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Two-Step Hybrid Block Method for Solving Second Order Initial Value Problem of Ordinary Differential Equations

AbdulAzeez K. Jimoh

Department of Mathematics and Statistics, Faculty of Pure and Applied Sciences,
Kwara State University, Malete, Nigeria
e-mail: abdulazeez.jimoh@kwasu.edu.ng

Abstract

A new zero-stable two-step hybrid block method for solving second order initial value problems of ordinary differential equations directly is derived and proposed. In the derivation of the method, the assumed power series solution is interpolated at the initial and the hybrid points while its second ordered derivative is collocated at all the nodal and selected off-step points in the interval of consideration. The relevant properties of the method were examined and the method was found to be zero-stable, consistent and convergent. A comparison of the results by the method with the exact solutions and other results in literature shows that the method is accurate, simple and effective in solving the class of problems considered.

1 Introduction

The importance of numerical methods in providing approximate solutions to scientific and engineering problems cannot be over-emphasised. The reason for this assertion is that, most of such problems if not all, certainly defy analytic approach due to the complex nature of the problem itself and the associated

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conditions under which it is being studied. With the advent of computers and other computational software, the use of numerical methods has become easy and it is gaining prominence day by day.

In this paper, the general explicit second order ordinary differential equation being considered is of the form

$$y'' = f(x, y, y') \qquad : x_0 \in [a,\ b] \tag{1}$$

together with the initial conditions

$$y(x_0) = y_0, \qquad y'(x_0) = y_1 \tag{2}$$

where $f(x, y, y')$ is a given real valued function in the strip $S = [a,\ b] \subset [-\infty,\ \infty]$ which is continuous and second variable Lipschitzian over its existence domain for the existence of solution of (1) to be guaranteed; y_0 and y_1 are constants.

Traditionally, equation (1) together with its associated conditions can be solved by reducing it to its first-order equivalent as reported by Dahlquist [9]. This reduced form is then solved using any appropriate numerical method. According to Awoyemi [4], this approach does not fully utilize all the information connected with certain ordinary differential equations such as oscillatory nature of solution. Sagir [16] and Jain *et al.* [11] reported that apart from enlarging the size of equations, this approach also causes computational burden for writing computer programmes which may jeopardize the performance of the method in terms of time and error. Some of the methods used by researchers in solving this class of problems include Taylor series method, Runge-Kutta methods, predictor-corrector method and the linear multi-step method. From application point of view, the Taylor series method has a major disadvantage in that it requires evaluation of partial derivatives of higher orders manually, Potta and Alabi [15]. Yakusak and Adeniyi [18] and Sunday *et al.* [17] researched into the Runge-Kutta and predictor-corrector methods and found out that the methods are very expensive to implement due to function evaluation per step and subroutines needed to supply the starting values. The need to overcome or circumvent these drawbacks prompted researchers to look for such methods that would solve this category of problems directly. Prominent among such direct methods are linear multistep block method and its variant otherwise called hybrid block method as implemented by Areo and Adeniyi [3], Bolaji [5], Yahaya and Tijjani [19] and Abdelrahim *et al.* [1].

2 Derivation of the Method

In this section, an illustration for the derivation of the two-step with two hybrid points block method for solving (1) is presented. The solution is approximated in the interval $[x_n,\ x_{n+1}]$ where the step length is given by $h = x_{n+1} - x_n$. The assumed solution adopted to solve (1) is of the form

$$y(x) = \sum_{j=0}^{p+q-1} a_j x^j, \tag{3}$$

where a_j's are continuous coefficients to be determined.

The two-step continuous hybrid multi-step method derived for the solution of (1) is of the form

$$\sum_{j=0}^{p+q-1} \alpha_j y_{n+j} = h^2 \left[\sum_{j=0}^{p+q-1} \beta_j f_{n+j} + \beta_{vi} f_{n+vi} \right], \tag{4}$$

where α_j and β_j are continuous coefficients and $vi = [\frac{1}{2}, \frac{3}{2}]$ are hybrid points.

Setting $p = 2$, $q = 6$ and $j = 0(1)7$ in (2), the assumed solution is written as

$$y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7. \tag{5}$$

Equation (3) is differentiated twice, to obtain

$$\sum_{j=0}^{7} j(j-1) a_j x^{j-2} = f(x, y, y'). \tag{6}$$

Equation (5) is interpolated at $x = x_n$, $x = x_{n+\frac{1}{2}}$ and $x = x_{n+\frac{3}{2}}$ to obtain

$$y_{n+vi}(x) = \sum_{j=0}^{2} a_j x_{n+vi}^j \tag{7}$$

while (6) is collocated at $x = x_{n+i}\ :\ i = 0,\ \frac{1}{2},\ 1,\ \frac{3}{2},\ 2$ to obtain

