Earthline Journal of Mathematical Sciences
E-ISSN: 2581-8147
Volume 14, Number 5, 2024, Pages 1031-1046
https://doi.org/10.34198/ejms.14524.10311046

Coefficients Estimates for a Subclass of Starlike Functions

Kaiyang Wang[1], Abbas Kareem Wanas[2,*] and Xiaoyuan Wang[3]

[1] School of Mathematics and Statistics, Nanjing University of Science and Technology, Nanjing 210094, Jiangsu, P. R. China
e-mail: 1807774771@qq.com

[2] Department of Mathematics, College of Science, University of Al-Qadisiyah, Iraq
e-mail: abbas.kareem.w@qu.edu.iq

[3] School of Mathematics and Statistics, Nanjing University of Science and Technology, Nanjing 210094, Jiangsu, P. R. China
e-mail: mewangxiaoyuan@163.com
School of Engineering and Management, Pingxiang University, Pingxiang 337055, Jiangxi, P. R. China

Abstract

The paper mainly investigates the initial coefficients for the subclasses of starlike functions defined by using the Cosine function involving α $(0 \leq \alpha < 1)$, we obtain upper bounds for initial order of Hankel determinants and symmetric Toeplitz determinants whose elements are the initial coefficients. Also, we obtain initial coefficient estimation of logarithmic coefficients for the subclass.

Received: May 22, 2024; Accepted: June 23, 2024; Published: June 30, 2024
2020 Mathematics Subject Classification: 30C45, 30C50, 30C80.
Keywords and phrases: univalent functions, starlike functions, logarithmic coefficients, Toeplitz determinant.
[*] Corresponding author

1 Introduction

1.1 A subclass of starlike functions involving Cosine functions

Let $\mathscr{A}$ denote the class of analytic functions f which in the open unit disk $\mathbb{D} = \{z : |z| < 1\}$ of the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \quad (z \in \mathbb{D}) \tag{1.1}$$

and let $\mathcal{S}$ be the subclass of $\mathscr{A}$ such that functions are univalent.

Let $\mathscr{P}$ denotes the Carathéodory class of analytic functions p normalized by

$$p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots \tag{1.2}$$

and meeting the condition $\Re(p(z)) > 0 \ \ (z \in \mathbb{D})$. Let f and g be analytic functions in $\mathbb{D}$. Then, we say that the function g is subordinate to the function f, and we write

$$g(z) \prec f(z) \quad (z \in \mathbb{D}),$$

if there exists a Schwarz function $\omega(z)$ with $\omega(0) = 0$ and $|\omega(z)| < 1$, such that (see [8, Chap. 6, p. 190])

$$g(z) = f(\omega(z)) \quad (z \in \mathbb{D}).$$

In a recent paper, Wang et al. [28] considered following definition of $\mathcal{ST}_c(\alpha)$ of starlike functions of order α $(0 \leq \alpha < 1)$, which belongs $\mathcal{S}$.

Definition 1. *Let $f \in \mathcal{A}$. Then $f \in \mathcal{ST}_c(\alpha)$ if it satisfies the following condition*

$$\frac{1}{1-\alpha}\left(\frac{zf'(z)}{f(z)} - \alpha\right) \prec 1 + \cos z.$$

Equivalently, let the functions $\psi_c^\alpha(z) = \alpha + (1-\alpha)(1 + z\cos z)$, that is

$$\psi_c^\alpha(c) = 1 + (1-\alpha) z \cos z,$$

then the subclasses $\mathcal{ST}_c(\alpha)$ can be defined as

$$\mathcal{ST}_c(\alpha) = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \psi_c^{\alpha}(z) \right\}.$$

As an immediate inference, we can see that

$$\mathcal{ST}_c(0) \equiv S_c^*.$$

Another subclasses of starlike functions defined by using the Sine function involving α $(0 \leq \alpha < 1)$, we see [7, 23, 28].

1.2 The logarithmic coefficients

The logarithmic coefficients γ_n for certain subclasses of analytic univalent function have been widely studied. Recently, the sharpness of third logarithmic coefficient for some analytic close-to-convex functions is studied by Ali and Vasudevarao [3] and Thomas [24], also see [6]. the logarithmic coefficients of convex and starlike functions studied by [11, 12], strongly starlike and strongly convex functions studied by [13], close-to-convex functions studied by [9]. Inverse functions in certain classes of univalent functions considered by [14, 29]. Other sharp inequalities for logarithmic coefficients problems and applications we can see recent paper by Obradović *et al*. [15, 18], Ponnusamy and Sugawa [19].

