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Cubic Spline Chebyshev Polynomial Approximation for Solving Boundary Value Problems

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Abstract

In this work, a Chebyshev polynomial spline function is derived and used to approximate the solution of the second order two-point boundary value problems of variable coefficients with the associated boundary conditions. In deriving the method, the cubic spline Chebyshev polynomial approximation, $S(x)$ is made to satisfy certain conditions for continuity and smoothness of functions. Numerical examples are presented to illustrate the applications of this method. The solution, $y(x)$ of these examples are obtained at some nodal points in the interval of consideration. The absolute errors in each example are estimated, and the comparison of exact values, and approximate values by the present method and other methods in literature at the nodal points are presented graphically. The comparison shows that the proposed method produces better results than Approaching Spline Techniques and collocation method.

1 Introduction

The basic idea behind cubic spline approximation/interpolation is based on the engineer's tool used for drawing smooth curves through a number of points [1].

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This spline consists of weights attached to a flat surface at the points to be connected. A flexible strip is then bent across each of these weights, resulting in a pleasingly smooth curve [2].

There are many linear and nonlinear problems in science and engineering, namely, second order differential equations with various types of boundary conditions, which are solved either analytically or numerically [3]. Numerical simulation in engineering science and in applied mathematics has become a powerful tool to model the physical phenomena particularly when analytical solutions are not available [8]. The numerical solution of two-point Boundary Value Problems (BVPs) is of great importance due to its wide applications in scientific research [8]. Several authors have considered the solution of boundary value problems in the past: Ravi and Reddy [3], Caglar et al. [4], Chang et al. [5], El-Gamel [6], Jang [7] and Chang et al. [8].

In interpolating problems, spline interpolation is often preferred to polynomial interpolation because it yields similar results to interpolating with higher degree polynomials while avoiding instability due to Runge's phenomenon [9]. In computer graphics, parametric curves whose coordinates are given by spline are popular because of the simplicity of their construction, ease of manipulation and accuracy of evaluation, and their capacity to approximate complex shapes through curve fitting and interactive curve design [9].

The linear two-point boundary value problem considered in this paper is of the form

$$y''(x) + p(x)y'(x) + q(x)y(x) = f(x), \quad a \le x \le b \tag{1}$$

with the boundary conditions

$$\begin{cases} y(a) = y_a \\ y(b) = y_b \end{cases} \tag{2}$$

where equation (1) together with the boundary conditions is assumed to have a unique solution $y(x)$ if $p(x) \in C'[a,b]$, $p > 0$, $q(x)$, $f(x) \in C[a,b]$, $f(x)$ is a given smooth function and y_a, y_b are real constants [2,8,9,10,11].

2 Methodology

Generally, cubic spline polynomial involves four constants, so there is sufficient flexibility in the procedure to ensure that the interpolant is not only continuously differentiable on the interval but also has a continuous second derivative.

The cubic spline Chebyshev polynomial $S(x)$ approximation for the solution of (1) together with (2) is derived and made to satisfy the following conditions:
1. $S(x) = S_i(x)$ is a cubic polynomial on each sub-interval $[x_i, x_{i+1}]$ for $i = 0, 1, ..., N-1$;
2. $S(x) = y_i$ for $i = 0, 1, ..., N$ and
3. S is smooth. That is, $S(x)$, $S'(x)$, and $S''(x)$ are continuous on $[a, b]$.

The Chebyshev polynomial of degree n of the first kind valid in the interval $[a, b]$ is defined as

$$T_n = cos\left[ncos^{-1}\left(\frac{2x-a-b}{b-a}\right)\right]. \tag{3}$$

In the interval $[-1, 1]$, equation (3) becomes

$$T_n = cos(ncos^{-1}(x) \tag{4}$$

and the recurrence relation of equation (4) is

$$T_{n+1} = 2T_1(x)T_n(x) - T_{n-1}(x) \ : \ T_0(x) = 1, T_1(x) = x. \tag{5}$$

Therefore, using (3)-(5)

$$T_2(x) = 2x^2 - 1 \tag{6}$$

and

$$T_3(x) = 4x^3 - 3x. \tag{7}$$

Using (5)-(7), the cubic spline Chebyshev polynomial approximation is then written as

$$S_\triangle = a_i + b_i(x - x_i) + c_i[2(x - x_i)^2 - 1] + d_i[4(x - x_i)^3 - 3(x - x_i)]. \tag{8}$$

Letting
$$S_\triangle(x_i) = y_i \qquad S_\triangle(x_{i+1}) = y_{i+1}$$
$$S''_\triangle(x_i) = M_i \qquad S''_\triangle(x_{i+1}) = M_{i+1}$$

$$S'_\triangle(x) = b_i + 4c_i(x - x_i) + 12d_i(x - x_i)^2 - 3d_i \tag{9}$$

$$S''_\triangle(x) = 4c_i + 24d_i(x - x_i). \tag{10}$$

Substitute x_i for x in (9) and (10) to obtain

$$a_i = y_i + \frac{M_i}{4} \tag{11}$$

$$c_i = \frac{M_i}{4}. \tag{12}$$

Substitute x_{i+1} for x in (8) to obtain

$$a_i+b_i(x_{i+1}-x_i)+c_i[2(x_{i+1}-x_i)^2-1]+d_i[4(x_{i+1}-x_i)^3-3(x_{i+1}-x_i)] = y_{i+1}. \tag{13}$$

Then
$$4c_i + 24d_i(x_{i+1} - x_i) = M_{i+1}. \tag{14}$$

Substituting for a_i and c_i (13) and (14) with the step length $h = x_{i+1} - x_i$, gives

$$a_i + hb_i + (2h^2 - 1)c_i + (4h^3 - 3h)d_i = y_{i+1} \tag{15}$$

$$d_i = \frac{M_{i+1} - M_i}{24h}. \tag{16}$$

Using equations (11)-(16),

$$b_i = \frac{y_{i+1} - y_i}{h} - \left(\frac{h}{3} + \frac{1}{8h}\right) M_i - \left(\frac{h}{6} - \frac{1}{8h}\right) M_{i+1}. \tag{17}$$

Setting
$$S'_{\triangle_{i-1}}(x_i) = S'_{\triangle_i}(x_i)$$

leads to
$$y_{i+1} - 2y_i + y_{i-1} = \frac{h^2}{6}(M_{i+1} + 4M_i + M_{i-1}). \tag{18}$$

Generally, equation (18) is written as

$$\alpha M_{i+1} + 2\beta M_i + \alpha M_{i-1} = \frac{1}{h^2}(y_{i+1} - 2y_i + y_{i-1}) \tag{19}$$

where $\alpha = \frac{1}{6}$ and $\beta = \frac{1}{3}$.

Equation (1) is rewritten as

$$\frac{d^2y}{dx^2} + q(x)\frac{dy}{dx} + r(x)y = f(x), \tag{20}$$

where $q(x) = \frac{p'(x)}{p(x)}$, $r(x) = -\frac{v(x)}{p(x)}$, $f(x) = -\frac{g(x)}{p(x)}$.

Equation (20) is discretized at the nodal points x_i by

$$y_i'' + q_i y_i' + r_i y_i = f_i, \tag{21}$$

where $q_i = q(x_i)$, $r_i = r(x_i)$, $f_i = f(x_i)$.

Taking the moment of the spline leads to

$$M_i = f_i - q_i y_i' - r_i y_i. \tag{22}$$

Equation (22) is substituted into (18) to obtain

$$\begin{aligned} y_{i+1} - 2y_i + y_{i-1} = \frac{h^2}{6}[&f_{i+1} - q_{i+1}y'_{i+1} - r_{i+1}y_{i+1} + 4(f_i - q_i y_i' - r_i y_i) \\ &+ f_{i-1} - q_{i-1}y'_{i-1} - r_{i-1}y_{i-1}]. \end{aligned} \tag{23}$$

Using the fourth order methods in (23)

$$y_i' = \frac{y_{i+1} - y_{i-1}}{2h},$$

$$y'_{i+1} = \frac{3y_{i+1} - 4y_i + y_{i-1}}{2h}$$

and

$$y'_{i-1} = \frac{-y_{i+1} + 4y_i - 3y_{i-1}}{2h}$$

where $i = 1, 2, 3, ...$ leads to

$$\left(-\frac{3h}{2}q_{i-1} - 2hq_i + \frac{h}{2}q_{i+1} + h^2 r_{i-1} + 6\right) y_{i-1} + (2hq_{i-1} - 2hq_{i+1} + 4h^2 r_i - 12)y_i$$

$$+\left(-\frac{h}{2}q_{i-1} + 2hq_i + \frac{3h}{2}q_{i+1} + h^2 r_{i+1} + 6\right) y_{i+1} = h^2(f_{i-1} + 4f_i + f_{i+1}) \quad (24)$$

with $y(a) = y_a$ and $y(b) = y_b$ with $i = 1(1)N - 1$.

Equation (24) is a tridiagonal system written in the form

$$A\underline{Y} = \underline{f}. \quad (25)$$

Equation (25) is then solved using MATLAB to obtain the values $y_i \ : \ i = 1, 2, 3, ..., n - 1$.

3 Numerical Examples

To compare the approximate solution, $y(x_i)$ with the exact solution, $Y(x_i)$ the absolute error between the two solutions is defined as

$$E_{rr} = \mid Y(x_i) - y(x_i) \mid, \quad i = 1, 2, 3, ... \quad (26)$$

The computations and programmes contained in the work are carried out with the aid of MATLAB software.

Example 1: Consider the second order Ordinary Differential Equation

$$y''(x) = \frac{2}{x^2}y(x) - \frac{1}{x} \ ; \ 2 < x < 3 \ \ y(2) = 0, \ \ y(3) = 0 \ : h = \frac{1}{10}. \quad (27)$$

The exact solution is

$$y(x) = \frac{1}{38}\left(-5x^2 + 19x - \frac{36}{x}\right).$$

Comparing equations (21) and (27)

$$x_i = 2(\tfrac{1}{10})3$$

$r_i = -\frac{2}{x_i^2}$

$q_i = 0 \Longrightarrow q_0 = q_1 = ... = q_9 = 0$

$f_i = -\frac{1}{x_i}.$

When $i = 1(2)9$, equation (27) leads to the following system

$$\frac{1199}{200}y_0 - \frac{5300}{441}y_1 + \frac{1451}{241}y_2 = -\frac{1321}{46200}$$

$$\frac{2644}{441}y_1 - \frac{1454}{121}y_2 + \frac{3172}{529}y_3 = -\frac{145}{5313}$$

$$\frac{1451}{242}y_2 - \frac{6356}{529}y_3 + \frac{1727}{288}y_4 = -\frac{317}{12144}$$

$$\frac{3172}{529}y_3 - \frac{865}{72}y_4 + \frac{3748}{625}y_5 = -\frac{863}{34500}$$

$$\frac{1727}{288}y_4 - \frac{7508}{625}y_5 + \frac{2027}{338}y_6 = -\frac{1873}{78000}$$

$$\frac{3748}{625}y_5 - \frac{2030}{169}y_6 + \frac{4372}{729}y_7 = -\frac{1013}{43875}$$

$$\frac{2027}{338}y_6 - \frac{8756}{729}y_7 + \frac{2351}{392}y_8 = -\frac{437}{19656}$$

$$\frac{4372}{729}y_7 - \frac{1177}{98}y_8 + \frac{5044}{841}y_9 = -\frac{235}{10962}$$

$$\frac{2351}{392}y_8 - \frac{10100}{841}y_9 + \frac{2699}{450}y_{10} = -\frac{2521}{121800}$$

Imposing the boundary conditions $y(2) = y(3) = 0$ results in the tridiagonal system $A\underline{Y} = \underline{f}$, where

$$A = \begin{pmatrix} -\frac{5300}{441} & \frac{1451}{242} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{2644}{441} & -\frac{1454}{121} & \frac{3172}{529} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1451}{242} & -\frac{6356}{529} & \frac{1727}{288} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{3172}{529} & -\frac{865}{72} & \frac{3748}{625} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1727}{288} & -\frac{7508}{625} & \frac{2027}{338} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{3748}{625} & -\frac{2030}{169} & \frac{4372}{729} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{2027}{338} & -\frac{8756}{729} & \frac{2351}{392} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{4372}{729} & -\frac{1177}{98} & \frac{5044}{841} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{10100}{841} & \frac{2699}{450} \end{pmatrix} \tag{28}$$

and

$$\underline{f} = \left(-\frac{1321}{46200}, -\frac{145}{5313}, -\frac{317}{12144}, -\frac{863}{34500}, -\frac{1873}{78000}, -\frac{1013}{43875}, -\frac{437}{19656}, -\frac{235}{10962}, -\frac{2521}{121800}\right)^T. \tag{29}$$

Example 2: Consider the second order Ordinary Differential Equation of variable coefficients

$$y'' - \frac{2}{x}y' + \frac{4}{x^2}y = 1, \quad y(10) = 0,\ y(20) = 100. \tag{30}$$

Remark: Closed form solution does not exist.

Example 3: Consider the stiff second order Ordinary Differential Equation of variable coefficients

$$y'' - (x+1)y = -e^{-x}(x^2 - 2x + 2), \quad y(2) = 0, \; y(4) = 0.036631. \tag{31}$$

The exact solution is

$$y(x) = e^{-x}(x-2).$$

Tables of Results

Table 1: Numerical Results for Example 1: Comparison between the absolute errors in the Approaching Spline Techniques and the present method

x	Exact Solution	Kalyani and Rao (2013)	Error	Present Method	Error
2.0	0.000000	0.000000	0.00000	0.0000000	0.00000
2.1	0.018609	0.018809	2.00E-4	0.0186217	1.27E-5
2.2	0.032536	0.032904	3.68E-4	0.0325566	2.06E-5
2.3	0.042048	0.042537	4.89E-4	0.0420733	2.53E-5
2.4	0.047368	0.047925	5.57E-4	0.0473952	2.72E-5
2.5	0.048684	0.049252	5.68E-4	0.0487103	2.63E-5
2.6	0.046154	0.046680	5.26E-4	0.0461773	2.33E-5
2.7	0.039912	0.040350	4.38E-4	0.0399316	1.96E-5
2.8	0.030075	0.030389	3.14E-4	0.0300891	1.41E-5
2.9	0.016742	0.016908	1.66E-4	0.0167497	7070E-6
3.0	0.000000	0.000000	0.00000	0.0000000	0.00000

Table 2: Numerical Results for Example 2: Comparison between the absolute errors in the Chebyshev Collocation method and Present method

x	Exact Solution Mathematica	Chebyshev Collocation Method	Error	Present Method	Error
10	$-1.11022 * 10^{-16}$	$1.06581 * 10^{-14}$	1.06692E-14	0.0000000	1.11022E-16
11	2.36996	2.33973	3.023E-2	2.37478	4.82E-3
12	6.22043	6.18761	3.282E-2	6.22791	7.48E-3
13	11.66800	11.6401	2.790E-2	11.6761	8.10E-3
14	18.80360	18.7802	2.340E-2	18.8102	6.60E-3
15	27.69750	27.6774	2.010E-2	27.7010	3.50E-3
16	38.40360	38.3873	1.630E-2	38.4021	1.50E-3
17	50.96200	50.9526	9.400E-2	50.9539	8.10E-3
18	65.40200	65.4018	2.000E-2	65.3858	1.62E-2
19	81.74370	81.7504	6.700E-2	81.7181	2.56E-2
20	100.0000	100.0000	0.000000	100.000	0.000000

Table 3: Numerical Results for Example 3: Comparison between the exact solution, the Power series and Chebyshev Collocation method and the Present method

x	Exact Solution	Power Series Collocation Method N=7	Error $h = 0.0001$	Chebyshev Collocation Method N=4	Error $h = 0.0001$	Present Method	Error $h = 0.2$
2.0	0.0000000	-0.00000986	9.86E-6	0.00000624	6.24E-6	0.000000	0.00000
2.2	0.0221606	0.0221302	3.04E-5	0.0217249	4.357E-4	0.0222135	5.29E-5
2.4	0.0362872	0.0362667	2.05E-5	0.0359400	3.472E-4	0.036342	5.48E-5
2.6	0.0445641	0.0445480	1.61E-5	0.0443943	1.698E-4	0.0445911	2.70E-5
2.8	0.0486481	0.0486357	1.24E-5	0.0485827	6.540E-5	0.0486412	6.90E-6
3.0	0.0497871	0.0497802	6.90E-6	0.0497527	3.440E-5	0.0497643	2.28E-5
3.2	0.0489146	0.0489098	4.80E-6	0.0489038	1.080E-5	0.0488513	6.33E-5
3.4	0.0467226	0.0467163	6.30E-6	0.0467879	6.530E-5	0.0465919	1.307E-4
3.6	0.0437180	0.0437188	8.00E-7	0.0439092	1.912E-4	0.0434841	2.339E-4
3.8	0.0402674	0.0402904	2.30E-5	0.0405241	2.567E-4	0.0398765	3.909E-4
4.0	0.0366313	0.0366287	2.60E-6	0.0366414	1.010E-5	0.0366313	0.000000

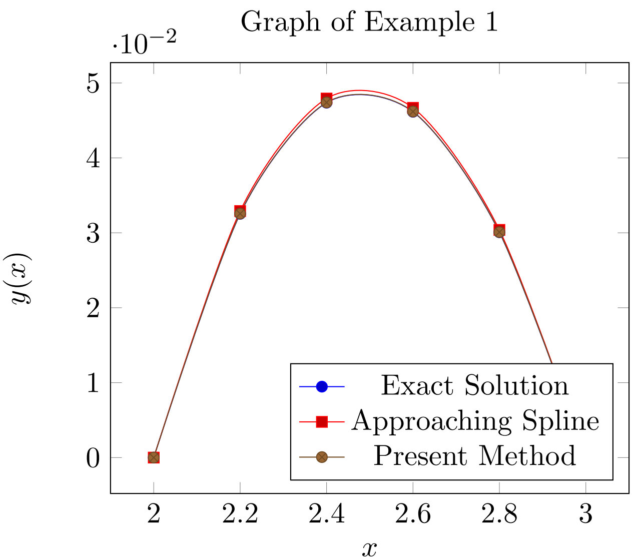

Figure 1: The relationship between the exact solution and the solutions by the Approaching Spline Techniques and the present method.

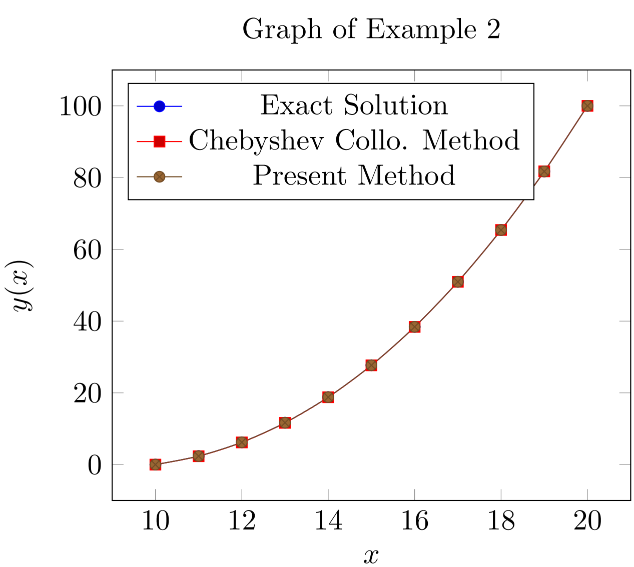

Figure 2: The relationship between the exact solution and the solutions by Chebyshev Collocation method and the Present Method.

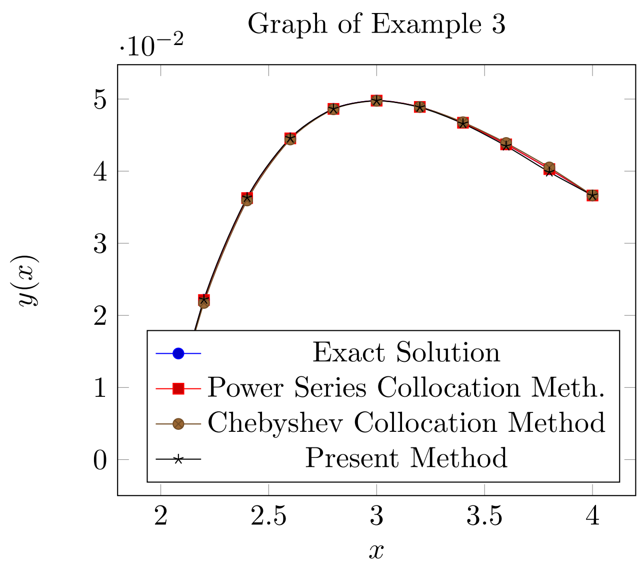

Figure 3: The relationship between the exact solution and the solutions by Power series collocation method, Chebyshev collocation method and Present Method.

4 Discussion of Results and Conclusion

In this paper, cubic spline Chebyshev approximation for solving second order boundary value problems of ordinary differential equations is derived. Three equations of variable coefficients are considered and the results obtained are presented graphically in Figures 1-3. In examples 1 and 3, the coefficient of y' is zero while the coefficients of y' and y are non-zero in example 2. The results obtained by the present method when compared with the exact solution and results by other methods in literature are generally better as indicated in Tables 1-3. Better results results can be obtained using the proposed method with h assuming smaller values.

In conclusion, the method proposed in this paper is a powerful tool for solving second order ordinary differential equations of variable coefficients due to its simplicity and accuracy.

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