$$\sum_{j=2}^{7} j(j-1)a_j x_{n+i}^{j-2} = f_{n+i}. \tag{8}$$

From equations (7) and (8), the following system of equations are generated

$$a_0 + a_1x_n + a_2x_n^2 + a_3x_n^3 + a_4x_n^4 + a_5x_n^5 + a_6x_n^6 + a_7x_n^7 = y_n \tag{9}$$

$$a_0+a_1x_{n+\frac{1}{2}}+a_2x_{n+\frac{1}{2}}^2+a_3x_{n+\frac{1}{2}}^3+a_4x_{n+\frac{1}{2}}^4+a_5x_{n+\frac{1}{2}}^5+a_6x_{n+\frac{1}{2}}^6+a_7x_{n+\frac{1}{2}}^7 = y_{n+\frac{1}{2}} \tag{10}$$

$$a_0+a_1x_{n+\frac{3}{2}}+a_2x_{n+\frac{3}{2}}^2+a_3x_{n+\frac{3}{2}}^3+a_4x_{n+\frac{3}{2}}^4+a_5x_{n+\frac{3}{2}}^5+a_6x_{n+\frac{3}{2}}^6+a_7x_{n+\frac{3}{2}}^7 = y_{n+\frac{3}{2}} \tag{11}$$

$$2a_2 + 6a_3x_n + 12a_4x_n^2 + 20a_5x_n^3 + 30a_6x_n^4 + 42a_7x_n^5 = f_n \tag{12}$$

$$2a_2x_{n+\frac{1}{2}}+6a_3x_{n+\frac{1}{2}}^2+12a_4x_{n+\frac{1}{2}}^3+20a_5x_{n+\frac{1}{2}}^4+30a_6x_{n+\frac{1}{2}}^5+42a_7x_{n+\frac{1}{2}}^6 = f_{n+\frac{1}{2}} \tag{13}$$

$$2a_2x_{n+1} + 6a_3x_{n+1}^2 + 12a_4x_{n+1}^3 + 20a_5x_{n+1}^4 + 30a_6x_{n+1}^5 + 42a_7x_{n+1}^6 = f_{n+1} \tag{14}$$

$$2a_2x_{n+\frac{3}{2}}+6a_3x_{n+\frac{3}{2}}^2+12a_4x_{n+\frac{3}{2}}^3+20a_5x_{n+\frac{3}{2}}^4+30a_6x_{n+\frac{3}{2}}^5+42a_7x_{n+\frac{3}{2}}^6 = f_{n+\frac{3}{2}} \tag{15}$$

$$2a_2x_{n+2} + 6a_3x_{n+2}^2 + 12a_4x_{n+2}^3 + 20a_5x_{n+2}^4 + 30a_6x_{n+2}^5 + 42a_7x_{n+2}^6 = f_{n+2}. \tag{16}$$

Equations (9)-(16) are written in matrix form to have

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} & \frac{1}{32} & \frac{1}{64} & \frac{1}{128} \\ 1 & \frac{3}{2} & \frac{9}{4} & \frac{27}{8} & \frac{81}{16} & \frac{243}{32} & \frac{729}{64} & \frac{2187}{128} \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 3 & 3 & \frac{5}{2} & \frac{15}{8} & \frac{21}{16} \\ 0 & 0 & 2 & 6 & 12 & 20 & 30 & 42 \\ 0 & 0 & 2 & \frac{9}{2} & 27 & \frac{135}{2} & \frac{1215}{8} & \frac{5103}{16} \\ 0 & 0 & 2 & 12 & 48 & 160 & 480 & 1344 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{pmatrix} = \begin{pmatrix} y_n \\ y_{n+\frac{1}{2}} \\ y_{n+\frac{3}{2}} \\ f_n \\ f_{n+\frac{1}{2}} \\ f_{n+1} \\ f_{n+\frac{3}{2}} \\ f_{n+2} \end{pmatrix} \tag{17}$$

Equation (17) is solved by Gaussian elimination method, to obtain the continuous variables a_i's as follows:

$$a_0 = y_n$$

$$a_1 = \frac{1}{127680}[-473920y_n + 583200y_{n+\frac{1}{2}} - 109280y_{n+\frac{3}{2}} - 16746f_n + 34236f_{n+\frac{1}{2}} + 31023y_{n+1} + 1500y_{n+\frac{3}{2}} + 27y_{n+2}]$$

$$a_2 = \frac{1}{684}[11520y_n - 17280y_{n+\frac{1}{2}} + 5760y_{n+\frac{3}{2}} - 347f_n - 2352f_{n+\frac{1}{2}} - 1512y_{n+1} - 112y_{n+\frac{3}{2}} + 3y_{n+2}]$$

$$a_3 = \frac{1}{2}f_n$$

$$a_4 = \frac{1}{1368}[-32640y_n + 48960y_{n+\frac{1}{2}} - 16320y_{n+\frac{3}{2}} - 331f_n + 8488f_{n+\frac{1}{2}} + 3600f_{n+1} + 520f_{n+\frac{3}{2}} - 37f_{n+2}]$$

