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Weakly Reich Type Cyclic Contraction Mapping Principle

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Abstract

In this paper we introduce the notion of Reich type cyclic weakly contraction and prove a fixed point theorem. Some Corollaries are consequences of the main result.

1 Introduction and Preliminaries

Theorem 1.1 ([1, 2]). *If $T : X \mapsto X$, where (X, d) is a complete metric space, satisfies*

$$d(Tx, Ty) \leq k[d(x, Tx) + d(y, Ty)]$$

where $0 < k < \frac{1}{2}$ and $x, y \in X$, then T has a unique fixed point.

Definition 1.2 ([3]). *Let X be a nonempty set and $T : X \mapsto X$ be an operator. By definition, $X = \bigcup_{i=1}^{m} X_i$ is a cyclic representation of X with respect to T if*

(a) $X_i, i = 1, \cdots, m$ are nonempty sets,

(b) $T(X_1) \subseteq X_2, \cdots, T(X_{m-1}) \subseteq X_m, T(X_m) \subseteq X_1$.

Notations 1.3 ([4]). *Φ will denote all monotone increasing continuous functions $\mu : [0, \infty) \mapsto [0, \infty)$ with $\mu(t) = 0$ if and only if $t = 0$.*

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Notations 1.4 ([4]). *Ψ will denote all lower semi-continuous functions $\psi : [0,\infty)^2 \mapsto [0,\infty)$ with $\psi(x_1,x_2) > 0$ for $x_1, x_2 \in (0,\infty)$ and $\psi(0,0)=0$.*

Definition 1.5 ([4]). *Let (X,d) be a metric space, $m \in \mathbb{N}$, $A_1, A_2, \cdots, A_m$ be nonempty subsets of X, and $Y = \bigcup_{i=1}^m A_i$. An operator $T : Y \mapsto Y$ is called a Kannan type cyclic weakly contraction if*

(a) $\bigcup_{i=1}^m A_i$ *is a cyclic representation of Y with respect to T,*

(b) $\mu(d(Tx,Ty)) \leq \mu\left(\frac{1}{2}[d(x,Tx)+d(y,Ty)]\right) - \psi(d(x,Tx), d(y,Ty))$ *for any $x \in A_i$, $y \in A_{i+1}, i = 1,2,\cdots,m$, where $A_{m+1} = A_1$, $\mu \in \Phi$, and $\psi \in \Psi$.*

Theorem 1.6 ([4]). *Let (X,d) be a complete metric space, $m \in \mathbb{N}, A_1, A_2, \cdots, A_m$ be nonempty closed subsets of X and $Y = \bigcup_{i=1}^m A_i$. Suppose that T is a Kannan type cyclic weakly contraction. Then, T has a fixed point $z \in \bigcap_{i=1}^m A_i$.*

Theorem 1.7 ([5]). *If (X,d) is a complete metric space and $T : X \mapsto X$ satisfies*

$$d(Tx,Ty) \leq k[d(x,Ty)+d(y,Tx)]$$

where $0 < k < \frac{1}{2}$ and $x, y \in X$, then T has a unique fixed point.

Definition 1.8 ([6]). *Let (X,d) be a metric space, $m \in \mathbb{N}$, $A_1, A_2, \cdots, A_m$ be nonempty subsets of X, and $Y = \bigcup_{i=1}^m A_i$. An operator $T : Y \mapsto Y$ is called a Chatterjea type cyclic weakly contraction if*

(a) $\bigcup_{i=1}^m A_i$ *is a cyclic representation of Y with respect to T,*

(b) $\mu(d(Tx,Ty)) \leq \mu\left(\frac{1}{2}[d(x,Ty)+d(y,Tx)]\right) - \psi(d(x,Ty), d(y,Tx))$ *for any $x \in A_i$, $y \in A_{i+1}$, $i = 1,2,\cdots,m$, where $A_{m+1} = A_1$, $\mu \in \Phi$, and $\psi \in \Psi$.*

Theorem 1.9 ([5]). *Let (X,d) be a complete metric space, $m \in \mathbb{N}, A_1, A_2, \cdots, A_m$ be nonempty closed subsets of X and $Y = \bigcup_{i=1}^m A_i$. Suppose that T is a Chatterjea type cyclic weakly contraction. Then, T has a fixed point $z \in \bigcap_{i=1}^m A_i$.*

2 Main Result

Notations 2.1. *Ω will denote all lower semi-continuous functions $\psi : [0,\infty)^3 \mapsto [0,\infty)$ with $\psi(x_1, x_2, x_3) > 0$ for $x_1, x_2, x_3 \in (0,\infty)$ and $\psi(0,0,0) = 0$.*

Definition 2.2. *Let (X, d) be a metric space, $m \in \mathbb{N}$, $A_1, A_2, \cdots, A_m$ be nonempty subsets of X, and $Y = \bigcup_{i=1}^{m} A_i$. An operator $T : Y \mapsto Y$ is called a Reich type cyclic weakly contraction if*

(a) $\bigcup_{i=1}^{m} A_i$ is a cyclic representation of Y with respect to T,

(b) $\mu(d(Tx, Ty)) \leq \mu\left(\frac{1}{3}[d(x, Tx) + d(y, Ty) + d(x, y)]\right) - \psi(d(x, Tx), d(y, Ty), d(x, y))$ for any $x \in A_i$, $y \in A_{i+1}$, $i = 1, 2, \cdots, m$, where $A_{m+1} = A_1$, $\mu \in \Phi$, and $\psi \in \Omega$.

Theorem 2.3. *Let (X, d) be a complete metric space, $m \in \mathbb{N}, A_1, A_2, \cdots, A_m$ be nonempty closed subsets of X and $Y = \bigcup_{i=1}^{m} A_i$. Suppose that T is a Reich type cyclic weakly contraction. Then, T has a fixed point $z \in \bigcap_{i=1}^{m} A_i$.*

Proof. Let $x_0 \in X$. We can construct a sequence $x_{n+1} = Tx_n, n = 0, 1, 2, \cdots$. If there exists $n_0 \in \mathbb{N}$ such that $x_{n(0)+1} = x_{n(0)}$, hence the result. Indeed, we have $Tx_{n(0)} = x_{n(0)+1} = x_{n(0)}$. So we assume that $x_{n+1} \neq x_n$ for any $n = 0, 1, 2, \cdots$. As $X = \bigcup_{i=0}^{m} A_i$ for any $n > 0$ there exists $i_n \in \{1, 2, 3, \cdots, m\}$ such that $x_{n-1} \in A_{i(n)}$ and $x_n \in A_{i(n+1)}$.

Since T is a Reich type cyclic weakly contraction, we have

$$\begin{aligned}\mu\left(d\left(x_{n+1},x_n\right)\right) &= \mu\left(d\left(Tx_n,Tx_{n-1}\right)\right)\\ &\leq \mu\left(\frac{1}{3}\left[d\left(x_n,Tx_n\right)+d\left(x_{n-1},Tx_{n-1}\right)+d\left(x_n,x_{n-1}\right)\right]\right)\\ &\quad -\psi\left(d\left(x_n,Tx_n\right),d\left(x_{n-1},Tx_{n-1}\right),d\left(x_n,x_{n-1}\right)\right)\\ &= \mu\left(\frac{1}{3}\left[d\left(x_n,x_{n+1}\right)+d\left(x_{n-1},x_n\right)+d\left(x_n,x_{n-1}\right)\right]\right)\\ &\quad -\psi\left(d\left(x_n,x_{n+1}\right),d\left(x_{n-1},x_n\right),d\left(x_n,x_{n-1}\right)\right)\\ &< \mu\left(\frac{1}{3}\left[3d\left(x_{n-1},x_n\right)\right]\right)\\ &= \mu\left(d\left(x_{n-1},x_n\right)\right).\end{aligned}$$

Since μ is a non-decreasing function, for all $n = 1, 2, \cdots$ we have

$$d\left(x_{n+1},x_n\right)) \leq d\left(x_{n-1},x_n\right).$$

Thus $\{d\left(x_{n+1},x_n\right)\}$ is a monotone decreasing sequence of non-negative real numbers and hence is convergent. Hence there exists $r \geq 0$ such that $d\left(x_{n+1},x_n\right) \to r$ as $n \to \infty$. Since

$$\begin{aligned}\mu\left(d\left(x_{n+1},x_n\right)\right) \leq \mu&\left(\frac{1}{3}\left[d\left(x_n,x_{n+1}\right)+d\left(x_{n-1},x_n\right)+d\left(x_n,x_{n-1}\right)\right]\right)\\ &-\psi\left(d\left(x_n,x_{n+1}\right),d\left(x_{n-1},x_n\right),d\left(x_n,x_{n-1}\right)\right).\end{aligned}$$

If we let $n \to \infty$ in the above inequality, using the continuity of μ and lower semi-continuity of ψ, we obtain $\mu(r) \leq \mu(r) - \psi(r,r,r)$. This implies $\mu(r,r,r) \leq 0$ by the continuity of ψ, which is a contradiction unless $r = 0$. Thus we proved that $d\left(x_{n+1},x_n\right) \to 0$. Now we show that $\{x_n\}$ is a Cauchy sequence. For this, we prove the following claim first

(A) For every $\epsilon > 0$, there exists $n \in \mathbb{N}$ such that if $r, q \geq n$ with $r - q \equiv 1(m)$ then $d\left(x_r, x_q\right) < \epsilon$.

Assume the contrary of (A). Thus, there exists $\epsilon > 0$ such that for any $n \in \mathbb{N}$, we can find $r_n > q_n \geq n$ with $r_n - q_n \equiv 1(m)$ satisfying $d\left(x_{r(n)}, x_{q(n)}\right) \geq \epsilon$. Now, we take $n > 2m$. Then corresponding to $q_n \geq n$, we can choose r_n in such a way that it is the smallest integer with $r_n > q_n$ satisfying $r_n - q_n \equiv 1(m)$ and $d\left(x_{r(n)}, x_{q(n)}\right) \geq \epsilon$. Therefore $d\left(x_{r(n-m)}, x_{q(n)}\right) < \epsilon$. By using the triangular inequality we have

$$\begin{aligned}\epsilon &\leq d\left(x_{q(n)}, x_{r(n)}\right) \\ &\leq d\left(x_{q(n)}, x_{r(n-m)}\right) + \sum_{i=1}^{m} d\left(x_{r(n-i)}, x_{r(n-i+1)}\right) \\ &< \epsilon + \sum_{i=1}^{m} d\left(x_{r(n-i)}, x_{r(n-i+1)}\right).\end{aligned}$$

Letting $n \to \infty$ and using $d\left(x_{n+1}, x_n\right) \to 0$, we have $\lim d\left(x_{q(n)}, x_{r(n)}\right) = \epsilon$. Again, by the triangular inequality we have

$$\begin{aligned}\epsilon &\leq d\left(x_{q(n)}, x_{r(n)}\right) \\ &\leq d\left(x_{q(n)}, x_{q(n+1)}\right) + d\left(x_{q(n+1)}, x_{r(n+1)}\right) + d\left(x_{r(n+1)}, x_{r(n)}\right) \\ &\leq d\left(x_{q(n)}, x_{q(n+1)}\right) + d\left(x_{q(n+1)}, x_{q(n)}\right) \\ &\quad + d\left(x_{q(n)}, x_{r(n)}\right) + d\left(x_{r(n)}, x_{r(n+1)}\right) + d\left(x_{r(n+1)}, x_{r(n)}\right)\end{aligned}$$

Letting $n \to \infty$ and using $d\left(x_{n+1}, x_n\right) \to 0$, we have $\lim d\left(x_{q(n+1)}, x_{r(n+1)} = \epsilon\right.$. As $x_{q(n)}$ and $x_{r(n)}$ lie in different adjacently labeled sets A_i and A_{i+1} for certain $1 \leq i \leq m$, using the fact that T is a Reich type cyclic weakly contraction, we have

$$\begin{aligned}\mu(\epsilon) &\leq \mu\left(d\left(x_{q(n+1)}, x_{r(n+1)}\right)\right) \\ &= \mu\left(d\left(Tx_{q(n)}, Tx_{r(n)}\right)\right) \\ &\leq \mu\left(\frac{1}{3}\left[d\left(x_{q(n)}, Tx_{q(n)}\right) + d\left(x_{r(n)}, Tx_{r(n)}\right) + d\left(x_{q(n)}, x_{r(n)}\right)\right]\right) \\ &\quad - \psi\left(d\left(x_{q(n)}, Tx_{q(n)}\right), d\left(x_{r(n)}, Tx_{r(n)}\right), d\left(x_{q(n)}, x_{r(n)}\right)\right) \\ &= \mu\left(\frac{1}{3}\left[d\left(x_{q(n)}, x_{q(n+1)}\right) + d\left(x_{r(n)}, x_{r(n+1)}\right) + d\left(x_{q(n)}, x_{r(n)}\right)\right]\right) \\ &\quad - \psi\left(d\left(x_{q(n)}, x_{q(n+1)}\right), d\left(x_{r(n)}, x_{r(n+1)}\right), d\left(x_{q(n)}, x_{r(n)}\right)\right).\end{aligned}$$

On letting $n \to \infty$, using continuity of μ, and lower semi-continuity of ψ we get that $\epsilon = 0$, which is a contradiction with $\epsilon > 0$. Hence (A) is proved. Using

(A), we shall show that $\{x_n\}$ is a Cauchy sequence in Y. Fix $\epsilon > 0$. By (A) we can find $n(0) \in \mathbb{N}$ such that $r, q \geq n(0)$ with $r - q \equiv 1(m)$ and $d(x_r, x_q) \leq \frac{\epsilon}{2}$. Since $\lim d(x_n, x_{n+1}) = 0$, we can also find $n_1 \in \mathbb{N}$ such that $d(x_n, x_{n+1}) \leq \frac{\epsilon}{2m}$ for any $n \geq n_1$. Assume that $r, s \geq \max\{n_0, n_1\}$ and $s > r$. Then there exists $k \in \{1, 2, \cdots, m\}$ such that $s - r \equiv k(m)$. Hence $s - r + t \equiv 1(m)$ for $t = m - k + 1$. So we have

$$d(x_r, x_s) \leq d(x_r, x_{s+j}) + d(x_{s+j}, x_{s+j-1}) + \cdots + d(x_{s+1}, x_s).$$

From which it follows that

$$\begin{aligned} d(x_r, x_s) &\leq \frac{\epsilon}{2} + j \times \frac{\epsilon}{2m} \\ &\leq \frac{\epsilon}{2} + m \times \frac{\epsilon}{2m} \\ &= \epsilon. \end{aligned}$$

Hence $\{x_n\}$ is a Cauchy sequence in Y. Since Y is closed in X, then Y is also complete and there exists $x \in Y$ such that $\lim x_n = x$. Now, we will prove that x is a fixed point of T. As $Y = \bigcup_{i=1}^m A_i$ is a cyclic representation of Y with respect to T, the sequence $\{x_n\}$ has infinite terms in each A_i for $i \in \{1, 2, \cdots, m\}$. Suppose that $x \in A_i, Tx \in A_{i+1}$, and we take a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ with $x_{n(k)} \in A_i$. By using the contractive condition we obtain

$$\begin{aligned} \mu\left(d\left(x_{n(k)+1}, Tx\right)\right) &= \mu\left(d\left(Tx_{n(k)}, Tx\right)\right) \\ &\leq \mu\left(\frac{1}{3}\left[d\left(x_{n(k)}, Tx_{n(k)}\right) + d(x, Tx) + d\left(x_{n(k)}, x\right)\right]\right) \\ &\quad - \psi\left(d\left(x_{n(k)}, Tx_{n(k)}\right), d(x, Tx), d\left(x_{n(k)}, x\right)\right) \\ &= \mu\left(\frac{1}{3}\left[d\left(x_{n(k)}, x_{n(k+1)}\right) + d(x, Tx) + d\left(x_{n(k)}, x\right)\right]\right) \\ &\quad - \psi\left(d\left(x_{n(k)}, x_{n(k+1)}\right), d(x, Tx), d\left(x_{n(k)}, x\right)\right). \end{aligned}$$

Letting $n \to \infty$ and using the continuity of μ and lower semi-continuity of ψ, we have

$$\mu(d(x, Tx)) \leq \mu\left(\frac{1}{3}[d(x, Tx)]\right) - \psi(0, d(x, Tx), 0)$$

which is a contradiction unless $d(x, Tx) = 0$. Hence, x is a fixed point of T. Now we will prove the uniqueness of the fixed point. Suppose that x_1 and x_2 $(x_1 \neq x_2)$ are two fixed points of T. Using the contractive condition and continuity of μ and lower semi-continuity of ψ, we have

$$\begin{aligned}
\mu\left(d\left(x_1, x_2\right)\right) &= \mu\left(d\left(Tx_1, Tx_2\right)\right) \\
&\leq \mu\left(\frac{1}{3}\left[d\left(x_1, Tx_1\right)+d\left(x_2, Tx_2\right)+d\left(x_1, x_2\right)\right]\right) \\
&\quad -\psi\left(d\left(x_1, Tx_1\right), d\left(x_2, Tx_2\right), d\left(x_1, x_2\right)\right) \\
&= \mu\left(\frac{1}{3}\left[d\left(x_1, x_1\right)+d\left(x_2, x_2\right)+d\left(x_1, x_2\right)\right]\right) \\
&\quad -\psi\left(d\left(x_1, x_1\right), d\left(x_2, x_2\right), d\left(x_1, x_2\right)\right) \\
&= \mu\left(\frac{1}{3}\left[0+0+d\left(x_1, x_2\right)\right]\right) \\
&\quad -\psi\left(0, 0, d\left(x_1, x_2\right)\right)
\end{aligned}$$

which is a contradiction unless $d\left(x_1, x_2\right) = 0$. Hence the result, and the proof is finished. □

If $\mu(a) = a$, then we have the following result

Corollary 2.4. *Let (X, d) be a complete metric space, $m \in \mathbb{N}$, $A_1, A_2, \cdots, A_m$ be nonempty closed subsets of X, and $Y = \bigcup_{i=1}^m A_i$. Suppose that $T : Y \mapsto Y$ is an operator such that*

(a) $\bigcup_{i=1}^m A_i$ is a cyclic representation of Y with respect to T,

(b) $d(Tx, Ty) \leq \frac{1}{3}[d(x, Tx) + d(y, Ty) + d(x, y)] - \psi(d(x, Tx), d(y, Ty), d(x, y))$

for any $x \in A_i$, $y \in A_{i+1}$, $i = 1, 2, \cdots, m$, where $A_{m+1} = A_1$, and $\psi \in \Psi$. Then T has a fixed point $z \in \bigcap_{i=1}^m A_i$.

If $\psi(x, y, z) = \left(\frac{1}{3} - k\right)(x + y + z)$, where $k \in \left[0, \frac{1}{3}\right)$, then we have the following result

Corollary 2.5. *Let (X, d) be a complete metric space, $m \in \mathbb{N}$, $A_1, A_2, \cdots, A_m$ be nonempty closed subsets of X, and $Y = \bigcup_{i=1}^{m} A_i$. Suppose that $T : Y \mapsto Y$ is an operator such that*

(a) $\bigcup_{i=1}^{m} A_i$ *is a cyclic representation of* Y *with respect to* T,

(b) there exists $k \in \left[0, \frac{1}{3}\right)$ *such that* $d(Tx, Ty) \leq k[d(x, Tx) + d(y, Ty) + d(x, y)]$

for any $x \in A_i$, $y \in A_{i+1}$, $i = 1, 2, \cdots, m$, where $A_{m+1} = A_1$. Then T has a fixed point $z \in \bigcap_{i=1}^{m} A_i$.

Notations 2.6. Γ *will denote the set of functions $\mu : [0, \infty) \mapsto [0, \infty)$ satisfying the following hypotheses*

(a) μ *is Lebesgue-integrable mapping on each compact of* $[0, \infty)$,

(b) for any $\epsilon > 0$, *we have* $\int_0^{\epsilon} \mu(t) > 0$.

The following result is immediate from the above Corollary.

Corollary 2.7. *Let (X, d) be a complete metric space, $m \in \mathbb{N}$, $A_1, A_2, \cdots, A_m$ be nonempty closed subsets of X, and $Y = \bigcup_{i=1}^{m} A_i$. Suppose that $T : Y \mapsto Y$ is an operator such that*

(a) $\bigcup_{i=1}^{m} A_i$ *is a cyclic representation of* Y *with respect to* T,

(b) there exists $k \in \left[0, \frac{1}{3}\right)$ *such that*

$$\int_0^{d(Tx,Ty)} \alpha(s)\, ds \leq k \int_0^{d(x,Tx)+d(y,Ty)+d(x,y)} \alpha(s)\, ds$$

for any $x \in A_i$, $y \in A_{i+1}$, $i = 1, 2, \cdots, m$, where $A_{m+1} = A_1$ and $\alpha \in \Gamma$. Then T has a fixed point $z \in \bigcap_{i=1}^{m} A_i$.

If we take $A_i = X, i = 1, 2, \cdots, m$, then we have the following result.

Corollary 2.8. *Let (X, d) be a complete metric space and $T : X \mapsto X$ be a mapping such that*

$$\int_0^{d(Tx,Ty)} \alpha(s)ds \leq k \int_0^{d(x,Tx)+d(y,Ty)+d(x,y)} \alpha(s)ds$$

for any $x, y \in X, k \in \left[0, \frac{1}{3}\right)$ and $\alpha \in \Gamma$. Then T has a fixed point $z \in X$.

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