The logarithmic coefficients of univalent functions is important in solving Bieberbach conjecture and Milin conjecture, see [8, p. 151]. Let $\gamma_n = \gamma_n(f)$, the logarithmic coefficients of $f \in \mathcal{S}$ are defined by

$$\log\left(\frac{f(z)}{z}\right) = 2\sum_{n=1}^{\infty} \gamma_n z^n. \tag{1.3}$$

For a function f given by (1.3), the logarithmic coefficients are as below

$$\gamma_1 = \frac{1}{2}a_2, \tag{1.4}$$

$$\gamma_2 = \frac{1}{2}\left(a_3 - \frac{1}{2}a_2^2\right), \tag{1.5}$$

$$\gamma_3 = \frac{1}{2}\left(a_4 - a_2a_3 + \frac{1}{3}a_2^2\right), \tag{1.6}$$

$$\gamma_4 = \frac{1}{2}\left(a_5 - a_2a_4 + a_2^2a_3 - \frac{1}{2}a_3^2 - \frac{1}{4}a_2^4\right). \tag{1.7}$$

In part of this paper, we will consider the bounds of γ_1 to γ_4 for $f \in \mathcal{ST}_c(\alpha)$.

1.3 The Hankel determinants and symmetric Toeplitz determinants

We first introduce two kinds of determinants whose elements are coefficients of subclasses of univalent functions.

The first one is Hankel determinant, which the elements are coefficients of the function $f \in \mathcal{S}$. We let the Hankel determinant $H_{\lambda,n}(f)$ with $n, \lambda \in \mathbb{N} = \{1, 2, \ldots\}$

$$\mathcal{H}_n(q)(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}. \tag{1.8}$$

Some specific determinants composed of elements with initial coefficients we call first-, second-and third-order Hankel determinants, respectively, are as following.

$$\mathcal{H}_2(1) = \begin{vmatrix} 1 & a_2 \\ a_2 & a_3 \end{vmatrix} = a_3 - a_2^2,$$

$$\mathcal{H}_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2a_4 - a_3^2,$$

and

$$H_3(1) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} \tag{1.9}$$
$$= a_3\left(a_2a_4 - a_3^2\right) - a_4\left(a_4 - a_2a_3\right) + a_5\left(a_3 - a_2^2\right)$$
$$= a_3H_2(2) + a_4(a_2a_3 - a_4) + a_5H_2(1).$$

Hankel matrices play important roles in Markov process theory, non-stationary signals, Hamburger moment problems and other purely mathematical and technical application problems. In univalent functions, the earliest research to study the upper bound of $H_n(q)(f)$ began from Pommerenke [16]. In recent years there has been a great deal of attention devoted to finding bounds for the modulus of the second Hankel determinant In recent years, people have paid great attention to finding the bound of the sharp Hankel determinant. For example, Srivastava *et al*. [22] studied the subclass of close-to-convex functions. Srivastava *et al*. [21], Raza *et al*. [20], Wanas *et al*. [25,26] studied the inverse coefficients for the subclass of univalent functions. The logarithmic coefficients of inverse functions in certain classes of univalent functions for Hankel determinant was considered by [5,14].

The second kind of determinant we call the Toeplitz determinant $T_q(n)$, which the elements are arrange diagonally symmetrically. Ali *et al*. [2] defined the symmetric Toeplitz determinant $T_q(n)$ as follows:

$$\mathcal{T}_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_n & \cdots & a_{n+q-2} \\ \vdots & & \vdots & \vdots \\ a_{n+q-1} & a_{n+q-2} & \cdots & a_n \end{vmatrix} \quad (n \geq 1, q \geq 1). \tag{1.10}$$

As a special case, the second order of $\mathcal{T}_q(n)$ is

$$\mathcal{T}_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_2 \end{vmatrix}. \tag{1.11}$$