$$a_5 = \frac{1}{380}[2240f_n - 3360f_{n+\frac{1}{2}} + 1120f_{n+\frac{3}{2}} + 331f_n - 1116f_{n+\frac{1}{2}} + 67f_{n+1} - 140f_{n+\frac{3}{2}} + 18f_{n+2}]$$

$$a_6 = \frac{1}{342}[1536y_n - 2340y_{n+\frac{1}{2}} + 768y_{n+\frac{3}{2}} - 197f_n - 40f_{n+\frac{1}{2}} - 384f_{n+1} + 56f_{n+\frac{3}{2}} - 11f_{n+2}]$$

$$a_7 = \frac{1}{1197}[-19206y_n + 2880y_{n+\frac{1}{2}} - 960y_{n+\frac{3}{2}} + 137f_n + 240f_{n+\frac{1}{2}} + 366f_{n+1} - 32f_{n+\frac{3}{2}} + 9f_{n+2}]$$

The parameters α_j's and β_j's are then written as the following functions of t.

$$\alpha_0(t) = 1 - \frac{1481}{399}t + \frac{320}{19}t^2 - \frac{1360}{57}t^4 + \frac{112}{19}t^5 + \frac{256}{57}t^6 - \frac{640}{399}t^7$$

$$\alpha_{\frac{1}{2}}(t) = \frac{1215}{266}t - \frac{480}{19}t^2 + \frac{680}{19}t^4 - \frac{168}{19}t^5 - \frac{128}{19}t^6 + \frac{320}{133}t^7$$

$$\alpha_{\frac{3}{2}}(t) = -\frac{683}{798}t + \frac{160}{19}t^2 - \frac{680}{57}t^4 + \frac{56}{19}t^5 + \frac{128}{57}t^6 - \frac{320}{399}t^7$$

$$\beta_0(t) = -\frac{2791}{21280}t - \frac{347}{684}t^2 + \frac{1}{2}t^3 - \frac{331}{1368}t^4 + \frac{331}{380}t^5 - \frac{197}{342}t^6 + \frac{137}{1197}t^7$$

$$\beta_{\frac{1}{2}}(t) = \frac{2853}{10640}t - \frac{196}{57}t^2 + \frac{1061}{171}t^4 - \frac{279}{95}t^5 - \frac{20}{171}t^6 + \frac{80}{399}t^7$$
$$\beta_1(t) = \frac{10341}{42560}t - \frac{42}{19}t^2 + \frac{50}{19}t^4 + \frac{67}{380}t^5 - \frac{64}{57}t^6 + \frac{122}{399}t^7$$
$$\beta_{\frac{3}{2}}(t) = \frac{25}{2128}t - \frac{28}{171}t^2 + \frac{65}{171}t^4 - \frac{7}{19}t^5 + \frac{28}{171}t^6 - \frac{32}{1197}t^7$$
$$\beta_2(t) = \frac{9}{42560}t + \frac{1}{228}t^2 - \frac{37}{1368}t^4 + \frac{9}{190}t^5 - \frac{11}{342}t^6 + \frac{1}{133}t^7$$

$\alpha_j(t)$, $\beta_j(t)$ are evaluated at $t = \frac{1}{2}$, 1, $\frac{3}{2}$ and $t = 2$, the values obtained are then substituted into (4) to obtain the implicit hybrid block method as follows:

$$\begin{aligned} y_{n+\frac{1}{2}} =& \frac{40}{19}y_n - \frac{41}{19}y_{n+\frac{1}{2}} + \frac{20}{19}y_{n+\frac{3}{2}} \\ &- h^2\left[\frac{689}{5472}f_n + \frac{49}{114}f_{n+\frac{1}{2}} + \frac{21}{76}f_{n+1} + \frac{7}{342}f_{n+\frac{3}{2}} - \frac{1}{1824}f_{n+2}\right] \end{aligned} \tag{18}$$

$$\begin{aligned} y_{n+1} =& -\frac{18}{19}y_n + \frac{73}{38}y_{n+\frac{1}{2}} + \frac{1}{38}y_{n+\frac{3}{2}} \\ &+ h^2\left[\frac{53}{1824}f_n + \frac{55}{304}f_{n+\frac{1}{2}} + \frac{85}{3648}f_{n+1} - \frac{1}{304}f_{n+\frac{3}{2}} + \frac{1}{3648}f_{n+2}\right] \end{aligned} \tag{19}$$

$$\begin{aligned} y_{n+\frac{3}{2}} =& -\frac{360}{19}y_n + \frac{540}{19}y_{n+\frac{1}{2}} - \frac{161}{19}y_{n+\frac{3}{2}} \\ &+ h^2\left[\frac{689}{608}f_n + \frac{147}{38}f_{n+\frac{1}{2}} + \frac{189}{76}f_{n+1} + \frac{7}{38}f_{n+\frac{3}{2}} - \frac{3}{608}f_{n+2}\right] \end{aligned} \tag{20}$$

and

$$\begin{aligned} y_{n+2} =& -\frac{951}{19}y_n + \frac{1417}{19}y_{n+\frac{1}{2}} - \frac{447}{19}y_{n+\frac{3}{2}} \\ &+ h^2\left[\frac{9563}{2736}f_n + \frac{1559}{152}f_{n+\frac{1}{2}} + \frac{4055}{608}f_{n+1} + \frac{983}{1368}f_{n+\frac{3}{2}} + \frac{3}{608}f_{n+2}\right]. \end{aligned} \tag{21}$$

