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Upper Bounds for Certain Families of m-Fold Symmetric Bi-Univalent Functions Associating Bazilevic Functions with λ-Pseudo Functions

Zainab Swayeh Ghali[1] and Abbas Kareem Wanas[2,*]

[1] Department of Mathematics, College of Education, University of Al-Qadisiyah, Diwaniya, Iraq
e-mail: sci.math.mas.22.3@qu.edu.iq

[2] Department of Mathematics, College of Education, University of Al-Qadisiyah, Diwaniya, Iraq
e-mail: abbas.kareem.w@qu.edu.iq

Abstract

In this paper, we introduce and study a new families $W_{\Sigma_m}(\lambda,\gamma,\delta;\alpha)$, $W^*_{\Sigma_m}(\lambda,\gamma,\delta;\beta)$, $M_{\Sigma_m}(\lambda,\gamma,\delta;\alpha)$ and $M^*_{\Sigma_m}(\lambda,\gamma,\delta;\beta)$ of holomorphic and m-fold symmetric bi-univalent functions associating the Bazilevic functions with λ-pseudo functions defined in the open unit disk U. We find upper bounds for the first two Taylor-Maclaurin $|a_{m+1}|$ and $|a_{2m+1}|$ for functions in these families. Further, we point out several special cases for our results.

1. Introduction

Let $\mathcal{A}$ be the family of functions f of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k , \tag{1.1}$$

which are holomorphic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by the conditions $f(0) = f'(0) - 1 = 0$.

We also denote by S the subfamily of $\mathcal{A}$ consisting of functions satisfying (1.1) which are also univalent in U.

A function $f \in \mathcal{A}$ is called starlike of order δ $(0 \le \delta < 1)$, if

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*Corresponding author

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \delta, \quad (z \in U).$$

Singh [25] introduced and studied Bazilevic function that is the function f such that

$$\operatorname{Re}\left\{\frac{z^{1-\gamma} f'(z)}{(f(z))^{1-\gamma}}\right\} > 0, \quad (z \in U, \gamma \geq 0).$$

On the other hand, a function $f \in \mathcal{A}$ is called a λ-Pseudo-starlike function in U if (see [5])

$$Re\left\{\frac{z(f'(z))^{\lambda}}{f(z)}\right\} > 0, \quad (\lambda \geq 1, z \in U).$$

According to the Koebe one-quarter theorem (see [9]), every function $f \in S$ has an inverse f^{-1} which satisfies

$$f^{-1}(f(z)) = z, \quad (z \in U)$$

and

$$f(f^{-1}(w)) = w, \ (|w| < r_0(f), r_0(f) \geq \tfrac{1}{4}),$$

where

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots . \quad (1.2)$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in U if both f and f^{-1} are univalent in U. We denote by Σ the family of bi-univalent functions in U given by (1.1). For a brief history and interesting examples in the family Σ see the pioneering work on this subject by Srivastava et al. [28], which actually revived the study of bi-univalent functions in recent years. In a considerably large number of sequels to the aforementioned work of Srivastava et al. [28], several different sub families of the bi-univalent function family Σ were introduced and studied analogously by the many authors (see, for example, [2,3,13,17,21,26,32,33,35,38]).

For each function $f \in S$, the function $h(z) = \sqrt[m]{f(z^m)}$, $(z \in U, m \in \mathbb{N})$ is univalent and maps the unit disk U into a region with m-fold symmetry. A function is said to be m-fold symmetric (see [14]) if it has the following normalized form:

$$f(z) = z + \sum_{k=1}^{\infty} a_{mk+1} z^{mk+1}, \qquad (z \in U, m \in \mathbb{N}). \quad (1.3)$$

We denote by S_m the family of m-fold symmetric univalent functions in U, which are normalized by the series expansion (1.3). In fact, the functions in the family S are one-fold symmetric.

In [26] Srivastava et al. defined m-fold symmetric bi-univalent functions analogues to the concept of m-fold symmetric univalent functions. They gave some important results, such as each function $f \in \Sigma$ generates an m-fold symmetric bi-univalent function for each $m \in \mathbb{N}$. Furthermore, for the normalized form of f given by (1.3), they obtained the series expansion for f^{-1} as follows:

$$g(w) = w - a_{m+1}w^{m+1} + [(m+1)a_{m+1}^2 - a_{2m+1}]w^{2m+1}$$
$$-\left[\frac{1}{2}(m+1)(3m+2)a_{m+1}^3 - (3m+2)a_{m+1}a_{2m+1} + a_{3m+1}\right]w^{3m+1} + \cdots, \quad (1.4)$$

where $f^{-1} = g$. We denote by Σ_m the family of m-fold symmetric bi-univalent functions in U. It is easily seen that for $m = 1$, the formula (1.4) coincides with the formula (1.2) of the family Σ. Some examples of m-fold symmetric bi-univalent functions are given as follows:

$$\left(\frac{z^m}{1-z^m}\right)^{\frac{1}{m}}, \ \left[\frac{1}{2}\log\left(\frac{1+z^m}{1-z^m}\right)\right]^{\frac{1}{m}} \ \text{ and } \ [-\log(1-z^m)]^{\frac{1}{m}}$$

with the corresponding inverse functions

$$\left(\frac{w^m}{1+w^m}\right)^{\frac{1}{m}}, \ \left(\frac{e^{2w^m}-1}{e^{2w^m}+1}\right)^{\frac{1}{m}} \ \text{ and } \ \left(\frac{e^{w^m}-1}{e^{w^m}}\right)^{\frac{1}{m}},$$

respectively.

Recently, many authors investigated bounds for various subfamilies of m-fold bi-univalent functions (see [1,4,7,15,20,22,24,27,29,30,33,36,37]).

In order to prove our main results, we require the following lemma.

Lemma 1.1 [3]. *If $h \in \mathcal{P}$, then $|c_k| \le 2$ for each $k \in \mathbb{N}$, where $\mathcal{P}$ is the family of all f all functions h holomorphic in U for which*

$$Re\big(h(z)\big) > 0, (z \in U),$$

where

$$h(z) = 1 + c_1 z + c_2 z^2 + \cdots, (z \in U).$$

2. Coefficient Estimates for the Function Family $W_{\Sigma_m}(\mu,\gamma,\lambda;\alpha)$

Definition 2.1. A function $f\in\Sigma_m$ given by (1.3) is said to be in the family $W_{\Sigma_m}(\mu,\gamma,\lambda;\alpha)$ $(0\le\mu\le1, 0\le\gamma\le1, \lambda\ge0; 0\le\alpha\le1, m\in\mathbb{N},\ z,w\in U)$ if it satisfies the following conditions:

$$\left|\arg\left((1-\mu)\frac{z^{1-\gamma}f'(z)}{\left(f(z)\right)^{1-\gamma}}+\mu\frac{z\left(f'(z)\right)^{\lambda}}{f(z)}\right)\right|<\frac{\alpha\pi}{2} \tag{2.1}$$

and

$$\left|\arg\left((1-\mu)\frac{w^{1-\gamma}g'(w)}{\left(g(w)\right)^{1-\gamma}}+\mu\frac{w(g'(w))^{\lambda}}{g(w)}\right)\right|<\frac{\alpha\pi}{2}, \tag{2.2}$$

where the function $g=f^{-1}$ is given by (1.4).

In particular, for one-fold symmetric bi-univalent functions, we denote the family $W_{\Sigma_1}(\mu,\gamma,\lambda;\alpha)=W_{\Sigma}(\mu,\gamma,\lambda;\alpha)$.

Remark 2.1. It should be remarked that the family $W_{\Sigma_m}(\mu,\gamma,\lambda;\alpha)$ is a generalization of well-known families consider earlier. These families are:

1. For $m=1$, we have

$$W_{\Sigma_m}(\mu,\gamma,\lambda;\alpha)=\mathcal{T}_{\Sigma}(\mu,\gamma,\lambda;\alpha),$$

where $\mathcal{T}_{\Sigma}(\mu,\gamma,\lambda;\alpha)$ is the bi-univalent function family studied recently by Srivastava et al. [31].

2. For $\mu=0$ and $m=1$, we have

$$W_{\Sigma_m}(0,\gamma,\lambda;\alpha)=P_{\Sigma}(\alpha,\gamma),$$

where $P_{\Sigma}(\alpha,\gamma)$ is the bi-univalent function family studied recently by Prema and Keerthi [19].

3. For $\mu=m=1$, we have

$$W_{\Sigma_m}(1,\gamma,\lambda;\alpha)=\mathcal{L}B_{\Sigma}^{\lambda}(\alpha),$$

where $\mathcal{L}B_{\Sigma}^{\lambda}(\alpha)$ denote the bi-univalent function family studied by Joshi et al. [12].

4. For $\mu=\gamma=0$ and $m=1$, we have

$$W_{\Sigma_m}(0,0,\lambda;\alpha)=:S_\Sigma^*(\alpha),$$

where $S_\Sigma^*(\alpha)$ is the bi-univalent function family introduced by Brannan and Taha [8].

5. For $\mu=0$ and $\gamma=m=1$, we have

$$W_{\Sigma_m}(0,1,\lambda;\alpha)=:\mathcal{H}_\Sigma^\alpha,$$

where $\mathcal{H}_\Sigma^\alpha$ denote the bi-univalent function family investigated in the aforementioned pioneering work by Srivastava et al. [28].

Theorem 2.1. *Let the function* $f\in W_{\Sigma_m}(\mu,\gamma,\lambda;\alpha)$ $(0\le\mu\le 1;\ 0\le\gamma\le 1;\ \lambda\ge 0;\ 0\le\alpha\le 1)$ *be given by* (1.1). *Then*

$$|a_{m+1}|\le\frac{2\alpha}{\sqrt{\left|\begin{array}{c}\alpha[(m+1)[(1-\mu)(\gamma+2m)+\mu(\lambda(2m+1)-1)]\\ +2\left[\frac{1}{2}(1-\mu)(\gamma-1)(\gamma+2m)+\left(\frac{1}{2}\lambda(m+1)\big((\lambda-1)(m+1)-2\big)+1\right)\right]\\ +(1-\alpha)[(1-\mu)(\gamma+m)+\mu(\lambda(m+1)-1)]^2\end{array}\right|}}$$

and

$$|a_{2m+1}|\le\frac{4(m+1)\alpha^2}{[(1-\mu)(\gamma+m)+\mu(\lambda(m+1)-1)]^2}+\frac{2\alpha}{[(1-\mu)(\gamma+2m)+\mu(\lambda(2m+1)-1)]}.$$

Proof. In light of the conditions (2.1) and (2.2), we have

$$(1-\mu)\frac{z^{1-\gamma}f'(z)}{\big(f(z)\big)^{1-\gamma}}+\mu\frac{z(f'(z))^\lambda}{f(z)}=[p(z)]^\alpha \tag{2.3}$$

and

$$(1-\mu)\frac{w^{1-\gamma}g'(w)}{\big(g(w)\big)^{1-\gamma}}+\mu\frac{w(g'(w))^\lambda}{g(w)}=[q(w)]^\alpha, \tag{2.4}$$

where $g=f^{-1}$ and the functions $p,q\in\mathcal{P}$ have the following series representations:

$$p(z)=1+p_m z^m+p_{2m}z^{2m}+p_{3m}z^{3m}+\cdots \tag{2.5}$$

$$q(w)=1+q_m w^m+q_{2m}w^{2m}+q_{3m}w^{3m}+\cdots. \tag{2.6}$$

By comparing the corresponding coefficient of (2.3) and (2.4), we find that

$$[(1-\mu)(\gamma+m)+\mu(\lambda(m+1)-1)]a_{m+1}=\alpha p_m, \tag{2.7}$$

$$[(1-\mu)(\gamma+2m)+\mu(\lambda(2m+1)-1)]a_{2m+1}$$
$$+\left[\frac{1}{2}(1-\mu)(\gamma-1)(\gamma+2m)+\mu\left(\frac{1}{2}\lambda(m+1)\big((\lambda-1)(m+1)-2\big)+1\right)\right]a_{m+1}^2$$
$$=\alpha p_{2m}+\frac{\alpha(\alpha-1)}{2}p_m^2, \tag{2.8}$$

$$-[(1-\mu)(\gamma+m)+\mu(\lambda(m+1)-1)]a_{m+1}=\alpha q_m\,, \tag{2.9}$$

and

$$[(1-\mu)(\gamma+2m)+\mu(\lambda(2m+1)-1)]\left((m+1)a_{m+1}^2-a_{2m+1}\right)$$
$$+\left[\frac{1}{2}(1-\mu)(\gamma-1)(\gamma+2m)+\mu\left(\frac{1}{2}\lambda(m+1)\big((\lambda-1)(m+1)-2\big)+1\right)\right]a_{m+1}^2$$
$$=\alpha q_{2m}+\frac{\alpha(\alpha-1)}{2}q_m^2. \tag{2.10}$$

Making use of (2.7) and (2.9), we conclude that

$$p_m=-q_m \tag{2.11}$$

and

$$2[(1-\mu)(\gamma+m)+\mu(\lambda(m+1)-1)]^2a_{m+1}^2=\alpha^2(p_m^2+q_m^2). \tag{2.12}$$

If we add (2.8) to (2.10), we obtain

$$(m+1)[(1-\mu)(\gamma+2m)+\mu(\lambda(2m+1)-1)]$$
$$+2\left[\frac{1}{2}(1-\mu)(\gamma-1)(\gamma+2m)+\mu(\frac{1}{2}\lambda(m+1)\big((\lambda-1)(m+1)-2\big)+1\right]a_{m+1}^2$$
$$=\alpha(p_{2m}+q_{2m})+\frac{\alpha(\alpha-1)}{2}(p_m^2+q_m^2). \tag{2.13}$$

Subsisting the value of $(p_m^2+q_m^2)$ from (2.12) in to the right-hand side of (2.13), and after some computations, we deduce that

$$a_{m+1}^2$$
$$=\frac{\alpha^2(p_{2m}+q_{2m})}{\begin{array}{c}\alpha[(m+1)[(1-\mu)(\gamma+2m)+\mu(\lambda(2m+1)-1)]\\ +2\left[\frac{1}{2}(1-\mu)(\gamma-1)(\gamma+2m)+\mu\left(\frac{1}{2}\lambda(m+1)\big((\lambda-1)(m+1)-2\big)+1\right)\right]\\ +(1-\alpha)[(1-\mu)(\gamma+m)+\mu(\lambda(m+1)-1)]^2\end{array}}. \tag{2.14}$$

Now, taking the absolute value of both sides of (2.14), and applying Lemma 1.1 for the coefficients p_{2m} and q_{2m}, we have

$$|a_{m+1}| \leq \frac{2\alpha}{\sqrt{\begin{array}{c}|\alpha[(m+1)[(1-\mu)(\gamma+2m)+\mu(\lambda(2m+1)-1)] \\ +2\left[\frac{1}{2}(1-\mu)(\gamma-1)(\gamma+2m)+\left(\frac{1}{2}\lambda(m+1)\big((\lambda-1)(m+1)-2\big)+1\right)\right] \\ +(1-\alpha)[(1-\mu)(\gamma+m)+\mu(\lambda(m+1)-1)]^2|\end{array}}}.$$

This gives the desired estimate for $|a_{m+1}|$.

Next, in order to determinate the bound on $|a_{2m+1}|$, by subtracting (2.10) from (2.8), we get

$$[(1-\mu)(\gamma+2m)+\mu(\lambda(2m+1)-1)](2a_{2m+1}-(m+1)a_{m+1}^2)$$
$$= \alpha(p_{2m}-q_{2m})+\frac{\alpha(\alpha-1)}{2}(p_m^2-q_m^2). \tag{2.15}$$

Now, upon substituting the value of a_{m+1}^2 from (2.12) in to (2.15) and using (2.11), we find that

$$a_{2m+1} = \frac{(m+1)\alpha^2(p_m^2+q_m^2)}{2[(1-\mu)(\gamma+m)+\mu(\lambda(m+1)-1)]^2}$$
$$+\frac{\alpha^2(p_{2m+1}-q_{2m+1})}{2[(1-\mu)(\gamma+2m)+\mu(\lambda(2m+1)-1)]}. \tag{2.16}$$

Finally, by taking the absolute value of (2.16) and applying Lemma 1.1 once again for the coefficients p_m, p_{2m}, q_m and q_{2m}, we obtain

$$|a_{2m+1}| \leq \frac{4(m+1)\alpha^2}{[(1-\mu)(\gamma+m)+\mu(\lambda(m+1)-1)]^2}$$
$$+\frac{2\alpha}{[(1-\mu)(\gamma+2m)+\mu(\lambda(2m+1)-1)]}.$$

This completes the proof of Theorem 2.1.

Remark 2.2. By selecting particular value of μ, m and γ in our main results (Theorem 2.1), we can derive a number of known results. Some of these special cases are recorded below.

1. Taking $m = 1$ in Theorem 2.1, we obtain the results which were proven by Srivastava et al. [31, Theorem 2.1].

2. Taking $\mu = 0$ and $m = 1$ in Theorem 2.1, we obtain the results which were proven by Prema and Keerthi [19, Theorem 2.2].

3. Taking $\mu = m = 1$ in Theorem 2.1, we obtain the results which were given by Joshi et al. [12, Theorem 1].

4. Taking $\mu = \gamma = 0$ and $m = 1$ in Theorem 2.1, we obtain the results which were derived by Murugusundaramoorthy et al. [18, Corollaries 6].

5. Taking $\mu = 0$ and $\gamma = m = 1$ in Theorem 2.1, we obtain the results which were obtained by Srivastava et al. [28, Theorem 1].

3. Coefficient Estimates for the Function Family $W^*_{\Sigma_m}(\mu, \gamma, \lambda; \beta)$

Definition 3.1. A function $f \in \Sigma_m$ given by (1.3), is said to be in the family $W^*_{\Sigma_m}(\mu, \gamma, \lambda; \beta)$ $(0 \le \mu \le 1,\ 0 \le \gamma \le 1,\ \lambda \ge 0;\ 0 \le \beta \le 1,\ m \in \mathbb{N},\ z, w \in U)$ if it satisfies the following conditions:

$$\operatorname{Re}\left((1-\mu)\frac{z^{1-\gamma} f'(z)}{\left(f(z)\right)^{1-\gamma}} + \mu \frac{z\left(f'(z)\right)^{\lambda}}{f(z)}\right) > \beta \tag{3.1}$$

and

$$\operatorname{Re}\left((1-\mu)\frac{w^{1-\gamma} g'(w)}{\left(g(w)\right)^{1-\gamma}} + \mu \frac{w\left(g'(w)\right)^{\lambda}}{g(w)}\right) > \beta, \tag{3.2}$$

where the function $g = f^{-1}$ is given by (1.4).

In particular, for one-fold symmetric bi-univalent functions, we denote the family $W^*_{\Sigma_1}(\mu, \gamma, \lambda; \beta) = W^*_{\Sigma}(\mu, \gamma, \lambda; \beta)$.

Remark 3.1. It should be remarked that the family $W^*_{\Sigma_m}(\mu, \gamma, \lambda; \beta)$ is a generalization of well-known families consider earlier. These families are:

1. For $m = 1$, we have

$$W^*_{\Sigma_m}(\mu, \gamma, \lambda; \beta) = \mathcal{T}^*_{\Sigma}(\mu, \gamma, \lambda; \beta),$$

where $\mathcal{T}_\Sigma^*(\mu,\gamma,\lambda;\alpha)$ are the bi-univalent function classes studied recently by Srivastava et al. [31].

2. For $\mu = 0$ and $m = 1$, we have

$$W_{\Sigma_m}^*(0,\gamma,\lambda;\beta) =: P_\Sigma(\beta,\gamma),$$

where $P_\Sigma(\beta,\gamma)$ are the bi-univalent function classes studied recently by Prema and Keerthi [19].

3. For $\mu = 0$ and $m = 1$, we have

$$W_{\Sigma_m}^*(1,\gamma,\lambda;\beta) =: \mathcal{L}B_\Sigma(\lambda,\beta),$$

where $\mathcal{L}B_\Sigma(\lambda,\beta)$ denote the bi-univalent function classes studied by Joshi et al. [12].

4. For $\mu = \gamma = 0$ and $m = 1$, we have

$$W_{\Sigma_m}^*(0,0,\lambda;\beta) =: S_\Sigma^*(\beta).$$

where $S_\Sigma^*(\beta)$ are the bi-univalent function classes introduced by Brannan and Taha [8].

5. For $\mu = 0$ and $\gamma = m = 1$, we have

$$W_{\Sigma_m}^*(0,1,\lambda;\beta) =: \mathcal{H}_\Sigma(\beta).$$

where $\mathcal{H}_\Sigma(\beta)$ denote the bi-univalent function classes investigated in the aforementioned pioneering work by Srivastava et al. [28].

Our second main result is asserted by Theorem 3.1.

Theorem 3.1. *Let the function* $f \in W_{\Sigma_m}^*(\mu,\gamma,\lambda;\beta)$ $(0 \le \beta < 1;\ 0 \le \mu \le 1;\ \gamma \ge 0;\ \lambda \ge 1)$ *be given by* (1.1). *Then*

$$|a_{m+1}| \le 2\sqrt{\frac{1-\beta}{\begin{array}{c}(m+1)[(1-\mu)(\gamma+2m)+\mu(\lambda(2m+1)-1)]\\ +2[\frac{1}{2}(1-\mu)(\gamma-1)(\gamma+2m)+\mu\left(\frac{1}{2}\lambda(m+1)\big((\lambda-1)(m+1)-2\big)+1\right]\end{array}}} \tag{3.3}$$

and

$$|a_{2m+1}| \le \frac{4(1-\beta)^2}{[(1-\mu)(\gamma+m)+\mu(\lambda(m+1)-1)]^2}$$

$$+\frac{2(1-\beta)}{[(1-\mu)(\gamma+2m)+\mu(\lambda(2m+1)-1)]}. \tag{3.4}$$

Proof. In view of the conditions (3.1) and (3.2), we have

$$(1-\mu)\frac{z^{1-\gamma}f'(z)}{\left(f(z)\right)^{1-\gamma}}+\mu\frac{z(f'(z))^{\lambda}}{f(z)}=\beta+(1-\beta)p(z) \tag{3.5}$$

and

$$(1-\mu)\frac{w^{1-\gamma}g'(w)}{\left(g(w)\right)^{1-\gamma}}+\mu\frac{w(g'(w))^{\lambda}}{g(w)}=\beta+(1-\beta)q(w), \tag{3.6}$$

where $g=f^{-1}$ and the functions $p,q \in \mathcal{P}$ have the series expansions given by (2.5) and (2.6), respectively. Thus, by comparing the corresponding coefficient in (3.5) and (3.6), we find that

$$[(1-\mu)(\gamma+m)+\mu(\lambda(m+1)-1)]a_{m+1}=(1-\beta)p_m, \tag{3.7}$$

$$\begin{aligned}&[(1-\mu)(\gamma+2m)+\mu(\lambda(2m+1)-1)]a_{2m+1}\\ &\quad+\left[\frac{1}{2}(1-\mu)(\gamma-1)(\gamma+2m)+\mu\left(\frac{1}{2}\lambda(m+1)\big((\lambda-1)(m+1)-2\big)+1\right)\right]a_{m+1}^2\\ &\qquad=(1-\beta)p_{2m},\end{aligned} \tag{3.8}$$

$$-[(1-\mu)(\gamma+m)+\mu(\lambda(m+1)-1]a_{m+1}=(1-\beta)q_m \tag{3.9}$$

and

$$\begin{aligned}&[(1-\mu)(\gamma+2m)+\mu(\lambda(2m+1)-1)]\left((m+1)a_{m+1}^2-a_{2m+1}\right)\\ &\quad+\left[\frac{1}{2}(1-\mu)(\gamma-1)(\gamma+2m)+\mu\left(\frac{1}{2}\lambda(m+1)\big((\lambda-1)(m+1)-2\big)+1\right)\right]a_{m+1}^2\\ &\qquad=(1-\beta)q_{2m}.\end{aligned} \tag{3.10}$$

Making use of (3.7) and (3.9), we conclude that

$$p_m=-q_m \tag{3.11}$$

and

$$2[(1-\mu)(\gamma+m)+\mu(\lambda(m+1)-1)]^2a_{m+1}^2=(1-\beta)^2(p_m^2+q_m^2). \tag{3.12}$$

If we add (3.8) to (3.10), we obtain

$$(m+1)[(1-\mu)(\gamma+2m)+\mu(\lambda(2m+1)-1)]$$
$$+2\left[\frac{1}{2}(1-\mu)(\gamma-1)(\gamma+2m)+\mu\left(\frac{1}{2}\lambda(m+1)\big((\lambda-1)(m+1)-2\big)+1\right)\right]a_{m+1}^2$$
$$=(1-\beta)(p_{2m}+q_{2m}). \tag{3.13}$$

Consequently, we have

$$a_{m+1}^2=\frac{(1-\beta)(p_{2m}+q_{2m})}{\begin{array}{c}(m+1)[(1-\mu)(\gamma+2m)+\mu(\lambda(2m+1)-1)]\\ +2\left[\frac{1}{2}(1-\mu)(\gamma-1)(\gamma+2m)+\mu\left(\frac{1}{2}\lambda(m+1)\big((\lambda-1)(m+1)-2\big)+1\right)\right]\end{array}}.$$

Next, by applying the Lemma 1.1 for the coefficients p_{2m} and q_{2m}, we have

$$|a_{m+1}|\le 2\sqrt{\frac{1-\beta}{\begin{array}{c}(m+1)[(1-\mu)(\gamma+2m)+\mu(\lambda(2m+1)-1)]\\ +2\left[\frac{1}{2}(1-\mu)(\gamma-1)(\gamma+2m)+\mu\left(\frac{1}{2}\lambda(m+1)\big((\lambda-1)(m+1)-2\big)+1\right)\right]\end{array}}}.$$

In order to determinate the bound on $|a_{2m+1}|$, by subtracting (3.10) from (3.8), we get

$$[(1-\mu)(\gamma+2m)+\mu(\lambda(2m+1)-1)](2a_{2m+1}-(m+1))a_{m+1}^2)=(1-\beta)(p_{2m}-q_{2m}).$$

or, equivalently,

$$a_{2m+1}=\frac{(m+1)}{2}a_{m+1}^2+\frac{(1-\beta)(p_{2m}-q_{2m})}{2[(1-\mu)(\gamma+2m)+\mu(\lambda(2m+1)-1)]}. \tag{3.14}$$

Now, upon substituting the value of a_{m+1}^2 from (3.12) in to (3.14), it follows that

$$a_{2m+1}=\frac{(m+1)(1-\beta)^2(p_m^2+q_m^2)}{2[(1-\mu)(\gamma+m)+\mu(\lambda(m+1)-1)]^2}+\frac{(1-\beta)(p_{2m}-q_{2m})}{2[(1-\mu)(\gamma+2m)+\mu(\lambda(2m+1)-1)]}.$$

Finally, by applying Lemma 1.1 once again for the coefficients p_m, p_{2m}, q_m and q_{2m}, we get

$$|a_{2m+1}|\le\frac{4(m+1)(1-\beta)^2}{[(1-\mu)(\gamma+m)+\mu(\lambda(m+1)-1)]^2}+\frac{2(1-\beta)}{[(1-\mu)(\gamma+2m)+\mu(\lambda(2m+1)-1)]}.$$

This completes the proof of Theorem 3.1.

Remark 3.2. By selecting particular value of μ and γ in our main results (Theorem 3.1), we can derive a number of known results. Some of these special cases are recorded below.

1. Taking $m = 1$ in Theorem 3.1, we obtain the results which were proven by Srivastava et al. [31, Theorem 2.1].

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4. Coefficient Estimates for the Function Family $M_{\Sigma_m}(\mu, \gamma, \lambda; \alpha)$

Definition 4.1. A function $f \in \Sigma_m$ given by (1.3) is said to be in the family $M_{\Sigma_m}(\mu, \gamma, \lambda; \alpha)$ $(0 \leq \mu \leq 1,\ 0 \leq \gamma \leq 1,\ \lambda \geq 0;\ 0 < \alpha \leq 1,\ m \in \mathbb{N},\ z, w \in U)$ if it satisfies the following conditions:

$$\left| \arg \left((1-\mu)(1 + \frac{z^{2-\gamma} f''(z)}{\left(zf'(z)\right)^{1-\gamma}}) + \mu \frac{\left(\left(zf'(z)\right)'\right)^{\lambda}}{f'(z)} \right) \right| < \frac{\alpha\pi}{2} \tag{4.1}$$

and

$$\left| \arg \left((1-\mu)(1 + \frac{w^{2-\gamma} g''(w)}{\left(zg'(w)\right)^{1-\gamma}}) + \mu \frac{\left(\left(wg'(w)\right)'\right)^{\lambda}}{g'(w)} \right) \right| < \frac{\alpha\pi}{2}, \tag{4.2}$$

where the function $g = f^{-1}$ is given by (1.4).

In particular, for one-fold symmetric bi-univalent functions, we denote the family $M_{\Sigma_1}(\mu, \gamma, \lambda; \alpha) = M_{\Sigma}(\mu, \gamma, \lambda; \alpha)$.

Remark 4.1. For $\mu = 0$, we have

$$M_{\Sigma}(0, \gamma, \lambda; \alpha) =: \mathfrak{B}_{\Sigma}(\gamma, \alpha),$$

where $\mathfrak{B}_{\Sigma}(\gamma, \alpha)$ is the bi-univalent function family studied recently by Sakar and Wanas [24].

Theorem 4.1. *Let the function* $f \in M_{\Sigma_m}(\mu, \gamma, \lambda; \alpha)$ $(0 \leq \mu \leq 1;\ 0 \leq \gamma \leq 1;\ \lambda \geq 0;\ 0 < \alpha \leq 1)$ *be given by* (1.1). *Then*

$$|a_{m+1}| \leq \frac{2\alpha}{\sqrt{\begin{array}{c}|\alpha[(m+1)(2m+1)[2m-\mu(2m-\lambda(2m+1)+1)] \\ +2(m+1)^2\left[\mu\lambda(m+1)\left(\frac{1}{2}(\lambda-1)(m+1)-1\right)+\mu(m-m\gamma+1)+m(\gamma-1)\right]\Big] \\ +(1-\alpha)\left[(1-\mu)m(m+1)+\mu\big(m(\lambda m+3\lambda-2)\big)\right]^2\Big|\end{array}}}. \tag{4.3}$$

and

$$|a_{2m+1}| \leq \frac{4\alpha^2}{\left[(1-\mu)m(m+1)+\mu\big(m(\lambda m+3\lambda-2)\big)\right]^2} + \frac{2\alpha}{(2m+1)[2m-\mu(2m-\lambda(2m+1)+1)]}. \tag{4.4}$$

Proof. In light of the conditions (4.1) and (4.2), we have

$$(1-\mu)\frac{z^{2-\gamma}f''(z)}{\big(zf'(z)\big)^{1-\gamma}} + \mu\frac{((zf'(z))')^{\lambda}}{f'(z)} = [p(z)]^{\alpha} \tag{4.5}$$

and

$$(1-\mu)\frac{w^{2-\gamma}g''(w)}{\big(zg'(w)\big)^{1-\gamma}} + \mu\frac{((wg'(w))')^{\lambda}}{g'(w)} = [q(w)]^{\alpha}, \tag{4.6}$$

where $g = f^{-1}$ and the functions $p, q \in \mathcal{P}$ have the series expansions given by (2.5) and (2.6), respectively. Thus, by comparing the corresponding coefficient in (4.5) and (4.6), yields

$$\left[(1-\mu)m(m+1)+\mu\big(m(\lambda m+3\lambda-2)\big)\right]a_{m+1} = \alpha p_m, \tag{4.7}$$

$$(2m+1)[2m-\mu(2m-\lambda(2m+1)+1)]a_{2m+1}$$
$$+(m+1)^2\left[\mu\lambda(m+1)\left(\frac{1}{2}(\lambda-1)(m+1)-1\right)+\mu(m-m\gamma+1)+m(\gamma-1)\right]a_{m+1}^2$$
$$=\alpha p_{2m}+\frac{\alpha(\alpha-1)}{2}p_m^2, \tag{4.8}$$

$$-[(1-\mu)m(m+1)+\mu(m(\lambda m+3\lambda-2))]a_{2m+1}=\alpha q_m \tag{4.9}$$

and

$$(2m+1)[2m-\mu(2m-\lambda(2m+1)+1)]\left((m+1)a_{m+1}^2-a_{2m+1}\right)$$
$$+(m+1)^2\left[\mu\lambda(m+1)\left(\frac{1}{2}(\lambda-1)(m+1)-1\right)+\mu(m-m\gamma+1)+m(\gamma-1)\right]a_{m+1}^2$$
$$=\alpha q_{2m}+\frac{\alpha(\alpha-1)}{2}q_m^2, \tag{4.10}$$

Making use of (4.7) and (4.9), we conclude that

$$p_m=-q_m \tag{4.11}$$

and

$$2[(1-\mu)m(m+1)+\mu(m(\lambda m+3\lambda-2))]^2a_{m+1}^2=\alpha^2(p_m^2+q_m^2). \tag{4.12}$$

If we add (4.8) to (4.10), we obtain

$$(m+1)(2m+1)[2m-\mu(2m-\lambda(2m+1)+1)]$$
$$+2(m+1)^2\left[\mu\lambda(m+1)\left(\frac{1}{2}(\lambda-1)(m+1)-1\right)+\mu(m-m\gamma+1)+m(\gamma-1)\right]a_{m+1}^2$$
$$=\alpha(p_{2m}+q_{2m})+\frac{\alpha(\alpha-1)}{2}(p_m^2+q_m^2). \tag{4.13}$$

Subsisting the value of $(p_m^2+q_m^2)$ from (4.12) in to the right-hand side of (4.13), and after some computations, we deduce that

$$a_{m+1}^2=\frac{\alpha^2(p_{2m}+q_{2m})}{\begin{array}{c}\alpha[(m+1)(2m+1)[2m-\mu(2m-\lambda(2m+1)+1)]\\ +2(m+1)^2\left[\mu\lambda(m+1)\left(\frac{1}{2}(\lambda-1)(m+1)-1\right)+\mu(m-m\gamma+1)+m(\gamma-1)\right]\Big]\\ +(1-\alpha)\left[(1-\mu)m(m+1)+\mu\left(m(\lambda m+3\lambda-2)\right)\right]^2\end{array}}. \tag{4.14}$$

By taking the absolute value of both sides of (4.14), and applying the Lemma 1.1 for the coefficients p_{2m} and q_{2m}, we have

$$|a_{m+1}| \le \frac{2\alpha}{\sqrt{\left|\begin{array}{c}\alpha[(m+1)(2m+1)[2m-\mu(2m-\lambda(2m+1)+1)] \\ +2(m+1)^2\left[\mu\lambda(m+1)\left(\frac{1}{2}(\lambda-1)(m+1)-1\right)+\mu(m-m\gamma+1)+m(\gamma-1)\right]\Big] \\ +(1-\alpha)\left[(1-\mu)m(m+1)+\mu\big(m(\lambda m+3\lambda-2)\big)\right]^2\end{array}\right|}}.$$

This gives the desired estimate for $|a_{m+1}|$ asserted in (4.3).

Next, in order to determinate the bound on $|a_{2m+1}|$, by subtracting (4.10) from (4.8), we get

$$(2m+1)[2m-\mu(2m-\lambda(2m+1)+1)](2a_{2m+1}-(m+1)a_{m+1}^2) = \alpha(p_{2m}-q_{2m})+\frac{\alpha(\alpha-1)}{2}(p_m^2-q_m^2). \tag{4.15}$$

Now, upon substituting the value of a_{m+1}^2 from (4.12) into (4.15) and using (4.12), we find that

$$a_{2m+1} = \frac{(m+1)\alpha^2(p_m^2+q_m^2)}{2\left[(1-\mu)m(m+1)+\mu\big(m(\lambda m+3\lambda-2)\big)\right]^2} + \frac{\alpha(p_{2m}-q_{2m})}{2(2m+1)[2m-\mu(2m-\lambda(2m+1)+1)]}. \tag{4.16}$$

Taking the absolute value of (4.16) and applying Lemma 1.1 once again for the coefficients p_m, p_{2m}, q_m and q_{2m}, we obtain

$$|a_{2m+1}| \le \frac{4\alpha^2}{\left[(1-\mu)m(m+1)+\mu\big(m(\lambda m+3\lambda-2)\big)\right]^2} + \frac{2\alpha}{(2m+1)[2m-\mu(2m-\lambda(2m+1)+1)]}.$$

This completes the proof of Theorem 4.1.

Remark 4.2. Taking $\mu = 0$ and $m = 1$ in Theorem 4. 1, we obtain the results which were proven by Sakar and Wanas [24, Theorem 2.1]

5. Coefficient Estimates for the Function Family $M^*_{\Sigma_m}(\mu,\gamma,\lambda;\beta)$

Definition 5.1. A function $f \in \Sigma_m$ given by (1.3) is said to be in the family $M^*_{\Sigma_m}(\mu,\gamma,\lambda;\beta)$ $(0 \le \mu \le 1,\ 0 \le \gamma \le 1,\ \lambda \ge 0;\ 0 \le \beta < 1,\ m \in \mathbb{N},\ z,w \in U)$ if it satisfies the following conditions:

$$Re\left((1-\mu)(1+\frac{z^{2-\gamma}f''(z)}{\left(zf'(z)\right)^{1-\gamma}})+\mu\frac{\left(\left(zf'(z)\right)'\right)^{\lambda}}{f'(z)}\right)>\beta \tag{5.1}$$

and

$$Re\left((1-\mu)(1+\frac{w^{2-\gamma}g''(w)}{\left(zg'(w)\right)^{1-\gamma}})+\mu\frac{\left(\left(wg'(w)\right)'\right)^{\lambda}}{g'(w)}\right)>\beta, \tag{5.2}$$

where the function $g = f^{-1}$ is given by (1.4).

In particular, for one-fold symmetric bi-univalent functions, we denote the family $M^*_{\Sigma_1}(\mu,\gamma,\lambda;\beta) = M^*_{\Sigma}(\mu,\gamma,\lambda;\beta)$.

Remark 5.1. For $\mu = 0$, we have

$$M^*_{\Sigma}(0,\gamma,\lambda;\beta) =: \mathfrak{B}^*_{\Sigma}(\gamma,\beta),$$

where $\mathfrak{B}^*_{\Sigma}(\gamma,\beta)$ are the bi-univalent function classes studied recently by Sakar and Wanas [24].

Theorem 5.1. *Let the function* $f \in M^*_{\Sigma_m}(\mu,\gamma,\lambda;\beta)$ $(0 \le \mu \le 1;\ 0 \le \gamma \le 1;\ \lambda \ge 0;0\ \le \beta < 1\ )$ *be given by* (1.1). *Then*

$$|a_{m+1}| \le 2\sqrt{\frac{(1-\beta)}{\left|\begin{matrix}(m+1)(2m+1)[2m-\mu(2m-\lambda(2m+1)+1)] \\ +2(m+1)^2\left[\mu\lambda(m+1)\left(\frac{1}{2}(\lambda-1)(m+1)-1\right)+\mu(m-m\gamma+1)+m(\gamma-1)\right]\end{matrix}\right|}}. \tag{5.3}$$

and

$$|a_{2m+1}| \leq \frac{4(m+1)(1-\beta)^2}{\left[(1-\mu)m(m+1)+\mu\big(m(\lambda m+3\lambda-2)\big)\right]^2} + \frac{2(1-\beta)}{(2m+1)[2m-\mu(2m-\lambda(2m+1)+1)]}. \tag{5.4}$$

Proof. In light of the conditions (5.1) and (5.2), we have

$$(1-\mu)(1+\frac{z^{2-\gamma}f''(z)}{\big(zf'(z)\big)^{1-\gamma}})+\mu\frac{\Big(\big(zf'(z)\big)'\Big)^{\lambda}}{f'(z)} = \beta+(1-\beta)p(z) \tag{5.5}$$

and

$$(1-\mu)\left(1+\frac{w^{2-\gamma}g''(w)}{\big(zg'(w)\big)^{1-\gamma}}\right)+\mu\frac{\Big(\big(wg'(w)\big)'\Big)^{\lambda}}{g'(w)} = \beta+(1+\beta)q(w), \tag{5.6}$$

where $g = f^{-1}$ and the functions $p, q \in \mathcal{P}$ have the series expansions given by (2.5) and (2.6), respectively. Thus, by comparing the corresponding coefficient in (5.5) and (5.6), yields.

$$\left[(1-\mu)m(m+1)+\mu\big(m(\lambda m+3\lambda-2)\big)\right]a_{m+1} = (1-\beta)p_m\,, \tag{5.7}$$

$$(2m+1)[2m-\mu(2m-\lambda(2m+1)+1)]a_{2m+1}$$
$$+(m+1)^2\left[\mu\lambda(m+1)\left(\frac{1}{2}(\lambda-1)(m+1)-1\right)+\mu(m-m\gamma+1)+m(\gamma-1)\right]a_{m+1}^2$$
$$= (1-\beta)p_{2m}, \tag{5.8}$$

$$-\left[(1-\mu)m(m+1)+\mu\big(m(\lambda m+3\lambda-2)\big)\right]a_{2m+1} = (1-\beta)q_m \tag{5.9}$$

and

$$(2m+1)[2m-\mu(2m-\lambda(2m+1)+1)]\left((m+1)a_{m+1}^2-a_{2m+1}\right)$$
$$+(m+1)^2\left[\mu\lambda(m+1)\left(\frac{1}{2}(\lambda-1)(m+1)-1\right)+\mu(m-m\gamma+1)+m(\gamma-1)\right]a_{m+1}^2$$
$$= (1-\beta)q_{2m}, \tag{5.10}$$

Making use of (5.7) and (5.9), we conclude that

$$p_m = -q_m \tag{5.11}$$

and

$$2\left[(1-\mu)m(m+1)+\mu\big(m(\lambda m+3\lambda-2)\big)\right]^2 a_{m+1}^2=(1-\beta)^2(p_m^2+q_m^2). \quad (5.12)$$

If we add (5.8) to (5.10), we obtain

$$(m+1)(2m+1)[2m-\mu(2m-\lambda(2m+1)+1)]$$
$$+2(m+1)^2\left[\mu\lambda(m+1)\left(\frac{1}{2}(\lambda-1)(m+1)-1\right)+\mu(m-m\gamma+1)+m(\gamma-1)\right]a_{m+1}^2$$
$$=(1-\beta)(p_{2m}+q_{2m}). \quad (5.13)$$

Therefore, we have

$$a_{m+1}^2=\frac{(1-\beta)(p_{2m}+q_{2m})}{\begin{array}{c}(m+1)(2m+1)[2m-\mu(2m-\lambda(2m+1)+1)]\\+2(m+1)^2\left[\mu\lambda(m+1)\left(\frac{1}{2}(\lambda-1)(m+1)-1\right)+\mu(m-m\gamma+1)+m(\gamma-1)\right]\end{array}}. \quad (5.14)$$

By taking the absolute value of both sides of (5.14), and applying the Lemma 1.1 for the coefficients p_{2m} and q_{2m}, we have

$$|a_{m+1}|\le 2\sqrt{\frac{(1-\beta)}{\left|\begin{array}{c}(m+1)(2m+1)[2m-\mu(2m-\lambda(2m+1)+1)]\\+2(m+1)^2\left[\mu\lambda(m+1)\left(\frac{1}{2}(\lambda-1)(m+1)-1\right)+\mu(m-m\gamma+1)+m(\gamma-1)\right]\end{array}\right|}}.$$

This gives the desired estimate for $|a_{m+1}|$ asserted in (5.3).

Next, in order to determinate the bound on $|a_{2m+1}|$, by subtracting (5.10) from (5.8), we get

$$(2m+1)[2m-\mu(2m-\lambda(2m+1)+1)](2a_{2m+1}-(m+1)a_{m+1}^2)$$
$$=(1-\beta)(p_{2m}-q_{2m}). \quad (5.15)$$

Now, upon substituting the value of a_{m+1}^2 from (5.12) in to (5.15) and using (5.11), we find that

$$a_{2m+1}=\frac{(m+1)(1-\beta)^2(p_m^2+q_m^2)}{2\left[(1-\mu)m(m+1)+\mu\big(m(\lambda m+3\lambda-2)\big)\right]^2}$$

$$+\frac{(1-\beta)(p_{2m}-q_{2m})}{(2m+1)[2m-\mu(2m-\lambda(2m+1)+1)]}. \tag{5.16}$$

Taking the absolute value of (5.16) and applying Lemma 1.1 once again for the coefficients p_m, p_{2m}, q_m and q_{2m}, we obtain

$$|a_{2m+1}| \leq \frac{4(m+1)(1-\beta)^2}{\left[(1-\mu)m(m+1)+\mu\big(m(\lambda m+3\lambda-2)\big)\right]^2} + \frac{2(1-\beta)}{(2m+1)[2m-\mu(2m-\lambda(2m+1)+1)]}.$$

This completes the proof of Theorem 5.1.

Remark 5.2. Taking $\mu = 0$ and $m = 1$ in Theorem 5.1, we obtain the results which were proven by Sakar and Wanas [24, Theorem 3.1].

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