The third order is

$$\mathcal{T}_3(1) = \begin{vmatrix} 1 & a_2 & a_3 \\ a_2 & 1 & a_2 \\ a_3 & a_2 & 1 \end{vmatrix} = 1 - 2a_2^2 + 2a_2^2 a_3 - a_3^2 \tag{1.12}$$

and the forth order

$$\begin{aligned} \mathcal{T}_4(1) &= \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ a_2 & a_1 & a_2 & a_3 \\ a_3 & a_2 & a_1 & a_2 \\ a_4 & a_3 & a_2 & a_1 \end{vmatrix} (a_1 = 1, n = 1, q = 4) \\ &= \left(a_2^2 + 2a_2a_3 - a_2a_4 - a_2 + a_3^2 - a_4 - 1\right) \times \\ &\qquad \left(a_2^2 - 2a_2a_3 - a_2a_4 + a_2 + a_3^2 + a_4 - 1\right). \end{aligned} \tag{1.13}$$

Ali *et al.* [2] obtained the sharp estimates for the Toeplitz determinants whose elements are subclasses of univalent function, such as starlike functions, convex functions and close-to-convex functions, etc. About Toeplitz determinants for subclasses of univalent functions, we see recent research [1, 4, 10, 27].

In this paper, we will consider the Hankel determinants and symmetric Toeplitz determinants whose elements are subclasses of starlike functions of order α $(0 \leq \alpha < 1)$, that is $f \in \mathcal{ST}_c(\alpha)$.

2 Preliminary Results

To prove the desired result, we need the following lemmas.

Lemma 1. (*Carathéodory's Lemma, see [8, p. 41]*) *If $p \in \mathscr{P}$ and is given by (1.2) with $c_1 \geq 0$, then*

$$|c_n| \leq 2 \quad (n \geq 1). \tag{2.1}$$

This inequality is sharp for each n.

Lemma 2. *(see [17, p. 166]) Let $p(z) \in \mathscr{P}$, then*

$$\left| c_2 - \frac{c_1^2}{2} \right| \le 2 - \frac{|c_1|^2}{2}.$$

3 Main Results

Theorem 1. *Let $f \in \mathcal{ST}_c(\alpha)$, then*

$$|a_2| \le 1 - \alpha, \tag{3.1}$$

$$|a_3| \le \frac{1}{2}(1-\alpha)^2, \tag{3.2}$$

$$|a_4| \le \frac{1}{6}(1-\alpha)(2-\alpha)\alpha, \tag{3.3}$$

$$|a_5| \le \frac{1}{24}(1-\alpha)\left(11 - 7\alpha + 3\alpha^2 - \alpha^3\right). \tag{3.4}$$

Proof. Because $f \in \mathcal{ST}_c(\alpha)$, by the definition of subordination, so there exists a Schwarz function $w(z)$ with $w(0) = 0$ and $|w(z)| < 1$, such that

$$\frac{zf'(z)}{f(z)} = 1 + (1-\alpha)z\cos(w(z)). \tag{3.5}$$

Let

$$p(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + c_1 z + c_2 z^2 + \cdots.$$

It is obvious that $p(z) \in \mathscr{P}$ and

$$\begin{aligned} w(z) &= \frac{p(z) - 1}{p(z) + 1} \\ &= \frac{c_1 z + c_2 z^2 + c_3 z^3 + \cdots}{2 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots} \\ &= \frac{c_1}{2} z + \frac{1}{4}(2c_2 - c_1^2) z^2 + \frac{1}{8}\left(c_1^3 - 4c_1 c_2 + 4c_3\right) z^3 \\ &\quad + \frac{1}{16}\left(-c_1^4 + 6c_1^2 c_2 - 8c_1 c_3 - 4c_2^2 + 8c_4\right) z^4 + \cdots. \end{aligned} \tag{3.6}$$

Thus, the right hand side of (3.5) is

$$
\begin{aligned}
&1+(1-\alpha)z\cos(w(z)) \\
&= 1+(1-\alpha)z+\frac{1}{8}(\alpha-1)c_1^2z^3+(1-\alpha)\left(\frac{5c_1^3}{48}+\frac{c_3-c_1c_2}{2}\right)z^3 \\
&\quad -(1-\alpha)\left(\frac{c_4-c_1c_3}{2}+\frac{5c_1^2c_2}{16}-\frac{c_2^2}{4}-\frac{c_1^4}{32}\right)z^4+\cdots. \qquad (3.7)
\end{aligned}
$$