Differentiating (4) once with $x = x_n + th$ such that $\frac{dt}{dx} = \frac{1}{h}$ to have

$$y'(x) = \sum_{j=0}^{k} \alpha'_j(x) y_{n+j} + \sum_{vi} \alpha'_{vi} y_{n+vi} + h^2\left[\sum_{j=0}^{k} \beta'_j(x) f_{n+j} + \sum_{vi} \beta'_{vi} f_{n+vi}\right]. \tag{22}$$

The derivatives of the parameters α_j's and β_j's are written as the following functions of t.

$$\alpha_0'(t) = \frac{1481}{399} + \frac{640}{19}t - \frac{5440}{57}t^3 + \frac{560}{19}t^4 + \frac{512}{19}t^5 - \frac{640}{57}t^6$$
$$\alpha_{\frac{1}{2}}'(t) = \frac{1215}{266} - \frac{960}{19}t + \frac{2720}{19}t^3 - \frac{840}{19}t^4 - \frac{768}{19}t^5 + \frac{320}{19}t^6$$
$$\alpha_{\frac{3}{2}}'(t) = -\frac{683}{798} + \frac{320}{19}t - \frac{2720}{57}t^3 + \frac{280}{19}t^4 + \frac{256}{19}t^5 - \frac{320}{57}t^6$$
$$\beta_0'(t) = -\frac{2791}{21280} - \frac{347}{342}t + \frac{3}{2}t^2 - \frac{331}{342}t^3 + \frac{331}{76}t^4 - \frac{197}{57}t^5 + \frac{137}{171}t^6$$
$$\beta_{\frac{1}{2}}'(t) = \frac{2853}{10640} - \frac{392}{57}t + \frac{4244}{171}t^3 - \frac{279}{19}t^4 - \frac{40}{57}t^5 + \frac{80}{57}t^6$$
$$\beta_1'(t) = \frac{10341}{42560} - \frac{84}{19}t + \frac{200}{19}t^3 + \frac{67}{76}t^4 - \frac{128}{19}t^5 + \frac{122}{57}t^6$$
$$\beta_{\frac{3}{2}}'(t) = \frac{25}{2128} - \frac{56}{171}t + \frac{260}{171}t^3 - \frac{35}{19}t^4 + \frac{56}{57}t^5 - \frac{32}{171}t^6$$
$$\beta_2'(t) = \frac{9}{42560} + \frac{1}{114}t - \frac{37}{342}t^3 + \frac{9}{38}t^4 - \frac{11}{57}t^5 + \frac{1}{19}t^6$$

$\alpha_j'(t)$, $\beta_j'(t)$, $\alpha_{vi}'(t)$ and $\beta_{vi}'(t)$ are evaluated at $t = 0$, $\frac{1}{2}$, 1, $\frac{3}{2}$ and 2, the values obtained are then substituted into (22) to obtain the following implicit hybrid block scheme which are used together with the main method in (18)-(21) for the numerical solution of (1).

$$\begin{aligned} hy_n' =& -\frac{1481}{399}y_n + \frac{1215}{266}y_{n+\frac{1}{2}} - \frac{683}{798}y_{n+\frac{3}{2}} \\ & - h^2\left[\frac{2791}{21280}f_n - \frac{2853}{10640}f_{n+\frac{1}{2}} - \frac{10341}{42560}f_{n+1} - \frac{25}{2128}f_{n+\frac{3}{2}} - \frac{9}{42560}f_{n+2}\right] \end{aligned} \tag{23}$$

$$\begin{aligned} hy_{n+\frac{1}{2}}' =& \frac{1480}{399}y_n - \frac{873}{133}y_{n+\frac{1}{2}} + \frac{1139}{399}y_{n+\frac{3}{2}} \\ & - h^2\left[\frac{4973}{23940}f_n + \frac{11801}{11970}f_{n+\frac{1}{2}} + \frac{12349}{15960}f_{n+1} + \frac{59}{1197}f_{n+\frac{3}{2}} - \frac{4}{5985}f_{n+2}\right] \end{aligned} \tag{24}$$

$$hy'_{n+1}=-\frac{8089}{399}y_n+\frac{7823}{266}y_{n+\frac{1}{2}}-\frac{7291}{798}y_{n+\frac{3}{2}}$$
$$+h^2\left[\frac{208121}{191520}f_n+\frac{404797}{95760}f_{n+\frac{1}{2}}+\frac{336223}{127680}f_{n+1}+\frac{3025}{19152}f_{n+\frac{3}{2}}-\frac{1039}{383040}f_{n+2}\right] \quad (25)$$

$$hy'_{n+\frac{3}{2}}=-\frac{19688}{399}y_n+\frac{9711}{133}y_{n+\frac{1}{2}}-\frac{9445}{399}y_{n+\frac{3}{2}}$$
$$+h^2\left[\frac{108041}{31920}f_n+\frac{6673}{665}f_{n+\frac{1}{2}}+\frac{36297}{5320}f_{n+1}+\frac{523}{798}f_{n+\frac{3}{2}}-\frac{199}{10640}f_{n+2}\right] \quad (26)$$

and

$$hy'_{n+2}=-\frac{33737}{399}y_n+\frac{33471}{266}y_{n+\frac{1}{2}}-\frac{32939}{798}y_{n+\frac{3}{2}}$$
$$+h^2\left[\frac{1237121}{191520}f_n+\frac{1674317}{95760}f_{n+\frac{1}{2}}+\frac{1419823}{127680}f_{n+1}+\frac{28897}{19152}f_{n+\frac{3}{2}}+\frac{51601}{383040}f_{n+2}\right]. \quad (27)$$

3 Analysis of the Proposed Method

In this section, the main properties of the two-step hybrid block method for solving second order initial value problem are presented. The properties include the order and error constant, zero stability, consistency and interval of convergence of the method.

Consider the linear operator L associated with the implicit hybrid block method (18)-(21) defined as

$$L[y(x_n):h]=\sum_i\left[\alpha_i y(x_n+ih)-h^2\beta_i y''(x_n+ih)\right], \quad (28)$$

where $y(x)$ is an arbitrary test function that is continuous and differentiable in the interval $[a,\ b]$. Obtaining the Taylor series expansions of $y(x_n+ih)$ and $y''(x_n+ih)$ about x_n and collecting the coefficients of h^p lead to

$$L[y(x_n):h]=c_0y(x_n)+c_1hy'(x_n)+c_2h^2y''(x_n)+...+c_ph^py^{(p)}(x_n)+..., \quad (29)$$

where c_i's such that $i = 0,\ 1,\ 2, ...$ are vectors. From equation (29), if it is obtained that

$$c_0 = c_1 = c_2 = ... = c_p = 0 \quad : c_{p+1} \neq 0$$

then the block method (18)-(21) is said to be of order p and c_{p+1} is called the error constant.

3.1 Orders and Error constants

Following Henrici [10] and Lambert [13] and using (29), the order of the method is

$$(1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1)^T$$

and the error constants are

$$\left(-\frac{19}{16}, -1.622e-16,\ \frac{9}{16}, 2,\ \ -\frac{1}{8},\ 1,\ \frac{15}{8},\ 4\right)^T.$$

3.2 Zero stability of the Methods

Using the block method (18)-(21), the first characteristics polynomial of the method is given by

$$\rho(r) = Ar - B$$

where

$$A = \begin{pmatrix} 1 & 0 & -\frac{1}{3} & 0 \\ -\frac{73}{38} & 1 & -\frac{1}{38} & 0 \\ -\frac{540}{19} & 0 & 1 & 0 \\ -\frac{1417}{19} & 0 & \frac{447}{19} & 1 \end{pmatrix}$$

and

$$B = \begin{pmatrix} 0 & 0 & 0 & \frac{2}{3} - \frac{689}{5472}z \\ 0 & 0 & 0 & -\frac{18}{19} + \frac{53}{1824}z \\ 0 & 0 & 0 & -\frac{360}{19} + \frac{689}{5760}z \\ 0 & 0 & 0 & -\frac{951}{19} + \frac{9563}{2736}z \end{pmatrix}$$

$$Det(Ar-B) = \begin{vmatrix} r & 0 & -\frac{1}{3}r & -\frac{2}{3}+\frac{689}{5472}z \\ -\frac{73}{38}r & r & -\frac{1}{38}r & \frac{18}{19}-\frac{53}{1824}z \\ -\frac{540}{19}r & 0 & r & \frac{360}{19}-\frac{689}{5760}z \\ -\frac{1417}{19}r & 0 & \frac{447}{19}r & r+\frac{951}{19}-\frac{9563}{2736}z \end{vmatrix}$$

$$= -\frac{161}{57}r^3 - \frac{161}{19}r^4 - \frac{335885609r^3}{7407720}z.$$

Setting $Det(Ar-B)=0$ with $z=0$, gives

$$-\frac{161}{57}r^3 - \frac{161}{19}r^4 = 0.$$

Then,

$$r_1 = r_2 = r_3 = 0,\ r_4 = \frac{1}{3}.$$

The block method is said to be zero-stable if as $h \longrightarrow 0$ the roots r_j's, $j = 1(1)k$ of the characteristics polynomial $\rho(r) = 0$. That is,

$$\rho(r) = det\left[\sum A^{(i)}R^{k-1}\right] = 0$$

satisfies $|R| \leq 1$ and for those roots with $|R| \leq 1$, must have multiplicity not greater than 2.

3.3 Interval of absolute stability

The Routh-Hurwitz criterion as discussed by Jain *et al.* [11] is applied to determine the interval of absolute stability of the method. For the two-step method

$$\begin{aligned} y_{n+2} = &-\frac{951}{19}y_n + \frac{1417}{19}y_{n+\frac{1}{2}} - \frac{447}{19}y_{n+\frac{3}{2}} \\ &+ h^2\left[\frac{9563}{2736}f_n + \frac{1559}{152}f_{n+\frac{1}{2}} + \frac{4055}{608}f_{n+1} + \frac{983}{1368}f_{n+\frac{3}{2}} + \frac{3}{608}f_{n+2}\right]. \end{aligned}$$

The characteristics equation is

$$(1-\frac{3}{608}h^2)\lambda^2+(\frac{447}{19}-\frac{983}{1368}h^2)\lambda^{\frac{3}{2}}-\frac{4055}{608}h^2\lambda-(\frac{1417}{19}+\frac{1559}{152}h^2)\lambda^{\frac{1}{2}}+(\frac{951}{19}-\frac{9563}{2736}h^2)=0. \quad (30)$$

Taking the first, third and fifth terms in equation (30), gives

$$(1-\frac{3}{608}h^2)\lambda^2-\frac{4055}{608}h^2\lambda+(\frac{951}{19}-\frac{9563}{2736}h^2)=0.$$

Setting $\lambda=\frac{1+z}{1-z}$, leads to

$$(\frac{970}{19}+\frac{8671}{2736}h^2)z^2-(\frac{1864}{19}-\frac{19099}{2736}h^2)z+\frac{970}{19}-\frac{1739}{171}h^2=0. \qquad (31)$$

The Routh-Hurwitz criterion is satisfied if

$$\frac{970}{19}+\frac{8671}{2736}h^2z^2>0$$

$$-\frac{1864}{19}+\frac{19099}{2736}h^2>0$$

$$\frac{970}{19}-\frac{1739}{171}h^2>0.$$

Solving for h in (32), the interval of absolute stability is

$$-3.74885<h<-2.24056.$$

3.4 Consistency of the methods

A linear multistep method is said to be consistent if it has an order of convergence, $p\geq 1$, Lambert [13] and Henrici [10]. The derived hybrid block method is consistent since the orders are all greater than 1.

3.5 Convergence of the methods

Following Henrici [10], a hybrid block method is said to be convergent if and only if it is consistent and zero-stable. Since the proposed methods satisfy the two conditions, then the methods converge.

3.6 Numerical implementation of the scheme

In this section, the effectiveness and validity of the derived methods are tested and demonstrated by applying them to solve some second order ordinary differential equations. To compare the results obtained by the methods with those in literature, the values of h are chosen to be the same. For error calculation, the error formula is given by

$$E(x_n) = |y(x) - y(x_n)|. \tag{32}$$

In (32), $y(x)$ is the exact solution for the problem considered and $y(x_n)$ is the approximate solution obtained using the derived methods.

All computations and programmes are carried out with the aid of Mathematica software.

Example 1: Consider the second order Ordinary Differential Equation

$$y'' + 2y' - 8x = 0, \quad y(0) = 0, \quad y'(0) = 0 \quad : h = \frac{1}{320}. \tag{33}$$

The exact solution is $y(x) = 1 - 2x + 2x^2 - e^{-2x}$.

Table 1 shows the comparison between the proposed method and Laplace Transform Method.

Duffy (1998)

Example 2: Consider the second order Ordinary Differential Equation

$$y'' + \frac{6}{x}y' + \frac{4}{x^2}y = 0, \quad y(1) = 1, \quad y'(1) = 1 : \quad h = \frac{1}{320}. \tag{34}$$

The exact solution is

$$y(x) = \frac{5}{3x} - \frac{2}{3x^4}.$$

Table 2 shows the comparison between the proposed method and Uniform Order Single Step Hybrid Block Method.

Omar and Abdelrahim (2016)

Example 3: Consider the second order Ordinary Differential Equation

$$y'' - y + x = 0, \quad y(0) = 0, \quad y(1) = 0, \quad h = \frac{1}{320}. \tag{35}$$

The exact solution is

$$y(x) = x - \frac{e^x - e^{-x}}{e - e^{-1}}.$$

Table 3 shows the comparison between proposed method and Galerkin's method proposed in.

Cicelia (2014)

Example 4: Consider the second order Ordinary Differential Equation

$$y'' + y - 3x^2 = 0, \quad y(0) = 0, \quad y(2) = 3.5 \quad x \in [0,\ 2], \quad h = \frac{1}{320}. \tag{36}$$

The exact solution is

$$y(x) = 6Cosx + 3(x^2 - 2) + \frac{3}{10000}Sinx.$$

Table 4 shows the comparison between proposed method and Rayleigh-Ritz method proposed in.

Gerald and Wheatley (2004)

Example 5: Consider the second order Ordinary Differential Equation

$$y'' + 12y' + 100y = 0, \quad y(0) = \frac{1}{2}, \quad y'(0) = -10, \quad h = \frac{1}{320}. \tag{37}$$

The exact solution is

$$y(x) = e^{-6x}[\frac{1}{2}Cos(8x) - \frac{7}{8}Sin(8x)].$$

Table 5 shows the comparison between proposed method and Laplace transform method proposed in.

Duffy (1998)

4 Tables of Results

Table 1: Numerical Results for Example 1: Comparison between the absolute errors in our method and other methods in literature

x	Exact Solution by Mathematica	Results by the new	Error	Results by Duffy (1998)	Error
0.0000000	0.00000000000	0.0000000000	0.0000000	0.0000000000	0.00000000
0.0015625	$5.08229 * 10^{-9}$	$7.00649 * 10^{-6}$	7.00141E-6	$5.08229 * 10^{-9}$	0.0000000
0.0031250	$4.06266 * 10^{-8}$	$1.40109 * 10^{-5}$	1.39703E-5	$4.06266 * 10^{-8}$	0.0000000
0.0046875	$1.37008 * 10^{-7}$	$2.10195 * 10^{-5}$	2.08825E-5	$1.37008 * 10^{-7}$	0.0000000
0.0062500	$3.24506 * 10^{-7}$	$2.80397 * 10^{-5}$	2.77152E-5	$3.24506 * 10^{-7}$	0.0000000
0.0078125	$6.33307 * 10^{-7}$	$4.36108 * 10^{-5}$	1.29775E-5	$6.33307 * 10^{-7}$	0.0000000
0.0093750	$1.09350 * 10^{-6}$	$8.53164 * 10^{-5}$	8.42229E-5	$1.09350 * 10^{-6}$	0.0000000
0.0109400	$1.73509 * 10^{-6}$	$9.78125 * 10^{-5}$	9.60774E-5	$1.73509 * 10^{-6}$	0.0000000
0.0125000	$2.58797 * 10^{-6}$	$2.42106 * 10^{-4}$	2.39518E-4	$2.58797 * 10^{-6}$	0.0000000
0.0140600	$3.68196 * 10^{-6}$	$2.87521 * 10^{-4}$	2.83839E-4	$3.68196 * 10^{-6}$	0.0000000
0.0156300	$5.04677 * 10^{-6}$	$3.52169 * 10^{-4}$	3.47122E-4	$5.04677 * 10^{-6}$	0.0000000

Table 2: Numerical Results for Example 2: The exact solution and the approximate solution obtained by the new method

x	Exact Solution	Results by the new method
1.009400	1.0089449950888376	1.0089449950889365
1.012500	1.0117410181679887	1.0117410181682742
1.015625	1.0144475426864139	1.0144475426868062
1.018750	1.0170664942356729	1.0170664942362451
1.021875	1.0195997547562881	1.0195997547568893
1.025000	1.0220491636294322	1.0220491636301212
1.028125	1.0244165187384029	1.0244165187391280
1.031250	1.0267035775008062	1.0267035775016994

Table 3: Numerical Results for Example 2: Comparison between the absolute errors in our method and other methods in literature

x	Kayode and Adeyeye (2011)	Anake *et al.* (2012)	Omar and Abdelrahim (2016)	Error in the new Method
1.009400	8.5357E-10	2.0169E-10	5.6221E-13	9.8810E-14
1.012500	1.7846E-09	4.5540E-10	1.1082E-12	2.8555E-13
1.015625	2.9171E-09	7.9967E-10	1.8187E-12	3.9235E-13
1.018750	4.2420E-09	1.2305E-09	2.6851E-12	5.7221E-13
1.021875	5.7509E-09	1.7440E-09	3.6994E-12	6.0130E-13
1.025000	7.4341E-09	2.3365E-09	4.8541E-12	6.8900E-13
1.028125	9.2848E-09	3.0043E-09	6.1419E-12	7.2520E-13
1.031250	1.1295E-09	3.7441E-09	7.5557E-12	8.9329E-13

Table 4: Numerical Results for Example 3: Comparison between the absolute errors in our method and other methods in literature

x	Exact Solution	Results by the new Method	Error	Results by Cicelia (2014)	Error
0.0000000	0.00000000000	0.0000000000	0.0000000	0.0000000	0.00000000
0.0015625	$2.32940 * 10^{-4}$	$2.32910 * 10^{-4}$	7.00141E-6	$3.54559 * 10^{-4}$	1.21619E-4
0.0031250	$4.65877 * 10^{-4}$	$4.65874 * 10^{-4}$	1.39703E-5	$7.08008 * 10^{-4}$	2.42131E-4
0.0046875	$6.98807 * 10^{-4}$	$6.98803 * 10^{-4}$	2.08825E-5	$1.06035 * 10^{-3}$	3.61543E-4
0.0062500	$9.31727 * 10^{-4}$	$9.31725 * 10^{-4}$	2.77152E-5	$1.41158 * 10^{-3}$	4.79853E-4
0.0078125	$1.16463 * 10^{-3}$	$1.16454 * 10^{-3}$	4.29775E-5	$1.76170 * 10^{-3}$	5.97070E-4
0.0093750	$1.39753 * 10^{-3}$	$1.39751 * 10^{-3}$	8.42229E-5	$2.11071 * 10^{-3}$	7.13180E-4
0.0109400	$1.63040 * 10^{-3}$	$1.63030 * 10^{-3}$	9.60774E-5	$2.45861 * 10^{-3}$	8.28210E-4
0.0125000	$1.86325 * 10^{-3}$	$1.86323 * 10^{-3}$	2.39518E-4	$2.80540 * 10^{-3}$	9.42150E-4
0.0140600	$2.09607 * 10^{-3}$	$2.09606 * 10^{-3}$	2.83839E-4	$3.15108 * 10^{-3}$	1.05501E-3
0.0156300	$2.32886 * 10^{-3}$	$2.32885 * 10^{-3}$	3.47122E-4	$3.49565 * 10^{-3}$	1.16679E-3

Table 5: Numerical Results for Example 4: Comparison between the absolute errors in our method and other methods in literature

x	Exact Solution	Results by the new Method	Error	Results by Gerald and Wheatley (2004)	Error
0.0000000	0.00000000000	0.0000000000	0.0000000	0.000000000	0.00000000
0.0015625	$4.68751 * 10^{-7}$	$1.28668 * 10^{-6}$	8.17927E-6	$3.61245 * 10^{-4}$	3.60776E-4
0.0031250	$9.37522 * 10^{-7}$	$9.59136 * 10^{-6}$	8.65384E-6	$7.18568 * 10^{-4}$	7.17630E-4
0.0046875	$1.40637 * 10^{-6}$	$1.08329 * 10^{-5}$	9.42649E-6	$1.07199 * 10^{-3}$	1.07058E-3
0.0062500	$1.87537 * 10^{-6}$	$3.51402 * 10^{-5}$	3.32648E-5	$1.42152 * 10^{-3}$	1.41964E-3
0.0078125	$2.34466 * 10^{-6}$	$3.99931 * 10^{-5}$	3.76484E-5	$1.76718 * 10^{-3}$	1.76484E-3
0.0093750	$2.81439 * 10^{-6}$	$5.53450 * 10^{-5}$	5.25306E-5	$2.10899 * 10^{-3}$	2.10618E-3
0.0109400	$3.28476 * 10^{-6}$	$4.96552 * 10^{-5}$	6.77284E-5	$2.44697 * 10^{-3}$	2.44369E-3
0.0125000	$3.75601 * 10^{-6}$	$7.61634 * 10^{-5}$	7.24074E-5	$2.78113 * 10^{-3}$	2.77737E-3
0.0140600	$4.22839 * 10^{-6}$	$6.88503 * 10^{-5}$	7.92317E-5	$3.11150 * 10^{-3}$	3.11272E-3
0.0156300	$4.70221 * 10^{-6}$	$9.13259 * 10^{-5}$	8.66237E-5	$3.43809 * 10^{-3}$	3.43339E-3

Table 6: Numerical Results for Example 5: Comparison between the absolute errors in our method and other methods in literature

x	Exact Solution by Mathematica	Results by the new Method	Error	Results by Duffy (1998)	Error
0.0000000	0.500000	0.500000	0.000000	0.500000	0.0000000
0.0015625	0.484461	0.484436	0.000005	0.484461	0.0000000
0.0031250	0.469093	0.469084	0.000009	0.469093	0.0000000
0.0046875	0.453897	0.453884	0.000008	0.453897	0.0000000
0.0062500	0.438873	0.438867	0.000006	0.438873	0.0000000
0.0078125	0.424023	0.424017	0.000006	0.424023	0.0000000
0.0093750	0.409345	0.409338	0.000007	0.409345	0.0000000
0.0109400	0.394819	0.394813	0.000006	0.394819	0.0000000
0.0125000	0.380512	0.380507	0.000005	0.380512	0.0000000
0.0140600	0.366379	0.366372	0.000007	0.366379	0.0000000
0.0156300	0.352331	0.352328	0.000003	0.352331	0.0000000

5 Conclusion

A two-step hybrid block method for the numerical solution of second order ordinary differential equations is considered and proposed. The method was derived through interpolation of the assumed power series solution at the point $x = x_n$ and collocation of its first ordered derivative at equally spaced points in the interval of consideration. The consistency, zero-stability, interval of convergence and applicability of the proposed method were considered and well discussed. Numerical results as presented in Tables 1-5 show that the method performs better than most of the existing methods in literature. Furthermore, the method produces results that are very close to the exact solutions for all the problems considered.

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