.$$

*The result is sharp for the function* $k(s)$ *is given by* (3.1).

**Proof.** For $k \in \mathcal{S}_{\lambda,\alpha,b}^{z,m,t}(\beta, \gamma, \omega, \mu)$, we have

$$|k(s)'| = \left| 1 - \sum_{n=2}^{\infty} n d_n\, s^{n-1} \right| \leq 1 + |s| \sum_{n=2}^{\infty} n d_n$$

$$\leq 1 + \frac{2\mu\big(\omega + (1 - \gamma)\big)}{\big(2\beta + \gamma + \mu(2\omega + 1 - \gamma)\big)\varphi_2^{z,m,t}(\lambda, \alpha, b)} r.$$

On the other hand

$$|k(s)'| = \left|1 - \sum_{n=2}^{\infty} n d_n\, s^{n-1}\right|$$

$$\geq 1 - |s| \sum_{n=2}^{\infty} n d_n$$

$$\geq 1 - \frac{2\mu\big(\omega + (1-\gamma)\big)}{\big(2\beta + \gamma + \mu(2\omega + 1 - \gamma)\big)\varphi_2^{z,m,t}(\lambda, \alpha, b)} r.$$

This completes the proof.

## 4. Convex Set

**Theorem 4.1.** *The class $\mathcal{S}_{\lambda,\alpha,b}^{z,m,t}(\beta,\gamma,\omega,\mu)$ is convex set.*

**Proof.** Let functions $k$ and $g$ be in the class $\mathcal{S}_{\lambda,\alpha,b}^{z,m,t}(\beta,\gamma,\omega,\mu)$. Then for every $0 \leq \vartheta \leq 1$, we must show that

$$(1-\vartheta)k(s) + \vartheta g(s) \in \mathcal{S}_{\lambda,\alpha,b}^{z,m,t}(\beta,\gamma,\omega,\mu). \tag{4.1}$$

We have

$$(1-\vartheta)k(s) + \vartheta g(s) = s - \sum_{n=2}^{\infty} [(1-\vartheta)d_n + \vartheta f_n]\, s^n.$$

So by Theorem 2.1, we get

$$\sum_{n=2}^{\infty} [(n-1)(\beta n + \gamma) + \mu(n\omega + 1 - \gamma)]\varphi_n^{z,m,t}(\lambda, \alpha, b)\, [(1-\vartheta)d_n + \vartheta f_n]$$

$$= (1-\vartheta) \sum_{n=2}^{\infty} [(n-1)(\beta n + \gamma) + \mu(n\omega + 1 - \gamma)]\varphi_n^{z,m,t}(\lambda, \alpha, b)\, d_n$$

$$+\vartheta \sum_{n=2}^{\infty} [(n-1)(\beta n + \gamma) + \mu(n\omega + 1 - \gamma)]\varphi_n^{z,m,t}(\lambda, \alpha, b) f_n$$

$$\leq (1-\vartheta)\mu(\omega + 1 - \gamma) + \vartheta\mu(\omega + 1 - \gamma)$$

$$= \mu(\omega + 1 - \gamma).$$

## 5. Radii of Convexity and Starlikeness

In the next Theorems, we will find the radii of convexity and starlikeness for the functions in the class $\mathcal{S}_{\lambda,\alpha,b}^{z,m,t}(\beta,\gamma,\omega,\mu)$.

**Theorem 5.1.** *Let* $k(s) \in \mathcal{S}_{\lambda,\alpha,b}^{z,m,t}(\beta,\gamma,\omega,\mu)$. *Then the function* $k(s)$ *is univalent convex of order* $\xi$ $(0 \le \xi < 1)$ *in the disk* $|s| < R_1$, *where*

$$R_1 = \inf_n \left[\frac{(1-\xi)[(n-1)(\beta n+\gamma)+\mu(n\omega+1-\gamma)]\varphi_n^{z,m,t}(\lambda,\alpha,b)}{n\mu(n-\xi)(\omega+1-\gamma)}\right]^{\frac{1}{n-1}}, \quad (n \ge 2).$$

*The outcome is sharp for the function* $k(s)$ *given by* (2.2).

**Proof.** It is enough to show that

$$\left|\frac{sk''(s)}{k'(s)}\right| \le 1-\xi \qquad (0 \le \xi < 1),$$

for $|s| < R_1$, we get

$$\left|\frac{sk''(s)}{k'(s)}\right| \le \frac{\sum_{n=2}^{\infty} n(n-1)d_n|s|^{n-1}}{1-\sum_{n=2}^{\infty} nd_n|s|^{n-1}}.$$

Thus

$$\left|\frac{sk''(s)}{k'(s)}\right| \le 1-\xi,$$

if

$$\sum_{n=2}^{\infty} \frac{n(n-\xi)}{1-\xi} d_n|s|^{n-1} \le 1. \tag{5.1}$$

Therefore, by using Theorem 2.1, (5.1) will be true if

$$\frac{n(n-\xi)}{1-\xi}|s|^{n-1} \le \frac{[(n-1)(\beta n+\gamma)+\mu(n\omega+1-\gamma)]\varphi_n^{z,m,t}(\lambda,\alpha,b)}{\mu(\omega+1-\gamma)},$$

and hence

$$|s| \le \left[\frac{(1-\xi)[(n-1)(\beta n+\gamma)+\mu(n\omega+1-\gamma)]\varphi_n^{z,m,t}(\lambda,\alpha,b)}{n\mu(n-\xi)(\omega+1-\gamma)}\right]^{\frac{1}{n-1}}, \quad (n \ge 2).$$

Setting $|s| = R_1$, we get the desired result.

**Theorem 5.2.** *Let $k(s) \in \mathcal{S}_{\lambda,\alpha,b}^{z,m,t}(\beta,\gamma,\omega,\mu)$. Then the function $k(s)$ is univalent starlike of order $\xi$ $(0 \leq \xi < 1)$ in the disk $|s| < R_2$, where*

$$R_2 = \inf_n \left[\frac{(1-\xi)[(n-1)(\beta n+\gamma)+\mu(n\omega+1-\gamma)]\varphi_n^{z,m,t}(\lambda,\alpha,b)}{\mu(n-\xi)(\omega+1-\gamma)}\right]^{\frac{1}{n-1}}, (n \geq 2).$$

*The outcome is sharp for the function $k(s)$ given by* (2.2).

**Proof.** It is enough to show that

$$\left|\frac{sk'(s)}{k(s)} - 1\right| \leq 1-\xi \qquad (0 \leq \xi < 1),$$

for $|s| < R_2$, we get

$$\left|\frac{sk'(s)}{k(s)} - 1\right| \leq \frac{\sum_{n=2}^{\infty}(n-1)d_n|s|^{n-1}}{1-\sum_{n=2}^{\infty}d_n|s|^{n-1}}.$$

Thus

$$\left|\frac{sk'(s)}{k(s)} - 1\right| \leq 1-\xi,$$

if

$$\sum_{n=2}^{\infty}\frac{(n-\xi)}{(1-\xi)}d_n|s|^{n-1} \leq 1. \tag{5.2}$$

Therefore, by using Theorem 2.1, (5.2) will be true if

$$\frac{(n-\xi)}{(1-\xi)}|s|^{n-1} \leq \frac{[(n-1)(\beta n+\gamma)+\mu(n\omega+1-\gamma)]\varphi_n^{z,m,t}(\lambda,\alpha,b)}{\mu(\omega+1-\gamma)},$$

and hence

$$|s| \leq \left[\frac{(1-\xi)[(n-1)(\beta n+\gamma)+\mu(n\omega+1-\gamma)]\varphi_n^{z,m,t}(\lambda,\alpha,b)}{\mu(n-\xi)(\omega+1-\gamma)}\right]^{\frac{1}{n-1}}, \quad (n \geq 2).$$

Setting $|s| = R_2$, we get the desired result.

## 6. Weighted Mean and Arithmetic Mean

**Definition 1.2.** The weighted mean $D_\partial(s)$ of $k(s)$ and $g(s)$ defined by

$$D_\partial(s) = \frac{1}{2}[(1-\partial)k(s) + (1+\partial)g(s)], \qquad 0 < \partial < 1.$$

The next theorem shows the weighted mean with relation to this class.

**Theorem 6.1.** *Let $k(s)$ and $g(s)$ be in the class $\mathcal{S}_{\lambda,\alpha,b}^{z,m,t}(\beta,\gamma,\omega,\mu)$. Then the weighted mean of $k(s)$ and $g(s)$ is also in the class $\mathcal{S}_{\lambda,\alpha,b}^{z,m,t}(\beta,\gamma,\omega,\mu)$.*

**Proof.** By Definition of the weighted mean, we have

$$\begin{aligned} D_\partial(s) &= \frac{1}{2}[(1-\partial)k(s) + (1+\partial)g(s)] \\ &= \frac{1}{2}\left[(1-\partial)\left(s - \sum_{n=2}^{\infty} d_n\, s^n\right) + (1+\partial)\left(s - \sum_{n=2}^{\infty} f_n\, s^n\right)\right] \\ &= s - \sum_{n=2}^{\infty} \frac{1}{2}[(1-\partial)d_n + (1+\partial)f_n]\, s^n. \end{aligned}$$

Since $k(s)$ and $g(s)$ are in the class $\mathcal{S}_{\lambda,\alpha,b}^{z,m,t}(\beta,\gamma,\omega,\mu)$, so by Theorem 2.1, we get

$$\sum_{n=2}^{\infty}[(n-1)(\beta n+\gamma) + \mu(n\omega+1-\gamma)]\varphi_n^{z,m,t}(\lambda,\alpha,b)d_n \le \mu(\omega+1-\gamma)$$

and

$$\sum_{n=2}^{\infty}[(n-1)(\beta n+\gamma) + \mu(n\omega+1-\gamma)]\varphi_n^{z,m,t}(\lambda,\alpha,b)f_n \le \mu(\omega+1-\gamma).$$

Hence,

$$\sum_{n=2}^{\infty}[(n-1)(\beta n+\gamma) + \mu(n\omega+1-\gamma)]\varphi_n^{z,m,t}(\lambda,\alpha,b)\left[\frac{1}{2}(1-\partial)d_n + \frac{1}{2}(1+\partial)f_n\right]$$

$$= \frac{1}{2}(1-\partial)\sum_{n=2}^{\infty}[(n-1)(\beta n+\gamma) + \mu(n\omega+1-\gamma)]\varphi_n^{z,m,t}(\lambda,\alpha,b)d_n$$

$$+\frac{1}{2}(1+\partial)\sum_{n=2}^{\infty}[(n-1)(\beta n+\gamma)+\mu(n\omega+1-\gamma)]\varphi_n^{z,m,t}(\lambda,\alpha,b)\mathscr{f}_n$$

$$\leq \frac{1}{2}\mu(1-\partial)(\omega+1-\gamma)+\frac{1}{2}\mu(1+\partial)(\omega+1-\gamma)=\mu(\omega+1-\gamma).$$

This shows that $D_\partial(s)\in \mathcal{S}_{\lambda,\alpha,b}^{z,m,t}(\beta,\gamma,\omega,\mu)$.

In the following theorem, we will show that the class $\mathcal{S}_{\lambda,\alpha,b}^{z,m,t}(\beta,\gamma,\omega,\mu)$ is closed under arithmetic mean.

**Theorem 6.2.** *Let $k_1(s),k_2(s),k_3(s),\cdots,k_\vartheta(s)$ that defined by*

$$k_\ell(s)=s-\sum_{n=2}^{\infty}d_{n,\ell}s^n,\qquad \left(d_{n,\ell}\geq 0,\ell=1,2,\dots,\vartheta,n\geq 2\right),\tag{6.1}$$

*be a member of the class $\mathcal{S}_{\lambda,\alpha,b}^{z,m,t}(\beta,\gamma,\omega,\mu)$. Then the arithmetic mean of $k_\ell(s)(\ell=1,2,\dots,\vartheta)$ that defined by*

$$h(s)=\frac{1}{\vartheta}\sum_{\ell=1}^{\vartheta}k_\ell(s)\tag{6.2}$$

*is also in the class $\mathcal{S}_{\lambda,\alpha,b}^{z,m,t}(\beta,\gamma,\omega,\mu)$.*

**Proof.** By (6.1) and (6.2), we can write

$$h(s)=\frac{1}{\vartheta}\sum_{\ell=1}^{\vartheta}\left(s-\sum_{n=2}^{\infty}d_{n,\ell}s^n\right)=s-\sum_{n=2}^{\infty}\left(\frac{1}{\vartheta}\sum_{\ell=1}^{\vartheta}d_{n,\ell}\right)s^n.$$

Since $k_\ell(s)\in \mathcal{S}_{\lambda,\alpha,b}^{z,m,t}(\beta,\gamma,\omega,\mu)$ for every $(\ell=1,2,\dots,\vartheta)$, so by Theorem 2.1, we have

$$\sum_{n=2}^{\infty}[(n-1)(\beta n+\gamma)+\mu(n\omega+1-\gamma)]\varphi_n^{z,m,t}(\lambda,\alpha,b)\left(\frac{1}{\vartheta}\sum_{\ell=1}^{\vartheta}d_{n,\ell}\right)$$

$$=\frac{1}{\vartheta}\sum_{\ell=1}^{\vartheta}\left(\sum_{n=2}^{\infty}[(n-1)(\beta n+\gamma)+\mu(n\omega+1-\gamma)]\varphi_n^{z,m,t}(\lambda,\alpha,b)\,d_{n,\ell}\right)$$

$$\leq \frac{1}{\vartheta}\sum_{\ell=1}^{\vartheta}\mu(\omega+1-\gamma)=\mu(\omega+1-\gamma).$$

The proof is complete.

## 7. Partial Sums

This section will be examined the ratio of a function of the form (1.2) to its sequence of partial sums defined by $k_1(s)=s$ and $k_q(s)=s-\sum_{n=2}^{\infty}d_n\,s^n$, when the coefficients of $k$ are sufficiently small to satisfy the condition (2.1). We will establish sharp lower bounds for

$$Re\left(\frac{k(s)}{k_q(s)}\right), Re\left(\frac{k_q(s)}{k(s)}\right), Re\left(\frac{k'(s)}{k_q'(s)}\right) \text{ and } Re\left(\frac{k_q'(s)}{k'(s)}\right).$$

In what follows, we will use the well know outcome that

$$Re\left(\frac{1-\mathcal{Z}(s)}{1+\mathcal{Z}(s)}\right)>0 \qquad (s\in E),$$

if and only if

$$\mathcal{Z}(s)=\sum_{n=1}^{\infty}\mathscr{f}_n s^n,$$

satisfies the inequality $|\mathcal{Z}(s)|\leq|s|$.

**Theorem 7.1.** *If* $k(s)\in\mathcal{S}_{\lambda,\alpha,b}^{z,m,t}(\beta,\gamma,\omega,\mu)$, *then*

$$Re\left(\frac{k(s)}{k_q(s)}\right)\geq 1-\frac{1}{\mathscr{f}_{q+1}} \qquad (s\in E, q\in\mathbb{N}) \tag{7.1}$$

*and*

$$Re\left(\frac{k_q(s)}{k(s)}\right)\geq\frac{\mathscr{f}_{q+1}}{1+\mathscr{f}_{q+1}} \qquad (s\in E, q\in\mathbb{N}), \tag{7.2}$$

*where*

$$\mathscr{f}_n=\frac{[(n-1)(\beta n+\gamma)+\mu(n\omega+1-\gamma)]\varphi_n^{z,m,t}(\lambda,\alpha,b)}{\mu(\omega+1-\gamma)}.$$

*The estimations in* (7.1) *and* (7.2) *are sharp***.**

**Proof.** We utilize the same method used by Silverman [10]**.** The function $k(s) \in \mathcal{S}_{\lambda,\alpha,b}^{z,m,t}(\beta,\gamma,\omega,\mu)$ if and only if

$$\sum_{n=2}^{q} \not{f}_n d_n \leq 1.$$

It is easy to verify that $\not{f}_{q+1} > \not{f}_q > 1$. Thus

$$\sum_{n=2}^{q} d_n + \not{f}_{q+1} \sum_{n=q+1}^{\infty} d_n \leq \sum_{n=2}^{\infty} \not{f}_n d_n \leq 1. \tag{7.3}$$

We may write

$$\not{f}_{q+1}\left\{\frac{k(s)}{k_q(s)} - \left(1 - \frac{1}{\not{f}_{q+1}}\right)\right\} = \frac{1 - \sum_{n=2}^{q} d_n\, s^{n-1} - \not{f}_{q+1} \sum_{n=q+1}^{\infty} d_n\, s^{n-1}}{1 - \sum_{n=2}^{q} d_n\, s^{n-1}} = \frac{1 + D(s)}{1 + F(s)}.$$

Set

$$\frac{1 + D(s)}{1 + F(s)} = \frac{1 - \mathcal{Z}(s)}{1 + \mathcal{Z}(s)},$$

so that

$$\mathcal{Z}(s) = \frac{F(s) - D(s)}{2 + D(s) + F(s)}.$$

Then

$$\mathcal{Z}(s) = \frac{\not{f}_{q+1} \sum_{n=q+1}^{\infty} d_n\, s^{n-1}}{2 - 2\sum_{n=2}^{q} d_n\, s^{n-1} - \not{f}_{q+1} \sum_{n=q+1}^{\infty} d_n\, s^{n-1}}$$

and

$$|\mathcal{Z}(s)| \leq \frac{\not{f}_{q+1} \sum_{n=q+1}^{\infty} d_n}{2 - 2\sum_{n=2}^{q} d_n - \not{f}_{q+1} \sum_{n=q+1}^{\infty} d_n}.$$

Now $|\mathcal{Z}(s)| \leq 1$ if and only if

$$\sum_{n=2}^{q} d_n + \not{f}_{q+1} \sum_{n=q+1}^{\infty} d_n \leq 1,$$

this is supported by (7.3). Assertion (7.1) of Theorem (7.1) is easily produced by this. To see that

$$k(s) = s - \frac{s^{q+1}}{\not{f}_{q+1}}, \tag{7.4}$$

gives sharp results, we observe that

$$\frac{k(s)}{k_q(s)} = 1 - \frac{s^q}{\not{f}_{q+1}}.$$

Lettering $s \to 1^-$, we have

$$\frac{k(s)}{k_q(s)} = 1 - \frac{1}{\not{f}_{q+1}},$$

which shows that the bounds in (7.1) are the best possible for each $k \in \mathbb{N}$.

Similarly, we take

$$\begin{aligned}(1 + \not{f}_{q+1})\left(\frac{k_q(s)}{k(s)} - \frac{\not{f}_{q+1}}{1 + \not{f}_{q+1}}\right) \\ = \frac{1 - \sum_{n=2}^{q} d_n\, s^{n-1} + \not{f}_{q+1} \sum_{n=q+1}^{\infty} d_n\, s^{n-1}}{1 - \sum_{n=2}^{q} d_n\, s^{n-1}} \\ = \frac{1 - \mathcal{Z}(s)}{1 + \mathcal{Z}(s)},\end{aligned}$$

where

$$|\mathcal{Z}(s)| \leq \frac{(1 + \not{f}_{q+1}) \sum_{n=q+1}^{\infty} d_n}{2 - 2\sum_{n=2}^{q} d_n + (1 - \not{f}_{q+1}) \sum_{n=q+1}^{\infty} d_n}.$$

Now $|\mathcal{Z}(s)| \leq 1$ if and only if

$$\sum_{n=2}^{q} d_n + \not{f}_{q+1} \sum_{n=q+1}^{\infty} d_n \leq 1,$$

which is supported by (7.3). This instantly leads to the assertion (7.2) of Theorem (7.1). The estimate in (7.2) is sharp with the extremal function $k(s)$ given by (7.4). Proof of Theorem 7.1 is now complete.

**Theorem 7.2.** *If* $k(s) \in \mathcal{S}_{\lambda,\alpha,b}^{z,m,t}(\beta, \gamma, \omega, \mu)$, *then*

$$Re\left(\frac{k'(s)}{k_q'(s)}\right) \geq 1 - \frac{q+1}{\not{f}_{q+1}} \qquad (s \in E, q \in \mathbb{N}), \tag{7.5}$$

*and*

$$Re\left(\frac{k_q'(s)}{k'(s)}\right) \geq \frac{\mathfrak{f}_{q+1}}{q+1+\mathfrak{f}_{q+1}} \qquad (s \in E, q \in \mathbb{N}). \tag{7.6}$$

*The estimates in* (7.5) *and* (7.6) *are sharp with the extremal function given by* (7.4).

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