The left hand side of (3.5) is

$$
\begin{aligned}
\frac{zf'(z)}{f(z)} &= \left(1+\sum_{n=2}^{\infty}na_nz^{n-1}\right)\left(1-a_2z+\left(a_2^2-a_3\right)z^2-\left(a_2^3-2a_2a_3+a_4\right)z^3+\cdots\right) \\
&= 1+a_2z+\left(2a_3-a_2^2\right)z^2+\left(a_2^3-3a_2a_3+3a_4\right)z^3 \\
&\quad +\left(4a_5-a_2^4+4a_2^2a_3-4a_2a_4-2a_3^2\right)z^4+\cdots. \qquad (3.8)
\end{aligned}
$$

By combining (3.7) and (3.8), comparing the initial coefficients at both sides of the formula (3.5), we have

$$\alpha_2 = 1-\alpha, \qquad (3.9)$$

$$\alpha_3 = \frac{1}{2}(1-\alpha)^2, \qquad (3.10)$$

$$\alpha_4 = \frac{1}{24}(\alpha-1)\left(c_1^2-4(\alpha-1)^2\right), \qquad (3.11)$$

$$\alpha_5 = \frac{1}{96}(\alpha-1)\left(4(\alpha-1)^3-3c_1^3-4(\alpha-1)c_1^2+6c_1c_2\right). \qquad (3.12)$$

By applying Lemma 1, we obtain

$$
\begin{aligned}
|a_4| &\le \frac{1}{24}(1-\alpha)\left|c_1^2-4(\alpha-1)^2\right| \\
&\le \frac{1}{24}(1-\alpha)\left(4-4(1-\alpha)^2\right) \\
&= \frac{1}{6}(1-\alpha)\alpha(2-\alpha). \qquad (3.13)
\end{aligned}
$$

By applying Lemmas 1 and 2, we obtain

$$\begin{aligned}
|a_5| &= \frac{1}{96}(1-\alpha)\left|\left(6c_1\left(c_2-\frac{c_1^2}{2}\right)+4(1-\alpha)^3-4(\alpha-1)c_1^2\right)\right| \\
&\leq \frac{1}{96}(1-\alpha)\left(\left|6c_1\left(c_2-\frac{c_1^2}{2}\right)\right|+4(1-\alpha)^3+\left|4(\alpha-1)c_1^2\right|\right) \\
&\leq \frac{1}{24}(1-\alpha)\left(11-7\alpha+3\alpha^2-\alpha^3\right). \qquad (3.14)
\end{aligned}$$

□

Remark 1. *Let us talk about the sharpness of the coefficients. As it mentioned in [28], we see that the extremal function $f_c^\alpha(z) \in \mathcal{ST}_c(\alpha)$ for a_2, a_3, a_4, a_5 is given by*

$$\begin{aligned}
f_c^\alpha(z) &= z\exp\left(\int_0^z \frac{\psi_c^\alpha(t)-1}{t}\right)dt \\
&= z+(1-\alpha)z^2+\frac{1}{2}(1-\alpha)^2z^3+\frac{1}{6}(1-\alpha)(-2+\alpha)\alpha z^4 \\
&\quad +\frac{1}{24}(1-\alpha)^2(1+\alpha)(-3+\alpha)z^5+\cdots. \qquad (3.15)
\end{aligned}$$

We find that the coefficients except a_2,a_3 and a_4 are sharp while a_5 not sharp in Theorem 1.

Next, we will give the upper bounds of the Toeplitz determinant $T_2(2)$ and $T_3(1)$, Hermitian determinant $H_2(1)$, $H_2(2)$ and $H_3(1)$ for the function class $\mathcal{ST}_c(\alpha)$, which are connected with the cosine function and α.

Theorem 2. *Let $f \in \mathcal{ST}_c(\alpha)$, then*

$$\begin{aligned}
|H_2(1)(f)| &\leq \frac{1}{2}(1-\alpha)^2, \\
|H_2(2)(f)| &\leq \frac{1}{12}(1-\alpha)^2\left((1-\alpha)^2+2\right), \\
|H_3(1)(f)| &\leq \frac{1}{144}(1-\alpha)^2\left(75-158\alpha+138\alpha^2-68\alpha^3+17\alpha^4\right).
\end{aligned}$$

Proof. According to equation (3.9) and (3.10), we obtain

$$|H_2(1)(f)| = \left|a_3 - a_2^2\right| = \frac{1}{2}(1-\alpha)^2. \tag{3.16}$$

According to equation (3.9),(3.10), (3.11) and Lemma 1, we obtain

$$\begin{aligned} |H_2(2)(f)| &= \left|a_2a_4 - a_3^2\right| \\ &= \left|\frac{1}{24}(1-\alpha)^2\left(2(1-\alpha)^2 + c_1^2\right)\right| \\ &\leq \frac{1}{12}(1-\alpha)^2\left((1-\alpha)^2 + 2\right). \end{aligned} \tag{3.17}$$

For $H_3(1)(f)$, from equation (3.9) to (3.12), we have

$$\begin{aligned} |H_3(1)| &= |a_3H_2(2) + a_4(a_2a_3 - a_4) + a_5H_2(1)| \\ &= |a_3H_2(2)| + |a_4(a_2a_3 - a_4)| + |a_5H_2(1)| \\ &\leq |a_3|\,|H_2(2)| + |a_4|\,|a_2a_3 - a_4| + |a_5|\,|H_2(1)|. \end{aligned} \tag{3.18}$$

By Lemma 1, we see that

$$|(a_2a_3 - a_4)| = \left|\frac{1}{24}(1-\alpha)\left(8(1-\alpha)^2 + c_1^2\right)\right| \tag{3.19}$$

$$\leq \frac{1}{6}(1-\alpha)\left(2(1-\alpha)^2 + 1\right). \tag{3.20}$$

Combining (3.16), (3.17), (3.3), (3.4) and (3.18), we have that

$$|H_3(1)| \leq \frac{1}{144}(1-\alpha)^2\left(75 - 158\alpha + 138\alpha^2 - 68\alpha^3 + 17\alpha^4\right).$$

□

Theorem 3. *Let $f \in \mathcal{ST}_c(\alpha)$, then*

$$|\mathcal{T}_2(2)| \leq \frac{1}{4}(1-\alpha)^2\left(3 + 2\alpha - \alpha^2\right),$$

$$|\mathcal{T}_3(1)| \leq \begin{cases} -\frac{3}{4}(1-\alpha)^4 + 2(1-\alpha)^2 - 1, & 0 \leq \alpha < \frac{1}{3}\left(3-\sqrt{6}\right) \\ \\ \frac{3}{4}(1-\alpha)^4 - 2(1-\alpha)^2 + 1, & \frac{1}{3}\left(3-\sqrt{6}\right) \leq \alpha < 1, \end{cases}$$

$$|\mathcal{T}_4(1)| \leq \begin{cases} F_1(\alpha) & 0 \leq \alpha < 0.0711457 \text{ and } 0.193836 \leq \alpha < 1, \\ \\ -F_1(\alpha) & 0.0711457 \leq \alpha < 0.193836, \end{cases}$$

where

$$F_1(\alpha) := \frac{1}{144}(9 - 176\alpha + 704\alpha^2 - 72\alpha^3 - 646\alpha^4 + 376\alpha^5 - 44\alpha^6 - 8\alpha^7 + \alpha^8). \tag{3.21}$$

Proof. According to equation (3.9) and (3.10), directly calculation, we have

$$|\mathcal{T}_2(2)| = \frac{1}{4}(1-\alpha)^2\left(3 + 2\alpha - \alpha^2\right)$$

and

$$|\mathcal{T}_3(1)| = \begin{cases} -\frac{3}{4}(1-\alpha)^4 + 2(1-\alpha)^2 - 1 & 0 \leq \alpha < \frac{1}{3}\left(3 - \sqrt{6}\right), \\ \\ \frac{3}{4}(1-\alpha)^4 - 2(1-\alpha)^2 + 1 & \frac{1}{3}\left(3 - \sqrt{6}\right) \leq \alpha < 1. \end{cases}$$

For $|\mathcal{T}_4(1)|$, from equation (3.9) to (3.11), by Lemma 1, we obtain

$$\begin{aligned} |\mathcal{T}_4(1)| &= \frac{1}{576}\left|4\left(\alpha^8 - 8\alpha^7 - 48\alpha^6 + 400\alpha^5 - 710\alpha^4 + 24\alpha^3 + 616\alpha^2 - 128\alpha - 3\right)\right. \\ &\qquad \left. +(\alpha-2)(\alpha-1)^2\alpha c_1^4 + 4(\alpha-1)^4\left(\alpha^2 - 2\alpha + 3\right)c_1^2\right| \\ &\leq \frac{1}{576}\left(4\left|\alpha^8 - 8\alpha^7 - 48\alpha^6 + 400\alpha^5 - 710\alpha^4 + 24\alpha^3 + 616\alpha^2 - 128\alpha - 3\right|\right. \\ &\qquad \left. +16\alpha(2-\alpha)(\alpha-1)^2 + 16(\alpha-1)^4\left(\alpha^2 - 2\alpha + 3\right)\right) \\ &= \begin{cases} F_1(\alpha) & 0 \leq \alpha < 0.0711457 \text{ and } 0.193836 \leq \alpha < 1, \\ \\ -F_1(\alpha) & 0.0711457 \leq \alpha < 0.193836, \end{cases} \end{aligned}$$

where $F_1(\alpha)$ given by (3.21). We complete the proof. □

Next, we will consider the initial coefficient estimations of logarithmic functions.

Theorem 4. *Let $f \in \mathcal{ST}_c(\alpha)$, then*

$$|\gamma_1| \leq \frac{1}{2}(1-\alpha), \tag{3.22}$$

$$|\gamma_2| = 0, \tag{3.23}$$

$$|\gamma_3| \leq \frac{1}{6}(1-\alpha)(1+3\alpha), \tag{3.24}$$

$$|\gamma_4| \leq \frac{1}{24}(1-\alpha)(7-10\alpha+9\alpha^2-3\alpha^3). \tag{3.25}$$

Proof. Let $f \in \mathcal{ST}_c(\alpha)$, combining (1.4) to (1.7) and combining (3.9) to (3.12), we have

$$\gamma_1 = \frac{1}{2}(1-\alpha), \tag{3.26}$$

$$\gamma_2 = 0, \tag{3.27}$$

$$\gamma_3 = \frac{1}{48}(1-\alpha)\left(8(\alpha-1)\alpha + c_1^2\right), \tag{3.28}$$

$$\gamma_4 = \frac{1}{48}(1-\alpha)\left(2\left(5-8\alpha+9\alpha^2-3\alpha^3\right) + (1-\alpha)c_1^2\right). \tag{3.29}$$

By using Lemma 1, we obtain

$$|\gamma_3| \leq \frac{1}{12}(1+2\alpha-2\alpha^2) \tag{3.30}$$

and

$$|\gamma_4| \leq \frac{1}{24}(1-\alpha)(7-10\alpha+9\alpha^2-3\alpha^3), \tag{3.31}$$

respectively. □

Remark 2. *We see that $|\gamma_1|$ and $|\gamma_2|$ are sharp for the extremal function $f_c^\alpha(z)$ given by (3.15), while $|\gamma_3|$ and $|\gamma_4|$ are not. We can also studied the bounds of Hankel determinants and Toeplitz determinants of the logarithmic coefficients $f \in \mathcal{ST}_c(\alpha)$, we omit here.*

Acknowledgements. The present investigation was supported by the Science and Technology support project of Pingxiang city (Grant No. 2020C0102) and the National Natural Science Foundation of China (Grant No. 62063029).

Conflicts of interest. The authors declare that they have no conflict of interest.

Data availability statement. The authors declare that this research is purely theoretical and does not associate with any data.

References

[1] Ahuja, O. P., Khatter, K., & Ravichandran, V. (2021). Toeplitz determinants associated with Ma-Minda classes of starlike and convex functions. *Iran. J. Sci. Technol. Trans. A Sci., 45*(6), 2021-2027. https://doi.org/10.1007/s40995-021-01173-6

[2] Ali, M. F., Thomas, D. K., & Vasudevarao, A. (2018). Toeplitz determinants whose elements are the coefficients of analytic and univalent functions. *Bull. Aust. Math. Soc., 97*(2), 253-264. https://doi.org/10.1017/S0004972717001174

[3] Ali, M. F., & Vasudevarao, A. (2018). On logarithmic coefficients of some close-to-convex functions. *Proc. Amer. Math. Soc., 146*(3), 1131-1142. http://dx.doi.org/10.1090/proc/13817

[4] Allu, V., Lecko, A., & Thomas, D. K. (2022). Hankel, Toeplitz, and Hermitian-Toeplitz determinants for certain close-to-convex functions. *Mediterr. J. Math., 19*(1), Paper No. 22, 17 pp. https://doi.org/10.1007/s00009-021-01934-y

[5] Allu, V., Arora, V., & Shaji, A. (2023). On the second Hankel determinant of logarithmic coefficients for certain univalent functions. *Mediterr. J. Math., 20*(2), Paper No. 81, 10 pp. https://doi.org/10.1007/s00009-023-02272-x

[6] Cho, N. E., Kowalczyk, B., Kwon, O. S., Lecko, A., & Sim, Y. J. (2020). On the third logarithmic coefficient in some subclasses of close-to-convex functions. *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM, 114*(2), Paper No. 52, 14 pp. https://doi.org/10.1007/s13398-020-00786-7

[7] Cho, N. E., Kumar, V., Kumar, S. S., & Ravichandran, V. (2019). Radius problems for starlike functions associated with the sine function. *Bull. Iranian Math. Soc., 45*(1), 213-232. https://doi.org/10.1007/s41980-018-0127-5

[8] Duren, P. (1983). *Univalent functions.* Springer-Verlag, New York Inc.

[9] Futa, A., Jastrzębska, M., & Zaprawa, P. (2023). Bounds of the third and the fourth logarithmic coefficients for close-to-convex functions. *Proc. Rom. Acad. Ser. A Math. Phys. Tech. Sci. Inf. Sci., 24*(3), 205-216. https://doi.org/10.59277/PRA-SER.A.24.3.01

[10] Giri, S., & Kumar, S. S. (2023). Toeplitz determinants for a class of holomorphic mappings in higher dimensions. *Complex Anal. Oper. Theory, 17*(6), Paper No. 86, 16 pp. https://doi.org/10.1007/s11785-023-01394-0

[11] Kazımoğlu, S., Deniz, E., & Srivastava, H. M. (2024). Sharp coefficients bounds for starlike functions associated with Gregory coefficients. *Complex Anal. Oper. Theory, 18*(1), Paper No. 6, 19 pp. https://doi.org/10.1007/s11785-023-01445-6

[12] Kowalczyk, B., & Lecko, A. (2022). Second Hankel determinant of logarithmic coefficients of convex and starlike functions. *Bull. Aust. Math. Soc., 105*(3), 458-467. https://doi.org/10.1017/S0004972721000836

[13] Kowalczyk, B., & Lecko, A. (2023). The second Hankel determinant of the logarithmic coefficients of strongly starlike and strongly convex functions. *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM, 117*(2), Paper No. 91, 13 pp. https://doi.org/10.1007/s13398-023-01427-5

[14] Mandal, S., & Ahamed, M. B. (2024). Second Hankel determinant of logarithmic coefficients of inverse functions in certain classes of univalent functions. *Lith. Math. J., 64*(1), 67-79. https://doi.org/10.1007/s10986-024-09623-5

[15] Obradović, M., Ponnusamy, S., & Wirths, K.-J. (2018). Logarithmic coefficients and a coefficient conjecture for univalent functions. *Monatsh. Math., 185*(3), 489-501. https://doi.org/10.1007/s00605-017-1024-3

[16] Pommerenke, Ch. (1966). On the coefficients and Hankel determinants of univalent functions. *J. London Math. Soc., 41*, 111-122. https://doi.org/10.1112/jlms/s1-41.1.111

[17] Pommerenke, Ch. (1975). *Univalent Functions.* Vandenhoeck & Ruprecht, Gottingen.

[18] Ponnusamy, S., Sharma, N. L., & Wirths, K.-J. (2018). Logarithmic coefficients of the inverse of univalent functions. *Results Math., 73*(4), Paper No. 160, 20 pp. https://doi.org/10.1007/s00025-018-0921-7

[19] Ponnusamy, S., & Sugawa, T. (2021). Sharp inequalities for logarithmic coefficients and their applications. *Bull. Sci. Math., 166*, Paper No. 102931, 23 pp. https://doi.org/10.1016/j.bulsci.2020.102931

[20] Raza, M., Riaz, A., & Thomas, D. K. (2024). The third Hankel determinant for inverse coefficients of convex functions. *Bull. Aust. Math. Soc., 109*(1), 94-100. https://doi.org/10.1017/S0004972723000357

[21] Srivastava, H. M., Shaba, T. G., Ibrahim, M., Tchier, F., & Khan, B. (2024). Coefficient bounds and second Hankel determinant for a subclass of symmetric bi-starlike functions involving Euler polynomials. *Bull. Sci. Math., 192*, Paper No. 103405, 17 pp. https://doi.org/10.1016/j.bulsci.2024.103405

[22] Srivastava, H. M., Rath, B., Kumar, K. S., & Krishna, D. V. (2024). Some sharp bounds of the third-order Hankel determinant for the inverses of the Ozaki type close-to-convex functions. *Bull. Sci. