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New Class of Multivalent Functions Defined by Generalized (p, q)-Bernard Integral Operator

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Abstract

Making use of the generalized (p,q)-Bernardi integral operator, we introduce and study a new class $\mathcal{FJ}_{p,q}^{m}(\alpha,\delta,\lambda,\gamma)$ of multivalent analytic functions with negative coefficients in the open unit disk E. Several geometric characteristics are obtained, like, coefficient estimate, radii of convexity, close-to-convexity and starlikeness, closure theorems, extreme points, integral means inequalities, neighborhood property and convolution properties for functions belonging to the class $\mathcal{FJ}_{p,q}^{m}(\alpha,\delta,\lambda,\gamma)$.

1. Introduction

Let $\mathcal{A}_p$ represent the function class that has been normalised by

$$k(s) = s^p + \sum_{n=p+1}^{\infty} d_n s^n \quad (s \in E; n \geq p+1; p \in \mathbb{N} = \{1, 2, \cdots\}), \tag{1.1}$$

which in the open unit disk $E = \{s : s \in \mathbb{C} \text{ and } |s| < 1\}$ are analytic and p-valent.

Let $\mathcal{M}_p$ represent the function subclass of $\mathcal{A}_p$ made up of the following functions:

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$$k(s) = s^p - \sum_{n=p+1}^{\infty} d_n s^n \quad (s \in E;\ d_n \geq 0;\ n \geq p+1;\ p \in \mathbb{N} = \{1, 2, \cdots\}). \quad (1.2)$$

For the function $k(s) \in \mathcal{M}_p$, given by (1.2), and $g(s) \in \mathcal{M}_p$ given by

$$g(s) = s^p - \sum_{n=p+1}^{\infty} \not{f}_n s^n \quad (s \in E;\ \not{f}_n \geq 0;\ n \geq p+1;\ p \in \mathbb{N}), \quad (1.3)$$

The definition of the convolution between $k(s)$ and $g(s)$ is the convolution of $k(s)$ and $g(s)$ is defined by

$$(k * g)(s) = s^p - \sum_{n=p+1}^{\infty} d_n \not{f}_n s^n = (g * k)(s). \quad (1.4)$$

If the function $k(s) \in \mathcal{A}_p$ meets the following criteria, it is referred to as p-valent starlike of order ξ $(0 \leq \xi < p)$.

$$Re\left(\frac{sk'(s)}{k(s)}\right) > \xi \quad (s \in E; 0 \leq \xi < p; p \in \mathbb{N}). \quad (1.5)$$

If the function $k(s) \in \mathcal{A}_p$ meets the following criteria, it is referred to as p-valent convex of order ξ $(0 \leq \xi < p)$.

$$Re\left(1 + \frac{sk''(s)}{k'(s)}\right) > \xi \quad (s \in E; 0 \leq \xi < p;\ p \in \mathbb{N}). \quad (1.6)$$

Denote the class of p-valent starlikes of order ξ by $S_n^*(p, \xi)$ and represent the class of p-valent convex of order ξ by $C_n(p, \xi)$, which Owa [15] investigated. Observations show that

$$k(s) \in C_n(p, \xi) \leftrightarrow \frac{sk'(s)}{p} \in S_n^*(p, \xi).$$

If the function $k(s) \in \mathcal{A}_p$ meets the following criteria, it is referred to as p-valent close to convex of order ξ $(0 \leq \xi < p)$.

$$Re\left(\frac{k'(s)}{s^{p-1}}\right) > \xi \quad (s \in E; 0 \leq \xi < p;\ p \in \mathbb{N}). \quad (1.7)$$

More recently, Srivastava *et al.* [21] studied the generalized (p, q)-Bernardi integral operator of multivalent functions as follows:

Definition 1.1. For $k(s) \in \mathcal{A}_p$, the generalized (p, q)-Bernardi integral operator of multivalent functions $ß^p_{m,q}: \mathcal{A}_p \to \mathcal{A}_p$ is defined by

$$ß^p_{m,q} k(s) := \begin{cases} ß^p_{1,q}\left(ß^p_{m-1,q} k(s)\right), & (m \in \mathbb{N}) \\ k(s), & (m = 0), \end{cases} \tag{1.8}$$

where $ß^p_{1,q} k(s)$ is given by

$$\begin{aligned} ß^p_{1,q} k(s) &= \frac{[p+\omega]_q}{s^\omega} \int_0^s t^{\omega-1} k(t) d_q t \\ &= s^p + \sum_{n=p+1}^{\infty} \frac{[p+\omega]_q}{[n+\omega]_q} d_n s^n \quad (\omega > -p; s \in E), \end{aligned} \tag{1.9}$$

Thus, in the particular instance when $p = 1$, results in the well-known q-Bernardi integral operator indicated by (see to [14]).

$$\begin{aligned} ß^1_{1,q} k(s) &= \frac{[1+\omega]_q}{s^\omega} \int_0^s t^{\omega-1} k(t) d_q t \\ &= s + \sum_{n=2}^{\infty} \frac{[1+\omega]_q}{[n+\omega]_q} d_n s^n \quad (\omega > -1; s \in E). \end{aligned} \tag{1.10}$$

From $ß^p_{1,q} k(s)$, we conclude that

$$ß^p_{2,q} k(s) = ß^p_{1,q}\left(ß^p_{1,q} k(s)\right) = s^p + \sum_{n=p+1}^{\infty} \left(\frac{[p+\omega]_q}{[n+\omega]_q}\right)^2 d_n s^n, \quad (\omega > -p)$$

and

$$ß^p_{m,q} k(s) = s^p + \sum_{n=p+1}^{\infty} \left(\frac{[p+\omega]_q}{[n+\omega]_q}\right)^m d_n s^n \qquad (m \in \mathbb{N}, \omega > -p). \tag{1.11}$$

Definition 1.2. We say that class $\mathcal{FJ}^m_{p,q}(\alpha, \delta, \lambda, \gamma)$ contain function $k(s) \in \mathcal{M}_p$ if and only if satisfies the condition:

$$\Re\left\{\frac{s\left(ß^p_{m,q} k(s)\right)' + \delta s^2 \left(ß^p_{m,q} k(s)\right)''}{(1-\lambda)s^p + (1-\delta)\lambda ß^p_{m,q} k(s) + \delta\lambda s\left(ß^p_{m,q} k(s)\right)'}\right\}$$

$$> \alpha \left| \frac{s\left(ß_{m,q}^{p} k(s)\right)' + \delta s^2 \left(ß_{m,q}^{p} k(s)\right)''}{(1-\lambda)s^p + (1-\delta)\lambda ß_{m,q}^{p} k(s) + \delta \lambda s \left(ß_{m,q}^{p} k(s)\right)'} - p \right| + \gamma, \qquad (1.12)$$

wherever $s \in E,\ \alpha, q \geq 0,\ 0 \leq \gamma < p,\ 0 \leq \lambda \leq 1,\ 0 \leq \delta \leq 1,\ m \in \mathbb{N}, n \geq p+1;\ p \in \mathbb{N} = \{1, 2, 3, \cdots\}$.

Remark 1.1. The following different subclasses as studied by various authors when $m = 0$.

1) When $p = 1$, the class $\mathcal{FJ}_{p,q}^{m}(\alpha, \delta, \lambda, \gamma)$ abbreviates to the class $\mathcal{NA}(\beta, \lambda, \gamma, \alpha)$ that Ajil [1] presented and discussed.

2) When $\alpha = 0$ and $\lambda = 1$, the class $\mathcal{FJ}_{p,q}^{m}(\alpha, \delta, \lambda, \gamma)$ abbreviates to the class $\mathcal{T}_n(p, \alpha, \lambda)$ that Altintas *et al.* [4] presented and discussed.

3) When $\alpha = 0,\ \lambda = 1$ and $p = 1$, the class $\mathcal{FJ}_{p,q}^{m}(\alpha, \delta, \lambda, \gamma)$ abbreviates to the class $\mathrm{P}(n, \alpha, \lambda)$ that Altintas [3] presented and discussed.

4) When $\alpha = 0,\ \lambda = 1$ and $\delta = 1$, the class $\mathcal{FJ}_{p,q}^{m}(\alpha, \delta, \lambda, \gamma)$ abbreviates to the class $C_n(p, \alpha)$ that Owa [15] presented and discussed.

5) When $\alpha = 0,\ \lambda = 1, \delta = 1$ and $p = 1$, the class $\mathcal{FJ}_{p,q}^{m}(\alpha, \delta, \lambda, \gamma)$ abbreviates to the class $C(\alpha)$ that Silverman [20] presented and discussed.

6) When $\lambda = 1$ and $\delta = 0$, the class $\mathcal{FJ}_{p,q}^{m}(\alpha, \delta, \lambda, \gamma)$ abbreviates to the class $UST(\alpha, \beta, p)$ that Khaimar and More [12] presented and discussed.

7) When $\alpha = 0,\ \lambda = 1, \delta = 0$ and $p = 1$, the class $\mathcal{FJ}_{p,q}^{m}(\alpha, \delta, \lambda, \gamma)$ abbreviates to the class $\mathcal{T}^*(\alpha)$ that Silverman [20] presented and discussed.

8) When $p = 1$ and$\delta = 1$, the class $\mathcal{FJ}_{p,q}^{m}(\alpha, \delta, \lambda, \gamma)$ abbreviates to the class $\mathcal{G}_\lambda(\gamma, \alpha)$ that Janani and Murugusundaramoorthy [11] presented and discussed.

9) When $p = 1, \delta = 1$ and $\lambda = 1$, the class $\mathcal{FJ}_{p,q}^{m}(\alpha, \delta, \lambda, \gamma)$ abbreviates to the class $UCT(\alpha, \beta)$ that Bharati *et al*. [6] presented and discussed.

10) When $p = 1, \delta = 1, \lambda = 1$ and $\gamma = 0$, the class $\mathcal{FJ}_{p,q}^{m}(\alpha, \delta, \lambda, \gamma)$ abbreviates to the class $UCV(\alpha)$ that Subramanian *et al*. [23] presented and discussed.

11) When $p = 1$ and $\delta = 0$, the class $\mathcal{FJ}_{p,q}^{m}(\alpha, \delta, \lambda, \gamma)$ abbreviates to the class $\mathcal{J}_\lambda(\gamma, \alpha)$ that Janani and Murugusundaramoorthy [11] presented and discussed.

12) When $p=1,\delta=0$ and $\lambda=1$, the class $\mathcal{FJ}_{p,q}^{m}(\alpha,\delta,\lambda,\gamma)$ abbreviates to the class $TS_p(\gamma,\alpha)$ that Bharati *et al.* [6] presented and discussed.

13) When $\delta=0$, $\lambda=1,\gamma=0$ and $p=1$, the class $\mathcal{FJ}_{p,q}^{m}(\alpha,\delta,\lambda,\gamma)$ abbreviates to the class $TS_p(\alpha)$ that Subramanian *et al.* [24] presented and discussed.

14) When $p=1,\delta=0$ and$\lambda=0$, the class $\mathcal{FJ}_{p,q}^{m}(\alpha,\delta,\lambda,\gamma)$ abbreviates to the class $TR(\gamma,\alpha)$ that Rosy [17] and Stephen and Subramanian [22] presented and discussed.

15) When $p=1,\delta=0,\lambda=0$ and $\alpha=0$, the class $\mathcal{FJ}_{p,q}^{m}(\alpha,\delta,\lambda,\gamma)$ abbreviates to the class $p(\gamma)$ that Al-Amiri [2], Sarangi and Uralegaddi [19] and Gupta and Jain [8] presented and discussed.

16) When $\delta=1$ and$\lambda=1$, the class $\mathcal{FJ}_{p,q}^{m}(\alpha,\delta,\lambda,\gamma)$ abbreviates to the class $UCV(p,\alpha,\beta)$ that Khaimar and More [12] presented and discussed.

17) When $\alpha=0,\delta=0$ and $\lambda=1$, the class $\mathcal{FJ}_{p,q}^{m}(\alpha,\delta,\lambda,\gamma)$ abbreviates to the class $S^*(p,\alpha)$ that Owa [15] presented and discussed.

18) When $\alpha=0,\delta=1$ and $\lambda=1$, the class $\mathcal{FJ}_{p,q}^{m}(\alpha,\delta,\lambda,\gamma)$ abbreviates to the class $C(p,\alpha)$ that Owa [15] presented and discussed.

Following lemmas are required to be able to demonstrate our major results.

Lemma 1.1 [5]. *Let $y=v+iu$, is complex number and $\alpha,\beta\in\mathbb{R}$. Then $Re(y)\geq\beta$ if and only if $|y-(p+\beta)|\leq|y+(p-\beta)|$, where $\beta\geq 0$.*

Lemma 1.2 [5]. *Let $y=v+iu$, is complex number and $\alpha,\beta\in\mathbb{R}$. Then $Re(y)\geq\alpha|y-p|+\beta$ if and only if $Re\left(y\left(1+\alpha e^{i\theta}\right)-p\alpha e^{i\theta}\right)\geq\beta$.*

Lemma 1.3 [13]. *If k and g are analytic in E with $k\prec g$, then*

$$\int_0^{2\pi}\left|k\left(re^{i\theta}\right)\right|^{\mu}d\theta\leq\int_0^{2\pi}\left|g\left(re^{i\theta}\right)\right|^{\mu}d\theta,$$

where $\mu>0, s=re^{i\theta}$ $(0<r<1)$.

The following characteristics were investigated for another classes in [9,10,16,21,25].

2. Coefficient Estimate

In the following Theorem, we are going to present the fundamental and necessary condition regarding the function inside the class $\mathcal{FJ}_{p,q}^{m}(\alpha,\delta,\lambda,\gamma)$.

Theorem 2.1. *The class* $\mathcal{FJ}_{p,q}^{m}(\alpha,\delta,\lambda,\gamma)$ *contains a function* $k(s)$ *defined* by (1.2) *if and only if*

$$\sum_{n=p+1}^{\infty}(1-\delta+\delta n)\big(n(1+\alpha)-\lambda(\gamma+\alpha p)\big)\left(\frac{[p+\omega]_q}{[n+\omega]_q}\right)^m d_n$$

$$\leq(1+\alpha)(p+\delta p^2-\delta p)-(\gamma+\alpha p)(1-\delta\lambda+p\delta\lambda), \qquad (2.1)$$

wherever $s\in E,\ q,\alpha\geq 0,\ 0\leq\gamma<p,\ 0\leq\lambda\leq 1,\ 0\leq\delta\leq 1,\omega>-p,n\geq p+1;\ p\in\mathbb{N}$.

The outcome is sharp to function $k(s)$ *specified by*

$$k(s)=s^p-\frac{(1+\alpha)(p+\delta p^2-\delta p)-(\gamma+\alpha p)(1-\delta\lambda+p\delta\lambda)}{(1-\delta+\delta n)\big(n(1+\alpha)-\lambda(\gamma+\alpha p)\big)\left(\frac{[p+\omega]_q}{[n+\omega]_q}\right)^m}s^n,\ \ (n\geq p+1;p\in\mathbb{N}). \quad (2.2)$$

Proof. Let $k(s)\in\mathcal{FJ}_{p,q}^{m}(\alpha,\delta,\lambda,\gamma)$. After then, the inequality (1.12) is satisfied by $k(s)$.

By using Lemma 1.2, the inequality (1.12) which is equivalent

$$Re\left\{\frac{s\left(ß_{m,q}^{p}k(s)\right)'+\delta s^2\left(ß_{m,q}^{p}k(s)\right)''(1+\alpha e^{i\theta})}{(1-\lambda)s^p+(1-\delta)\lambda ß_{m,q}^{p}k(s)+\delta\lambda s\left(ß_{m,q}^{p}k(s)\right)'}-p\alpha e^{i\theta}\right\}>\gamma$$

$(q,\alpha\geq 0,0\leq\gamma<p,0\leq\delta\leq 1,0\leq\lambda\leq 1,\omega>-p,n\geq p+1$ and $-\pi\leq\theta\leq\pi)$.

Or equivalently,

$$Re\left\{\frac{s\left(ß_{m,q}^{p}k(s)\right)'+\delta s^2\left(ß_{m,q}^{p}k(s)\right)''(1+\alpha e^{i\theta})}{(1-\lambda)s^p+(1-\delta)\lambda ß_{m,q}^{p}k(s)+\delta\lambda s\left(ß_{m,q}^{p}k(s)\right)'}\right.$$

$$\left.-\frac{p\alpha e^{i\theta}(1-\lambda)s^p+(1-\delta)\lambda ß_{m,q}^{p}k(s)+\delta\lambda s\left(ß_{m,q}^{p}k(s)\right)'}{(1-\lambda)s^p+(1-\delta)\lambda ß_{m,q}^{p}k(s)+\delta\lambda s\left(ß_{m,q}^{p}k(s)\right)'}\right\}$$

$$\geq\gamma. \qquad (2.3)$$

Let

$$M(s) = \left[s\left(ß_{m,q}^{p}k(s)\right)' + \delta s^{2}\left(ß_{m,q}^{p}k(s)\right)''\right]\left(1+\alpha e^{i\theta}\right)$$

$$-p\alpha e^{i\theta}\left[(1-\lambda)s^{p} + (1-\delta)\lambda ß_{m,q}^{p}k(s) + \delta\lambda s\left(ß_{m,q}^{p}k(s)\right)'\right],$$

$$N(s) = (1-\lambda)s^{p} + (1-\delta)\lambda ß_{m,q}^{p}k(s) + \delta\lambda s\left(ß_{m,q}^{p}k(s)\right)'.$$

Then by Lemma 1.1, (2.3) is equivalent to

$$|M(s) + (p-\gamma)N(s)| \geq |M(s) - (p+\gamma)N(s)|.$$

Now

$$|M(s) + (p-\gamma)N(s)|$$

$$= \left| \begin{array}{c} \left(ps^{p} - \sum_{n=p+1}^{\infty}\left(\frac{[p+\omega]_q}{[n+\omega]_q}\right)^{m} nd_n s^{n} + \delta p(p-1)s^{p} - \sum_{n=p+1}^{\infty}\left(\frac{[p+\omega]_q}{[n+\omega]_q}\right)^{m}\delta n(n-1)d_n s^{n}\right)\left(1+\alpha e^{i\theta}\right) \\ -p\alpha e^{i\theta}\left((1-\lambda)s^{p} + (1-\delta)\lambda s^{p} - \sum_{n=p+1}^{\infty}\left(\frac{[p+\omega]_q}{[n+\omega]_q}\right)^{m}(1-\delta)\lambda d_n s^{n} + \delta\lambda p s^{p} - \sum_{n=p+1}^{\infty}\left(\frac{[p+\omega]_q}{[n+\omega]_q}\right)^{m}\delta\lambda n d_n s^{n}\right) \\ +(p-\gamma)\left((1-\lambda)s^{p} + (1-\delta)\lambda s^{p} - \sum_{n=p+1}^{\infty}\left(\frac{[p+\omega]_q}{[n+\omega]_q}\right)^{m}(1-\delta)\lambda d_n s^{n} + \delta\lambda p s^{p} - \sum_{n=p+1}^{\infty}\left(\frac{[p+\omega]_q}{[n+\omega]_q}\right)^{m}\delta\lambda n d_n s^{n}\right) \end{array} \right|.$$

$$= \left| \begin{array}{c} \left(ps^{p} - \sum_{n=p+1}^{\infty}\left(\frac{[p+\omega]_q}{[n+\omega]_q}\right)^{m} nd_n s^{n} + \delta p(p-1)s^{p} - \sum_{n=p+1}^{\infty}\left(\frac{[p+\omega]_q}{[n+\omega]_q}\right)^{m}\delta n(n-1)d_n s^{n}\right)\left(1+\alpha e^{i\theta}\right) \\ + \\ \left(p - p\alpha e^{i\theta} - \gamma\right) \\ \left((1-\lambda)s^{p} + (1-\delta)\lambda s^{p} - \sum_{n=p+1}^{\infty}\left(\frac{[p+\omega]_q}{[n+\omega]_q}\right)^{m}(1-\delta)\lambda d_n s^{n} + \delta\lambda p s^{p} - \sum_{n=p+1}^{\infty}\left(\frac{[p+\omega]_q}{[n+\omega]_q}\right)^{m}\delta\lambda n d_n s^{n}\right) \end{array} \right|$$

$$= \left| \begin{array}{c} \left(p + \delta p^{2} - \delta p\right)\left(1+\alpha e^{i\theta}\right)s^{p} + \left(p - \gamma - p\alpha e^{i\theta}\right)(1 + \delta\lambda p - \delta\lambda)s^{p} \\ -\sum_{n=p+1}^{\infty}\left(\frac{[p+\omega]_q}{[n+\omega]_q}\right)^{m}\left(n + \delta n^{2} - \delta n\right)\left(1+\alpha e^{i\theta}\right)d_n s^{n} \\ -\sum_{n=p+1}^{\infty}\left(\frac{[p+\omega]_q}{[n+\omega]_q}\right)^{m}\lambda\left(p - \gamma - p\alpha e^{i\theta}\right)(1 - \delta + \delta n)d_n s^{n} \end{array} \right|$$

$$\geq (p+\delta p^2-\delta p)(1+\alpha)+(p-\gamma-p\alpha)(1+\delta\lambda p-\delta\lambda)|s|^p$$

$$-\sum_{n=p+1}^{\infty}\left(\frac{[p+\omega]_q}{[n+\omega]_q}\right)^m (n+\delta n^2-\delta n)(1+\alpha)+\lambda(p-\gamma-p\alpha)(1-\delta+\delta n)d_n|s|^n.$$

Similarly,

$$|M(s)-(p+\gamma)N(s)|$$

$$=\left|\begin{array}{c}\left(ps^p-\sum_{n=p+1}^{\infty}\left(\frac{[p+\omega]_q}{[n+\omega]_q}\right)^m nd_ns^n+\delta p(p-1)s^p-\sum_{n=p+1}^{\infty}\left(\frac{[p+\omega]_q}{[n+\omega]_q}\right)^m \delta n(n-1)d_ns^n\right)(1+\alpha e^{i\theta})\\ -(p\alpha e^{i\theta}+p+\gamma)\\ \left((1-\lambda)s^p+(1-\delta)\lambda s^p-\sum_{n=p+1}^{\infty}\left(\frac{[p+\omega]_q}{[n+\omega]_q}\right)^m (1-\delta)\lambda d_ns^n+\delta\lambda ps^p-\sum_{n=p+1}^{\infty}\left(\frac{[p+\omega]_q}{[n+\omega]_q}\right)^m \delta\lambda nd_ns^n\right)\end{array}\right|$$

$$\leq (p+\gamma+p\alpha)(1+\delta\lambda p-\delta\lambda)-(p+\delta p^2-\delta p)(1+\alpha)|s|^p+$$

$$\sum_{n=p+1}^{\infty}\left(\frac{[p+\omega]_q}{[n+\omega]_q}\right)^m \big((n+\delta n^2-\delta n)(1+\alpha)-\lambda(p+\gamma+p\alpha)(1-\delta+\delta n)\big)d_n|s|^n.$$

Therefore

$$|M(s)+(p-\gamma)N(s)|-|M(s)-(p+\gamma)N(s)|$$

$$\geq (p+\delta p^2-\delta p)(1+\alpha)-(\gamma+\alpha p)(1+\delta\lambda p-\delta\lambda)$$

$$-\sum_{n=p+1}^{\infty}\left(\frac{[p+\omega]_q}{[n+\omega]_q}\right)^m (1-\delta+\delta n)\big(n(1+\alpha)-\lambda(\gamma+p\alpha)\big)d_n\geq 0.$$

Hence

$$\sum_{n=p+1}^{\infty}\left(\frac{[p+\omega]_q}{[n+\omega]_q}\right)^m (1-\delta+\delta n)\big(n(1+\alpha)-\lambda(\gamma+p\alpha)\big)d_n$$

$$\leq (1+\alpha)(p+\delta p^2-\delta p)-(\gamma+\alpha p)(1+\delta\lambda p-\delta\lambda).$$

Conversely, by considering (2.1), we must show

$$Re\left\{\frac{s\left(ß_{m,q}^{p}k(s)\right)'+\delta s^2\left(ß_{m,q}^{p}k(s)\right)''(1+\alpha e^{i\theta})}{(1-\lambda)s^p+(1-\delta)\lambda ß_{m,q}^{p}k(s)+\delta\lambda\left(ß_{m,q}^{p}k(s)\right)'} - \frac{\left(p\alpha e^{i\theta}+\gamma\right)(1-\lambda)s^p+(1-\delta)\lambda ß_{m,q}^{p}k(s)+\delta\lambda\left(ß_{m,q}^{p}k(s)\right)'}{(1-\lambda)s^p+(1-\delta)\lambda ß_{m,q}^{p}k(s)+\delta\lambda\left(ß_{m,q}^{p}k(s)\right)'}\right\} \geq 0. \tag{2.4}$$

After determining that $0\leq s=r<1$ on the positive real axis, $Re\left(-e^{i\theta}\right)\geq -\left|e^{i\theta}\right|=-1$ and allowing $r\to 1^{-}$, we deduce to (2.4) based on using (2.1) in left hand of (2.4).

Corollary 2.1. *Suppose the function* $k(s)$ *defined by* (1.2) *be in the class* $\mathcal{FJ}_{p,q}^{m}(\alpha,\delta,\lambda,\gamma)$. *Then*

$$d_n \leq \frac{(1+\alpha)(p+\delta p^2-\delta p)-(\gamma+\alpha p)(1+\delta\lambda p-\delta\lambda)}{(1-\delta+\delta n)\left(n(1+\alpha)-\lambda(\gamma+p\alpha)\right)\left(\frac{[p+\omega]_q}{[n+\omega]_q}\right)^m}, \tag{2.5}$$

wherever $(s\in E,\ q,\alpha\geq 0, 0\leq\gamma<p, 0\leq\delta\leq 1, 0\leq\lambda\leq 1,, \omega>-p, n\geq p+1;\ p\in\mathbb{N})$.

3. Radii of Convexity, Starlikeness and Close-to-Convexity

The radii of convexity, starlikeness, and close-to-convexity for the functions in the class $\mathcal{FJ}_{p,q}^{m}(\alpha,\delta,\lambda,\gamma)$ will be found in the following Theorems.

Theorem 3.1. *Let* $k(s)\in\mathcal{FJ}_{p,q}^{m}(\alpha,\delta,\lambda,\gamma)$. *Then the function* $k(s)$ *is p-valent convex of order* ξ $(0\leq\xi<p)$ *in the disk* $|s|<R_1$, *where*

$$R_1=\inf_n\left[\frac{p(p-\xi)\left((1+\alpha)(n+\delta n^2-\delta n)-\lambda(\gamma+p\alpha)(1-\delta+\delta n)\right)\left(\frac{[p+\omega]_q}{[n+\omega]_q}\right)^m}{n(n-\xi)\left((1+\alpha)(p+\delta p^2-\delta p)-(\gamma+\alpha p)(1+\delta\lambda p-\delta\lambda)\right)}\right]^{\frac{1}{n-p}},$$

where $(n\geq p+1; p\in\mathbb{N})$.

For the function $k(s)$ *provided by* (2.2), *the outcome is sharp.*

Proof. It is enough to demonstrate that

$$\left|1+\frac{sk''(s)}{k'(s)}-p\right| \le p-\xi \quad (0 \le \xi < p),$$

for $|s| < R_1$, we have

$$\left|1+\frac{sk''(s)}{k'(s)}-p\right| \le \frac{\sum_{n=p+1}^{\infty} n(n-p)d_n|s|^{n-p}}{p-\sum_{n=p+1}^{\infty} nd_n|s|^{n-p}}.$$

Thus

$$\left|1+\frac{sk''(s)}{k'(s)}-p\right| \le p-\xi,$$

if

$$\sum_{n=p+1}^{\infty} \frac{n(n-\xi)}{p(p-\xi)} d_n|s|^{n-p} \le 1. \tag{3.1}$$

Therefore, by using Theorem 2.1, equation (3.1) shall hold if

$$\frac{n(n-\xi)}{p(p-\xi)}|s|^{n-p} \le \frac{(1-\delta+\delta n)\big(n(1+\alpha)-\lambda(\gamma+p\alpha)\big)\left(\frac{[p+\omega]_q}{[n+\omega]_q}\right)^m}{(1+\alpha)(p+\delta p^2-\delta p)-(\gamma+\alpha p)(1+\delta\lambda p-\delta\lambda)},$$

and hence

$$|s| \le \left[\frac{p(p-\xi)(1-\delta+\delta n)\big(n(1+\alpha)-\lambda(\gamma+p\alpha)\big)\left(\frac{[p+\omega]_q}{[n+\omega]_q}\right)^m}{n(n-\xi)\big((1+\alpha)(p+\delta p^2-\delta p)-(\gamma+\alpha p)(1+\delta\lambda p-\delta\lambda)\big)}\right]^{\frac{1}{n-p}}, (n \ge p+1;\ p \in \mathbb{N}).$$

Putting $|s| = R_1$, we achieve the intended outcome.

Theorem 3.2. *Let* $k(s) \in \mathcal{FJ}_{p,q}^{m}(\alpha,\delta,\lambda,\gamma)$. *Then the function* $k(s)$ *is p-valent starlike of order* ξ $(0 \le \xi < p)$ *in the disk* $|s| < R_2$, *where*

$$R_2 = \inf_n \left[\frac{(p-\xi)(1-\delta+\delta n)\big(n(1+\alpha)-\lambda(\gamma+p\alpha)\big)\left(\frac{[p+\omega]_q}{[n+\omega]_q}\right)^m}{(n-\xi)\big((1+\alpha)(p+\delta p^2-\delta p)-(\gamma+\alpha p)(1+\delta\lambda p-\delta\lambda)\big)}\right]^{\frac{1}{n-p}}, \ (n \ge p+1; p \in \mathbb{N}).$$

For the function $k(s)$ *provided by* (2.2), *the outcome is sharp.*

Proof. It is enough to demonstrate that

$$\left|\frac{sk'(s)}{k(s)}-p\right| \leq p-\xi \quad (0 \leq \xi < p),$$

for $|s| < R_2,$ we get

$$\left|\frac{sk'(s)}{k(s)}-p\right| \leq \frac{\sum_{n=p+1}^{\infty}(n-p)d_n|s|^{n-p}}{1-\sum_{n=p+1}^{\infty}d_n|s|^{n-p}}.$$

Thus

$$\left|\frac{sk'(s)}{k(s)}-p\right| \leq p-\xi,$$

if

$$\sum_{n=p+1}^{\infty}\frac{(n-\xi)}{(p-\xi)}d_n|s|^{n-p} \leq 1. \tag{3.2}$$

Therefore, by using Theorem 2.1, equation (3.2) shall hold if

$$\frac{(n-\xi)}{(p-\xi)}|s|^{n-p} \leq \frac{(1-\delta+\delta n)\big(n(1+\alpha)-\lambda(\gamma+p\alpha)\big)\left(\frac{[p+\omega]_q}{[n+\omega]_q}\right)^m}{(1+\alpha)(p+\delta p^2-\delta p)-(\gamma+\alpha p)(1+\delta\lambda p-\delta\lambda)},$$

and hence

$$|s| \leq \left[\frac{(p-\xi)(1-\delta+\delta n)\big(n(1+\alpha)-\lambda(\gamma+p\alpha)\big)\left(\frac{[p+\omega]_q}{[n+\omega]_q}\right)^m}{(n-\xi)\big((1+\alpha)(p+\delta p^2-\delta p)-(\gamma+\alpha p)(1+\delta\lambda p-\delta\lambda)\big)}\right]^{\frac{1}{n-p}}, (n \geq p+1; p \in \mathbb{N}).$$

Putting $|s| = R_2$, we achieve the intended outcome.

Theorem 3.3. *Let $k(s) \in \mathcal{FJ}_{p,q}^m(\alpha,\delta,\lambda,\gamma)$. Then the function $k(s)$ is p-valent close to convex of order ξ $(0 \leq \xi < p)$ in the disk $|s| < R_3$, where*

$$R_3 = \inf_n\left[\frac{(p-\xi)(1-\delta+\delta n)\big(n(1+\alpha)-\lambda(\gamma+p\alpha)\big)\left(\frac{[p+\omega]_q}{[n+\omega]_q}\right)^m}{n\big((1+\alpha)(p+\delta p^2-\delta p)-(\gamma+\alpha p)(1+\delta\lambda p-\delta\lambda)\big)}\right]^{\frac{1}{n-p}}, (n \geq p+1; p \in \mathbb{N}).$$

For the function $k(s)$ provided by (2.2), *the outcome is sharp.*

Proof. It is enough to demonstrate that

$$\left|\frac{k'(s)}{s^{p-1}}-p\right| \le p-\xi \quad (0 \le \xi < p),$$

for $|s| < R_3$, we have

$$\left|\frac{k'(s)}{s^{p-1}}-p\right| \le \sum_{n=p+1}^{\infty} n d_n |s|^{n-p}.$$

Thus

$$\left|\frac{k'(s)}{s^{p-1}}-p\right| \le p-\xi,$$

if

$$\sum_{n=p+1}^{\infty} \frac{n d_n |s|^{n-p}}{p-\xi} \le 1. \tag{3.3}$$

Therefore, by using Theorem 2.1, equation (3.3) shall hold if

$$\frac{n}{p-\xi}|s|^{n-p} \le \frac{(1-\delta+\delta n)\big(n(1+\alpha)-\lambda(\gamma+p\alpha)\big)\left(\frac{[p+\omega]_q}{[n+\omega]_q}\right)^m}{(1+\alpha)(p+\delta p^2-\delta p)-(\gamma+\alpha p)(1+\delta\lambda p-\delta\lambda)},$$

and hence

$$|s| \le \left[\frac{(p-\xi)(1-\delta+\delta n)\big(n(1+\alpha)-\lambda(\gamma+p\alpha)\big)\left(\frac{[p+\omega]_q}{[n+\omega]_q}\right)^m}{n\big((1+\alpha)(p+\delta p^2-\delta p)-(\gamma+\alpha p)(1+\delta\lambda p-\delta\lambda)\big)}\right]^{\frac{1}{n-p}}, \quad (n \ge p+1; p \in \mathbb{N}).$$

Putting $|s| = R_3$, we achieve the intended outcome.

4. Extreme Points

The extreme points of the class $\mathcal{FJ}_{p,q}^m(\alpha,\delta,\lambda,\gamma)$ are obtained here.

Theorem 4.1. *Let* $k_p(s) = s^p$ *and*

$$k_n(s) = s^p - \frac{(1+\alpha)(p+\delta p^2-\delta p)-(\gamma+\alpha p)(1+\delta\lambda p-\delta\lambda)}{(1-\delta+\delta n)\big(n(1+\alpha)-\lambda(\gamma+p\alpha)\big)\left(\frac{[p+\omega]_q}{[n+\omega]_q}\right)^m} s^n, \tag{4.1}$$

where $s \in E$, $q, \alpha \geq 0, 0 \leq \gamma < p, 0 \leq \delta \leq 1, 0 \leq \lambda \leq 1, \omega > -p, n \geq p+1$ *and* $p \in \mathbb{N}$.

Then the function $k(s) \in \mathcal{FJ}_{p,q}^{m}(\alpha,\delta,\lambda,\gamma)$ *if and only if it may be stated as follows*:

$$k(s) = \varphi_p s^p + \sum_{n=p+1}^{\infty} \varphi_n k_n(s), \tag{4.2}$$

where $\left(\varphi_p \geq 0,\ \varphi_n \geq 0, n \geq p+1\right)$ *and* $\varphi_p + \sum_{n=p+1}^{\infty} \varphi_n = 1$.

Proof. Assume that the expression for $k(s)$ takes the form (4.2). Then

$$k(s) = \varphi_p s^p + \sum_{n=p+1}^{\infty} \varphi_n \left(s^p - \frac{(1+\alpha)(p+\delta p^2-\delta p)-(\gamma+\alpha p)(1+\delta\lambda p-\delta\lambda)}{(1-\delta+\delta n)\big(n(1+\alpha)-\lambda(\gamma+p\alpha)\big)\left(\frac{[p+\omega]_q}{[n+\omega]_q}\right)^m} s^n \right)$$

$$= s^p - \sum_{n=p+1}^{\infty} \frac{(1+\alpha)(p+\delta p^2-\delta p)-(\gamma+\alpha p)(1+\delta\lambda p-\delta\lambda)}{(1-\delta+\delta n)\big(n(1+\alpha)-\lambda(\gamma+p\alpha)\big)\left(\frac{[p+\omega]_q}{[n+\omega]_q}\right)^m} \varphi_n s^n.$$

Hence

$$\sum_{n=p+1}^{\infty} \frac{(1-\delta+\delta n)\big(n(1+\alpha)-\lambda(\gamma+p\alpha)\big)\left(\frac{[p+\omega]_q}{[n+\omega]_q}\right)^m}{(1+\alpha)(p+\delta p^2-\delta p)-(\gamma+\alpha p)(1+\delta\lambda p-\delta\lambda)} \times$$

$$\frac{\big((1+\alpha)(p+\delta p^2-\delta p)-(\gamma+\alpha p)(1+\delta\lambda p-\delta\lambda)\big)\varphi_n}{(1-\delta+\delta n)\big(n(1+\alpha)-\lambda(\gamma+p\alpha)\big)\left(\frac{[p+\omega]_q}{[n+\omega]_q}\right)^m}$$

$$= \sum_{n=p+1}^{\infty} \varphi_n = 1 - \varphi_p \leq 1.$$

Then $k(s) \in \mathcal{FJ}_{p,q}^{m}(\alpha,\delta,\lambda,\gamma)$.

Conversely, assume that $k(s) \in \mathcal{FJ}_{p,q}^{m}(\alpha,\delta,\lambda,\gamma)$. We could decide to

$$\varphi_n = \frac{(1-\delta+\delta n)\big(n(1+\alpha)-\lambda(\gamma+p\alpha)\big)\left(\frac{[p+\omega]_q}{[n+\omega]_q}\right)^m}{(1+\alpha)(p+\delta p^2-\delta p)-(\gamma+\alpha p)(1+\delta\lambda p-\delta\lambda)} d_n,$$

where d_n is specified based on (2.5). Then

$$k(s) = s^p - \sum_{n=p+1}^{\infty} d_n s^n$$

$$= s^p - \sum_{n=p+1}^{\infty} \frac{(1+\alpha)(p+\delta p^2-\delta p)-(\gamma+\alpha p)(1+\delta\lambda p-\delta\lambda)}{(1-\delta+\delta n)\big(n(1+\alpha)-\lambda(\gamma+p\alpha)\big)\left(\frac{[p+\omega]_q}{[n+\omega]_q}\right)^m} \varphi_n s^n$$

$$= s^p - \sum_{n=p+1}^{\infty} \big(s^p - k_n(s)\big)\varphi_n = \left(1 - \sum_{n=p+1}^{\infty} \varphi_n\right) s^p + \sum_{n=p+1}^{\infty} \varphi_n k_n(s)$$

$$= \varphi_p s^p + \sum_{n=p+1}^{\infty} \varphi_n k_n(s).$$

With this, Theorem 4.1 is fully proved.

5. Closure Theorems

Theorem 5.1. *Let the function* k_c *specified by*

$$k_c(s) = s^p - \sum_{n=p+1}^{\infty} d_{n,c}\, s^n \quad \big(d_{n,c} \geq 0, n \geq p+1; p \in \mathbb{N}, c = 1,2,\dots,t\big), \quad (5.1)$$

be in the class $\mathcal{FJ}_{p,q}^{m}(\alpha,\delta,\lambda,\gamma)$ *for each* $c = 1,2,\dots,t$.

Then the function $f_1(s)$ *specified by*

$$f_1(s) = s^p - \sum_{n=p+1}^{\infty} e_n\, s^n \quad (e_n \geq 0, n \geq p+1; p \in \mathbb{N}),$$

also belongs to the class $\mathcal{FJ}_{p,q}^{m}(\alpha,\delta,\lambda,\gamma)$, *wherever*

$$e_n = \frac{1}{t}\sum_{c=1}^{t} d_{n,c}\,, \quad (n \geq p+1; p \in \mathbb{N}).$$

Proof. Since $k_c \in \mathcal{FJ}_{p,q}^{m}(\alpha,\delta,\lambda,\gamma)$ it follows from Theorem 2.1 that

$$\sum_{n=p+1}^{\infty} (1-\delta+\delta n)\big(n(1+\alpha)-\lambda(\gamma+p\alpha)\big)\left(\frac{[p+\omega]_q}{[n+\omega]_q}\right)^m d_{n,c}$$

$$\leq (1+\alpha)(p+\delta p^2-\delta p)-(\gamma+\alpha p)(1+\delta\lambda p-\delta\lambda),$$

for every $c=1,2,\dots,t$. Hence

$$\sum_{n=p+1}^{\infty}(1-\delta+\delta n)\big(n(1+\alpha)-\lambda(\gamma+p\alpha)\big)\left(\frac{[p+\omega]_q}{[n+\omega]_q}\right)^m e_n$$

$$=\sum_{n=p+1}^{\infty}(1-\delta+\delta n)\big(n(1+\alpha)-\lambda(\gamma+p\alpha)\big)\left(\frac{[p+\omega]_q}{[n+\omega]_q}\right)^m\left(\frac{1}{t}\sum_{c=1}^{t}d_{n,c}\right)$$

$$=\frac{1}{t}\sum_{c=1}^{t}\left(\sum_{n=p+1}^{\infty}(1-\delta+\delta n)\big(n(1+\alpha)-\lambda(\gamma+p\alpha)\big)\left(\frac{[p+\omega]_q}{[n+\omega]_q}\right)^m d_{n,c}\right)$$

$$\leq (1+\alpha)(p+\delta p^2-\delta p)-(\gamma+\alpha p)(1+\delta\lambda p-\delta\lambda).$$

Then, $f_1(s)\in\mathcal{FJ}_{p,q}^m(\alpha,\delta,\lambda,\gamma)$ by Theorem 2.1.

Theorem 5.2. *Let the function k_c specified based on* (5.1) *be in the class $\mathcal{FJ}_{p,q}^m(\alpha,\delta,\lambda,\gamma)$ for each $c=1,2,\dots,t$. Then the function $f_2(s)$ specified based on*

$$f_2(s)=\sum_{c=1}^{t}v_c\,k_c(s)$$

is also in the class $\mathcal{FJ}_{p,q}^m(\alpha,\delta,\lambda,\gamma)$, wherever

$$\sum_{c=1}^{t}v_c=1,\quad (v_c\geq 0).$$

Proof. According to the definition of $f_2(s)$, it can be written as for every $c=1,2,\dots,t$, we have

$$f_2(s)=\sum_{c=1}^{t}v_c\,k_c(s)=\sum_{c=1}^{t}v_c\left(s^p-\sum_{n=p+1}^{\infty}d_{n,c}\,s^n\right)=s^p-\sum_{n=p+1}^{\infty}\left(\sum_{c=1}^{t}v_c\,d_{n,c}\right)s^n.$$

Furthermore, since the functions $k_c(s)$ $(c=1,2,\dots,t)$ are in the class $\mathcal{FJ}_{p,q}^m(\alpha,\delta,\lambda,\gamma)$, then

$$\sum_{n=p+1}^{\infty}(1-\delta+\delta n)\big(n(1+\alpha)-\lambda(\gamma+p\alpha)\big)\left(\frac{[p+\omega]_q}{[n+\omega]_q}\right)^m d_{n,c}$$

$$\leq (1+\alpha)(p+\delta p^2-\delta p)-(\gamma+\alpha p)(1+\delta\lambda p-\delta\lambda).$$

Hence

$$\sum_{n=p+1}^{\infty}(1-\delta+\delta n)\big(n(1+\alpha)-\lambda(\gamma+p\alpha)\big)\left(\frac{[p+\omega]_q}{[n+\omega]_q}\right)^m \left(\sum_{c=1}^{t} v_c d_{n,c}\right)$$

$$=\sum_{c=1}^{t} v_c\left(\sum_{n=p+1}^{\infty}(1-\delta+\delta n)\big(n(1+\alpha)-\lambda(\gamma+p\alpha)\big)\left(\frac{[p+\omega]_q}{[n+\omega]_q}\right)^m d_{n,c}\right)$$

$$\leq \sum_{c=1}^{t} v_c\,[(1+\alpha)(p+\delta p^2-\delta p)-(\gamma+\alpha p)(1+\delta\lambda p-\delta\lambda)]$$

$$=(1+\alpha)(p+\delta p^2-\delta p)-(\gamma+\alpha p)(1+\delta\lambda p-\delta\lambda),$$

which implies that $f_2(s)$ be in the class $\mathcal{FJ}_{p,q}^{m}(\alpha,\delta,\lambda,\gamma)$.

Corollary 5.1. *The class $\mathcal{FJ}_{p,q}^{m}(\alpha,\delta,\lambda,\gamma)$ is close under convex linear combination.*

6. Integral Means Inequalities

We demonstrate the following theorems using Lemma 1.3 and Theorem 2.1.

Theorem 6.1. *Let $\varepsilon>0$. If $k(s)\in\mathcal{FJ}_{p,q}^{m}(\alpha,\delta,\lambda,\gamma)$ and assume that $k_c(s)$ is specified by*

$$k_c(s)=s^p-\frac{(1+\alpha)(p+\delta p^2-\delta p)-(\gamma+\alpha p)(1+\delta\lambda p-\delta\lambda)}{(1-\delta+\delta c)\big(c(1+\alpha)-\lambda(\gamma+p\alpha)\big)\left(\frac{[p+\omega]_q}{[c+\omega]_q}\right)^m}s^c,\quad (c\geq p+1; p\in\mathbb{N}).$$

If there is an analytic function $\mathcal{Y}(s)$ defined by

$$\big(\mathcal{Y}(s)\big)^{c-p}=\frac{(1-\delta+\delta c)\big(c(1+\alpha)-\lambda(\gamma+p\alpha)\big)\left(\frac{[p+\omega]_q}{[c+\omega]_q}\right)^m}{(1+\alpha)(p+\delta p^2-\delta p)-(\gamma+\alpha p)(1+\delta\lambda p-\delta\lambda)}\sum_{n=p+1}^{\infty} d_n s^{n-p}.$$

Then, for $s=re^{i\theta}$ and $(0<r<1)$,

$$\int_0^{2\pi} |k(s)|^{\varepsilon}\, d\theta \le \int_0^{2\pi} |k_c(s)|^{\varepsilon}\, d\theta, \qquad (\varepsilon > 0). \tag{6.1}$$

Proof. We have to demonstrate that

$$\int_0^{2\pi} \left| 1 - \sum_{n=p+1}^{\infty} d_n s^{n-p} \right|^{\varepsilon} d\theta$$

$$\le \int_0^{2\pi} \left| 1 - \frac{(1+\alpha)(p+\delta p^2-\delta p)-(\gamma+\alpha p)(1+\delta\lambda p-\delta\lambda)}{(1-\delta+\delta c)\big(c(1+\alpha)-\lambda(\gamma+p\alpha)\big)\left(\frac{[p+\omega]_q}{[c+\omega]_q}\right)^m} s^{c-p} \right|^{\varepsilon} d\theta$$

By using Lemma 1.3, it is sufficient to demonstrate that

$$1 - \sum_{n=p+1}^{\infty} d_n s^{n-p} \prec 1 - \frac{(1+\alpha)(p+\delta p^2-\delta p)-(\gamma+\alpha p)(1+\delta\lambda p-\delta\lambda)}{(1-\delta+\delta c)\big(c(1+\alpha)-\lambda(\gamma+p\alpha)\big)\left(\frac{[p+\omega]_q}{[c+\omega]_q}\right)^m} s^{c-p}.$$

Put

$$1 - \sum_{n=p+1}^{\infty} d_n s^{n-p}$$

$$= 1 - \frac{(1+\alpha)(p+\delta p^2-\delta p)-(\gamma+\alpha p)(1+\delta\lambda p-\delta\lambda)}{(1-\delta+\delta c)\big(c(1+\alpha)-\lambda(\gamma+p\alpha)\big)\left(\frac{[p+\omega]_q}{[c+\omega]_q}\right)^m} \big(\mathcal{Y}(s)\big)^{c-p}.$$

We find that

$$\big(\mathcal{Y}(s)\big)^{c-p} = \frac{(1-\delta+\delta c)\big(c(1+\alpha)-\lambda(\gamma+p\alpha)\big)\left(\frac{[p+\omega]_q}{[c+\omega]_q}\right)^m}{(1+\alpha)(p+\delta p^2-\delta p)-(\gamma+\alpha p)(1+\delta\lambda p-\delta\lambda)} \sum_{n=p+1}^{\infty} d_n s^{n-p},$$

that yield easily $\mathcal{Y}(0) = 0$.

In addition by using (2.1), we get

$$|\mathcal{Y}(s)|^{c-p} = \left| \frac{(1-\delta+\delta c)\big(c(1+\alpha)-\lambda(\gamma+p\alpha)\big)\left(\frac{[p+\omega]_q}{[c+\omega]_q}\right)^m}{(1+\alpha)(p+\delta p^2-\delta p)-(\gamma+\alpha p)(1+\delta\lambda p-\delta\lambda)} \sum_{n=p+1}^{\infty} d_n s^{n-p} \right|$$

$$\leq |s|\left|\sum_{n=p+1}^{\infty}\frac{(1-\delta+\delta n)\big(n(1+\alpha)-\lambda(\gamma+p\alpha)\big)\left(\frac{[p+\omega]_q}{[n+\omega]_q}\right)^m}{(1+\alpha)(p+\delta p^2-\delta p)-(\gamma+\alpha p)(1+\delta\lambda p-\delta\lambda)}d_n\right|$$

$$\leq |s| < 1.$$

Afterwards, the first derivative's proof.

Theorem 6.2. *Let* $\varepsilon > 0$. *If* $k(s) \in \mathcal{FI}_{p,q}^{m}(\alpha,\delta,\lambda,\gamma)$ *and*

$$k_c(s) = s^p - \frac{(1+\alpha)(p+\delta p^2-\delta p)-(\gamma+\alpha p)(1+\delta\lambda p-\delta\lambda)}{(1-\delta+\delta c)\big(c(1+\alpha)-\lambda(\gamma+p\alpha)\big)\left(\frac{[p+\omega]_q}{[c+\omega]_q}\right)^m}s^c,\ \ (c \geq p+1; p \in \mathbb{N}).$$

Then, for $s = re^{i\theta}$ *and* $(0 < r < 1)$,

$$\int_0^{2\pi}|k'(s)|^{\varepsilon}\,d\theta \leq \int_0^{2\pi}|k_c'(s)|^{\varepsilon}\,d\theta, \qquad (\varepsilon > 0). \tag{6.2}$$

Proof. It is enough to prove that

$$1-\sum_{n=p+1}^{\infty}\frac{n}{p}d_n s^{n-p} \prec 1-\frac{c\big((1+\alpha)(p+\delta p^2-\delta p)-(\gamma+\alpha p)(1+\delta\lambda p-\delta\lambda)\big)}{p(1-\delta+\delta c)\big(c(1+\alpha)-\lambda(\gamma+p\alpha)\big)\left(\frac{[p+\omega]_q}{[c+\omega]_q}\right)^m}s^{c-p}.$$

This follows because

$$|\mathcal{Y}(s)|^{c-p} = \left|\frac{p(1-\delta+\delta c)\big(c(1+\alpha)-\lambda(\gamma+p\alpha)\big)\left(\frac{[p+\omega]_q}{[c+\omega]_q}\right)^m}{c\big((1+\alpha)(p+\delta p^2-\delta p)-(\gamma+\alpha p)(1+\delta\lambda p-\delta\lambda)\big)}\sum_{n=p+1}^{\infty}\frac{n}{p}d_n s^{n-p}\right|$$

$$\leq |s|\left|\sum_{n=p+1}^{\infty}\frac{(1-\delta+\delta n)\big(n(1+\alpha)-\lambda(\gamma+p\alpha)\big)\left(\frac{[p+\omega]_q}{[n+\omega]_q}\right)^m}{(1+\alpha)(p+\delta p^2-\delta p)-(\gamma+\alpha p)(1+\delta\lambda p-\delta\lambda)}d_n\right|$$

$$\leq |s| < 1.$$

Theorem 6.3. *Let* $g(s) = s^p - \sum_{n=p+1}^{\infty}\not{f}_n s^n$, $(s \in E; \not{f}_n \geq 0; n \geq p+1; p \in \mathbb{N})$ *and* $k(s) \in \mathcal{FI}_{p,q}^{m}(\alpha,\delta,\lambda,\gamma)$ *be of the shape* (1.2) *and let for some* $c \in \mathbb{N}$,

$$\frac{Q_c}{\not{f}_c} = \min_{n\geq p+1}\frac{Q_n}{\not{f}_n},$$

where

$$Q_n = \frac{(1-\delta+\delta n)\big(n(1+\alpha)-\lambda(\gamma+p\alpha)\big)\left(\frac{[p+\omega]_q}{[n+\omega]_q}\right)^m}{(1+\alpha)(p+\delta p^2-\delta p)-(\gamma+\alpha p)(1+\delta\lambda p-\delta\lambda)}.$$

Also, let for such $c \in \mathbb{N}$, the functions k_c and g_c be specified based on

$$k_c(s) = s^p - \frac{(1+\alpha)(p+\delta p^2-\delta p)-(\gamma+\alpha p)(1+\delta\lambda p-\delta\lambda)}{(1-\delta+\delta c)\big(c(1+\alpha)-\lambda(\gamma+p\alpha)\big)\left(\frac{[p+\omega]_q}{[c+\omega]_q}\right)^m} s^c,$$

$$g_c(s) = s^p - \mathfrak{f}_c s^c. \tag{6.3}$$

If there is an analytic function $\mathcal{Y}(s)$ specified based on

$$\big(\mathcal{Y}(s)\big)^{c-p} = \frac{(1-\delta+\delta c)\big(c(1+\alpha)-\lambda(\gamma+p\alpha)\big)\left(\frac{[p+\omega]_q}{[c+\omega]_q}\right)^m}{(1+\alpha)(p+\delta p^2-\delta p)-(\gamma+\alpha p)(1+\delta\lambda p-\delta\lambda)\mathfrak{f}_c} \sum_{n=p+1}^{\infty} d_n \mathfrak{f}_n s^{n-p},$$

then, for $\varepsilon > 0$, $s = re^{i\theta}$ and $(0 < r < 1)$,

$$\int_0^{2\pi} |(k*g)(s)|^{\varepsilon}\, d\theta \le \int_0^{2\pi} |(k_c*g_c)(s)|^{\varepsilon}\, d\theta, \qquad (\varepsilon > 0).$$

Proof. The definition of the convolution between $k(s)$ and $g(s)$ is

$$(k*g)(s) = s^p - \sum_{n=p+1}^{\infty} d_n \mathfrak{f}_n s^n.$$

Likewise, from (6.3), we obtain

$$(k_c*g_c)(s) = s^p - \frac{(1+\alpha)(p+\delta p^2-\delta p)-(\gamma+\alpha p)(1+\delta\lambda p-\delta\lambda)\mathfrak{f}_c}{(1-\delta+\delta c)\big(c(1+\alpha)-\lambda(\gamma+p\alpha)\big)\left(\frac{[p+\omega]_q}{[c+\omega]_q}\right)^m} s^c.$$

To prove the theorem, we must show that for $\varepsilon > 0$, $s = re^{i\theta}$ and $(0 < r < 1)$,

$$\int_0^{2\pi} \left|1 - \sum_{n=p+1}^{\infty} d_n \mathfrak{f}_n s^{n-p}\right|^{\varepsilon} d\theta$$

$$\leq \int_0^{2\pi} \left| 1 - \frac{\big((1+\alpha)(p+\delta p^2-\delta p)-(\gamma+\alpha p)(1+\delta\lambda p-\delta\lambda)\big)\phi_c}{(1-\delta+\delta c)\big(c(1+\alpha)-\lambda(\gamma+p\alpha)\big)\left(\frac{[p+\omega]_q}{[c+\omega]_q}\right)^m} s^{c-p} \right|^{\varepsilon} d\theta.$$

By using Lemma 1.3, it is sufficient to show that

$$1 - \sum_{n=p+1}^{\infty} d_n \phi_n s^{n-p}$$
$$\prec 1 - \frac{\big((1+\alpha)(p+\delta p^2-\delta p)-(\gamma+\alpha p)(1+\delta\lambda p-\delta\lambda)\big)\phi_c}{(1-\delta+\delta c)\big(c(1+\alpha)-\lambda(\gamma+p\alpha)\big)\left(\frac{[p+\omega]_q}{[c+\omega]_q}\right)^m} s^{c-p}. \qquad (6.4)$$

If the subordination (6.4) is valid, then there is an analytic function $\mathcal{Y}(s)$ with $|\mathcal{Y}(s)| < 1$ and $\mathcal{Y}(0) = 0$ such that

$$1 - \sum_{n=p+1}^{\infty} d_n \phi_n s^{n-p}$$
$$= 1 - \frac{\big((1+\alpha)(p+\delta p^2-\delta p)-(\gamma+\alpha p)(1+\delta\lambda p-\delta\lambda)\big)\phi_c}{(1-\delta+\delta c)\big(c(1+\alpha)-\lambda(\gamma+p\alpha)\big)\left(\frac{[p+\omega]_q}{[c+\omega]_q}\right)^m} \big(\mathcal{Y}(s)\big)^{c-p}.$$

According to the assumption of the theorem, there is an analytic function $\mathcal{Y}(s)$ given by

$$\big(\mathcal{Y}(s)\big)^{c-p} = \frac{(1-\delta+\delta c)\big(c(1+\alpha)-\lambda(\gamma+p\alpha)\big)\left(\frac{[p+\omega]_q}{[n+\omega]_q}\right)^m}{\big((1+\alpha)(p+\delta p^2-\delta p)-(\gamma+\alpha p)(1+\delta\lambda p-\delta\lambda)\big)\phi_c} \sum_{n=p+1}^{\infty} d_n \phi_n s^{n-p},$$

which readily yield $\mathcal{Y}(0) = 0$. So for such function $\mathcal{Y}(s)$, using the assumption in the coefficient inequality for the class $\mathcal{FJ}_{p,q}^m(\alpha, \delta, \lambda, \gamma)$, we have

$$|\mathcal{Y}(s)|^{c-p}$$
$$= \left| \frac{(1-\delta+\delta c)\big(c(1+\alpha)-\lambda(\gamma+p\alpha)\big)\left(\frac{[p+\omega]_q}{[c+\omega]_q}\right)^m}{\big((1+\alpha)(p+\delta p^2-\delta p)-(\gamma+\alpha p)(1+\delta\lambda p-\delta\lambda)\big)\phi_c} \sum_{n=p+1}^{\infty} d_n \phi_n s^{n-p} \right|$$

$$\leq |s| \left| \frac{(1-\delta+\delta c)\big(c(1+\alpha)-\lambda(\gamma+p\alpha)\big)\left(\frac{[p+\omega]_q}{[c+\omega]_q}\right)^m}{\big((1+\alpha)(p+\delta p^2-\delta p)-(\gamma+\alpha p)(1+\delta\lambda p-\delta\lambda)\big)∱_c} \sum_{n=p+1}^{\infty} d_n ∱_n \right| \leq |s| < 1.$$

Therefore, the subordination (6.4) holds true.

7. Neighbourhoods Property

We now define the $(n-\xi)$-neighborhoods for the function $k(s) \in \mathcal{M}_p$ by

$$N_{n,\xi}(k) = \left\{ g \in \mathcal{M}_p : g(s) = s^p - \sum_{n=p+1}^{\infty} ∱_n s^n \text{ and } \sum_{n=p+1}^{\infty} n|d_n - ∱_n| \leq \xi, 0 \leq \xi < 1 \right\}. \quad (7.1)$$

For identity function $l(s) = s^p, (p \in \mathbb{N})$

$$N_{n,\xi}(l) = \left\{ g \in \mathcal{M}_p : g(s) = s^p - \sum_{n=p+1}^{\infty} ∱_n s^n \text{ and } \sum_{n=p+1}^{\infty} n|∱_n| \leq \xi, 0 \leq \xi < 1 \right\}. \quad (7.2)$$

Goodman [7] established the idea of neighborhoods originally, and Ruscheweyh [18] later generalized it.

Definition 7.1. A function $k(s) \in \mathcal{M}_p$ is said to be in the class $\mathcal{FJ}_{p,q}^m(\alpha,\delta,\lambda,\gamma)$, if there exist a function $g(s) \in \mathcal{FJ}_{p,q}^m(\alpha,\delta,\lambda,\gamma)$, such that

$$\left| \frac{k(s)}{g(s)} - 1 \right| < p - \vartheta \qquad (s \in E, 0 \leq \vartheta < 1).$$

Theorem 7.1. *If* $g(s) \in \mathcal{FJ}_{p,q}^m(\alpha,\delta,\lambda,\gamma)$ *and*

$$\vartheta = p - \frac{\xi(1+\delta p)[(p+1)(1+\alpha)-\lambda(\gamma+p\alpha)]\left(\frac{[p+\omega]_q}{[p+1+\omega]_q}\right)^m}{(p+1)\begin{bmatrix} (1+\delta p)[(p+1)(1+\alpha)-\lambda(\gamma+p\alpha)]\left(\frac{[p+\omega]_q}{[p+1+\omega]_q}\right)^m \\ -[(1+\alpha)(p+\delta p^2-\delta p)-(\gamma+\alpha p)(1+\delta\lambda p-\delta\lambda)] \end{bmatrix}}. \quad (7.3)$$

Then $N_{n,\xi}(g) \subset \mathcal{FJ}_{p,q}^m(\alpha,\delta,\lambda,\gamma)$.

Proof. Let $k(s) \in N_{n,\xi}(g)$. Then, from (7.1), we have

$$\sum_{n=p+1}^{\infty} n|d_n - ∱_n| \leq \xi,$$

it obviously indicates the coefficient inequality below

$$\sum_{n=p+1}^{\infty} |d_n - \not{f}_n| \leq \frac{\xi}{p+1}.$$

Next, because $g(s) \in \mathcal{FJ}_{p,q}^m(\alpha, \delta, \lambda, \gamma)$, we get from Theorem 2.1

$$\sum_{n=p+1}^{\infty} \not{f}_n \leq \frac{(1+\alpha)(p+\delta p^2-\delta p)-(\gamma+\alpha p)(1+\delta\lambda p-\delta\lambda)}{(1+\delta p)[(p+1)(1+\alpha)-\lambda(\gamma+p\alpha)]\left(\frac{[p+\omega]_q}{[p+1+\omega]_q}\right)^m}.$$

So that

$$\left|\frac{k(s)}{g(s)}-1\right| \leq \frac{\sum_{p+1}^{\infty}|d_n - \not{f}_n|}{1-\sum_{p+1}^{\infty}\not{f}_n}$$

$$\leq \frac{\xi(1+\delta p)[(p+1)(1+\alpha)-\lambda(\gamma+p\alpha)]\left(\frac{[p+\omega]_q}{[p+1+\omega]_q}\right)^m}{(p+1)\begin{bmatrix}(1+\delta p)[(p+1)(1+\alpha)-\lambda(\gamma+p\alpha)]\left(\frac{[p+\omega]_q}{[p+1+\omega]_q}\right)^m \\ -(1+\alpha)(p+\delta p^2-\delta p)-(\gamma+\alpha p)(1+\delta\lambda p-\delta\lambda)\end{bmatrix}}$$

$$= p - \vartheta.$$

Then based on Definition 7.1, $k(s) \in \mathcal{FJ}_{p,q}^m(\alpha, \delta, \lambda, \gamma)$ for each ϑ specified by (7.3).

8. Convolution Properties

Theorems showing the convolution characteristics for functions belonging to the class $\mathcal{FJ}_{p,q}^m(\alpha, \delta, \lambda, \gamma)$ are given below.

Theorem 8.1. *Let the functions $k_j (j = 1,2)$ defined by*

$$k_j(s) = s^p - \sum_{n=p+1}^{\infty} d_{n,j} s^n, \qquad (d_{n,j} \geq 0,\ j = 1,2), \tag{8.1}$$

be in the class $\mathcal{FJ}_{p,q}^m(\alpha, \delta, \lambda, \gamma)$. Then we have $(k_1 * k_2)(s) \in \mathcal{FJ}_{p,q}^m(\alpha, \delta, \beta, \gamma)$, where

$$\beta \leq \frac{\begin{array}{c}\big((1+\alpha)(p+\delta p^2-\delta p)-(\gamma+\alpha p)\big)(1+\delta n-\delta)[n(1+\alpha)-\lambda(\gamma+\alpha p)]^2\left(\frac{[p+\omega]_q}{[n+\omega]_q}\right)^m \\ -n(1+\alpha)[(1+\alpha)(p+\delta p^2-\delta p)-(\gamma+\alpha p)(1+\delta\lambda p-\delta\lambda)]^2\end{array}}{\begin{array}{c}\delta(p-1)(\gamma+\alpha p)(1+\delta n-\delta)[n(1+\alpha)-\lambda(\gamma+\alpha p)]^2\left(\frac{[p+\omega]_q}{[n+\omega]_q}\right)^m \\ -(\gamma+\alpha p)[(p+\delta p^2-\delta p)(1+\alpha)-(\gamma+\alpha p)(1+\delta\lambda p-\delta\lambda)]^2\end{array}}.$$

Proof. We need to find the largest β such that

$$\sum_{n=p+1}^{\infty}\frac{(1+\delta n-\delta)\big(n(1+\alpha)-\beta(\gamma+\alpha p)\big)\left(\frac{[p+\omega]_q}{[n+\omega]_q}\right)^m}{(1+\alpha)(p+\delta p^2-\delta p)-(\gamma+\alpha p)\big(1+\delta\beta(p-1)\big)}d_{n,1}d_{n,2}\leq 1.$$

Since the functions $k_j\in\mathcal{FJ}_{p,q}^m(\alpha,\delta,\lambda,\gamma)$, $(j=1,2)$, then from Theorem 2.1, we have

$$\sum_{n=p+1}^{\infty}\frac{(1+\delta n-\delta)\big(n(1+\alpha)-\lambda(\gamma+p\alpha)\big)\left(\frac{[p+\omega]_q}{[n+\omega]_q}\right)^m}{(1+\alpha)(p+\delta p^2-\delta p)-(\gamma+\alpha p)\big(1+\delta\lambda(p-1)\big)}d_{n,j}\leq 1,\quad (j=1,2). \tag{8.2}$$

By the Cauchy-Schwarz inequality, we have

$$\sum_{n=p+1}^{\infty}\frac{(1+\delta n-\delta)\big(n(1+\alpha)-\lambda(\gamma+p\alpha)\big)\left(\frac{[p+\omega]_q}{[n+\omega]_q}\right)^m}{(1+\alpha)(p+\delta p^2-\delta p)-(\gamma+\alpha p)\big(1+\delta\lambda(p-1)\big)}\sqrt{d_{n,1}d_{n,2}}\leq 1. \tag{8.3}$$

Thus, it is sufficient to show that

$$\frac{(1+\delta n-\delta)[n(1+\alpha)-\beta(\gamma+p\alpha)]\left(\frac{[p+\omega]_q}{[n+\omega]_q}\right)^m}{(1+\alpha)(p+\delta p^2-\delta p)-(\gamma+\alpha p)\big(1+\delta\beta(p-1)\big)}d_{n,1}d_{n,2}$$

$$\leq\frac{(1+\delta n-\delta)[n(1+\alpha)-\lambda(\gamma+p\alpha)]\left(\frac{[p+\omega]_q}{[n+\omega]_q}\right)^m}{(1+\alpha)(p+\delta p^2-\delta p)-(\gamma+\alpha p)\big(1+\delta\lambda(p-1)\big)}\sqrt{d_{n,1}d_{n,2}}.$$

That is

$$\sqrt{d_{n,1}d_{n,2}}\leq\frac{\big(n(1+\alpha)-\lambda(\gamma+p\alpha)\big)[(1+\alpha)(p+\delta p^2-\delta p)-(\gamma+\alpha p)(1+\delta\beta p-\delta\beta)]}{\big(n(1+\alpha)-\beta(\gamma+p\alpha)\big)\big((1+\alpha)(p+\delta p^2-\delta p)-(\gamma+\alpha p)(1+\delta\lambda p-\delta\lambda)\big)}. \tag{8.4}$$

But from (8.3), we get

$$\sqrt{d_{n,1}d_{n,2}} \leq \frac{(1+\alpha)(p+\delta p^2-\delta p)-(\gamma+\alpha p)\big(1+\delta\lambda(p-1)\big)}{(1+\delta n-\delta)\big(n(1+\alpha)-\lambda(\gamma+p\alpha)\big)\left(\frac{[p+\omega]_q}{[n+\omega]_q}\right)^m},$$

Consequently, we need only to prove that

$$\frac{(1+\alpha)(p+\delta p^2-\delta p)-(\gamma+\alpha p)\big(1+\delta\lambda(p-1)\big)}{(1+\delta n-\delta)\big(n(1+\alpha)-\lambda(\gamma+p\alpha)\big)\left(\frac{[p+\omega]_q}{[n+\omega]_q}\right)^m}$$

$$\leq \frac{\big(n(1+\alpha)-\lambda(\gamma+p\alpha)\big)\big[(1+\alpha)(p+\delta p^2-\delta p)-(\gamma+\alpha p)\big(1+\delta\beta(p-1)\big)\big]}{\big(n(1+\alpha)-\beta(\gamma+p\alpha)\big)\big[(1+\alpha)(p+\delta p^2-\delta p)-(\gamma+\alpha p)\big(1+\delta\lambda(p-1)\big)\big]}, \quad (8.5)$$

or equivalently, that

$$\beta \leq \frac{\begin{array}{c}\big((1+\alpha)(p+\delta p^2-\delta p)-(\gamma+\alpha p)\big)(1+\delta n-\delta)[n(1+\alpha)-\lambda(\gamma+\alpha p)]^2\left(\frac{[p+\omega]_q}{[n+\omega]_q}\right)^m\\ -n(1+\alpha)[(1+\alpha)(p+\delta p^2-\delta p)-(\gamma+\alpha p)(1+\delta\lambda p-\delta\lambda)]^2\end{array}}{\begin{array}{c}\delta(p-1)(\gamma+\alpha p)(1+\delta n-\delta)[n(1+\alpha)-\lambda(\gamma+\alpha p)]^2\left(\frac{[p+\omega]_q}{[n+\omega]_q}\right)^m\\ -(\gamma+\alpha p)[(1+\alpha)(p+\delta p^2-\delta p)-(\gamma+\alpha p)(1+\delta\lambda p-\delta\lambda)]^2\end{array}}.$$

The proof is complete.

Theorem 8.2. *Let the functions* $k_j (j = 1,2)$ *defined by* (8.1) *be in the class* $\mathcal{FI}^m_{p,q}(\alpha,\delta,\lambda,\gamma)$. *Then the function*

$$h(s) = s^p - \sum_{n=p+1}^{\infty}\big(d_{n,1}^2+d_{n,2}^2\big)s^n, \quad (8.6)$$

is in the class $\mathcal{FI}^m_{p,q}(\alpha,\delta,\varepsilon,\gamma)$, *where*

$$\varepsilon \leq \frac{\begin{array}{c}\big((1+\alpha)(p+\delta p^2-\delta p)-(\gamma+\alpha p)\big)\big[(1+\delta n-\delta)\big(n(1+\alpha)-\lambda(\gamma+\alpha p)\big)\big]^2\left(\frac{[p+\omega]_q}{[n+\omega]_q}\right)^m\\ -2n(1+\alpha)(1+\delta n-\delta)\big[(1+\alpha)(p+\delta p^2-\delta p)-(\gamma+\alpha p)\big(1+\lambda\delta(p-1)\big)\big]^2\end{array}}{\begin{array}{c}\big(\delta(p-1)(\gamma+\alpha p)\big)\big[(1+\delta n-\delta)\big(n(1+\alpha)-\lambda(\gamma+\alpha p)\big)\big]^2\left(\frac{[p+\omega]_q}{[n+\omega]_q}\right)^m\\ -2(\gamma+\alpha p)(1+\delta n-\delta)\big[(1+\alpha)(p+\delta p^2-\delta p)-(\gamma+\alpha p)\big(1+\lambda\delta(p-1)\big)\big]^2\end{array}}.$$

Proof. From Theorem 2.1, we are seeking the largest ε so that

$$\sum_{n=p+1}^{\infty} \frac{(1+\delta n-\delta)\big(n(1+\alpha)-\varepsilon(\gamma+p\alpha)\big)\left(\frac{[p+\omega]_q}{[n+\omega]_q}\right)^m}{(1+\alpha)(p+\delta p^2-\delta p)-(\gamma+\alpha p)\big(1+\varepsilon\delta(p-1)\big)}\big(d_{n,1}^2+d_{n,2}^2\big) \le 1.$$

Since $k_j (j=1,2)$ belong to the class $\mathcal{FI}_{p,q}^m(\alpha,\delta,\lambda,\gamma)$, we have

$$\sum_{n=p+1}^{\infty} \left(\frac{(1+\delta n-\delta)\big(n(1+\alpha)-\lambda(\gamma+p\alpha)\big)\left(\frac{[p+\omega]_q}{[n+\omega]_q}\right)^m}{(1+\alpha)(p+\delta p^2-\delta p)-(\gamma+\alpha p)\big(1+\lambda\delta(p-1)\big)}\right)^2 d_{n,1}^2$$

$$\le \sum_{n=p+1}^{\infty} \left(\frac{(1+\delta n-\delta)\big(n(1+\alpha)-\lambda(\gamma+p\alpha)\big)\left(\frac{[p+\omega]_q}{[n+\omega]_q}\right)^m}{(1+\alpha)(p+\delta p^2-\delta p)-(\gamma+\alpha p)\big(1+\lambda\delta(p-1)\big)} d_{n,1}\right)^2 \le 1, \quad (8.7)$$

and

$$\sum_{n=p+1}^{\infty} \left(\frac{(1+\delta n-\delta)\big(n(1+\alpha)-\lambda(\gamma+p\alpha)\big)\left(\frac{[p+\omega]_q}{[n+\omega]_q}\right)^m}{(1+\alpha)(p+\delta p^2-\delta p)-(\gamma+\alpha p)\big(1+\lambda\delta(p-1)\big)}\right)^2 d_{n,2}^2$$

$$\le \sum_{n=p+1}^{\infty} \left(\frac{(1+\delta n-\delta)\big(n(1+\alpha)-\lambda(\gamma+p\alpha)\big)\left(\frac{[p+\omega]_q}{[n+\omega]_q}\right)^m}{(1+\alpha)(p+\delta p^2-\delta p)-(\gamma+\alpha p)\big(1+\lambda\delta(p-1)\big)} d_{n,2}\right)^2 \le 1. \quad (8.8)$$

Hence, we have

$$\sum_{n=p+1}^{\infty} \frac{1}{2}\left(\frac{(1+\delta n-\delta)\big(n(1+\alpha)-\lambda(\gamma+p\alpha)\big)\left(\frac{[p+\omega]_q}{[n+\omega]_q}\right)^m}{(1+\alpha)(p+\delta p^2-\delta p)-(\gamma+\alpha p)\big(1+\lambda\delta(p-1)\big)}\right)^2 \big(d_{n,1}^2+d_{n,2}^2\big) \le 1. \quad (8.9)$$

$k(s) \in \mathcal{FI}_{p,q}^m(\alpha,\delta,\lambda,\gamma)$ if and only if

$$\sum_{n=p+1}^{\infty} \frac{(1+\delta n-\delta)\big(n(1+\alpha)-\varepsilon(\gamma+p\alpha)\big)\left(\frac{[p+\omega]_q}{[n+\omega]_q}\right)^m}{(1+\alpha)(p+\delta p^2-\delta p)-(\gamma+\alpha p)\big(1+\varepsilon\delta(p-1)\big)}\big(d_{n,1}^2+d_{n,2}^2\big) \le 1. \quad (8.10)$$

Therefore, we need to find the largest ε such that

$$\frac{(1+\delta n-\delta)\big(n(1+\alpha)-\varepsilon(\gamma+\alpha p)\big)\left(\frac{[p+\omega]_q}{[n+\omega]_q}\right)^m}{(1+\alpha)(p+\delta p^2-\delta p)-(\gamma+\alpha p)\big(1+\varepsilon\delta(p-1)\big)}$$

$$\leq \frac{1}{2}\left(\frac{(1+\delta n-\delta)\big(n(1+\alpha)-\lambda(\gamma+\alpha p)\big)\left(\frac{[p+\omega]_q}{[n+\omega]_q}\right)^m}{(1+\alpha)(p+\delta p^2-\delta p)-(\gamma+\alpha p)\big(1+\lambda\delta(p-1)\big)}\right)^2. \qquad (8.11)$$

From (8.11), we get

$$\varepsilon \leq \frac{\begin{array}{c}\big((1+\alpha)(p+\delta p^2-\delta p)-(\gamma+\alpha p)\big)\big[(1+\delta n-\delta)\big(n(1+\alpha)-\lambda(\gamma+\alpha p)\big)\big]^2\left(\frac{[p+\omega]_q}{[n+\omega]_q}\right)^m \\ -2n(1+\alpha)(1+\delta n-\delta)\big[(1+\alpha)(p+\delta p^2-\delta p)-(\gamma+\alpha p)\big(1+\lambda\delta(p-1)\big)\big]^2\end{array}}{\begin{array}{c}\big(\delta(p-1)(\gamma+\alpha p)\big)\big[(1+\delta n-\delta)\big(n(1+\alpha)-\lambda(\gamma+\alpha p)\big)\big]^2\left(\frac{[p+\omega]_q}{[n+\omega]_q}\right)^m \\ -2(\gamma+\alpha p)(1+\delta n-\delta)\big[(1+\alpha)(p+\delta p^2-\delta p)-(\gamma+\alpha p)\big(1+\lambda\delta(p-1)\big)\big]^2\end{array}}.$$

The proof is complete.

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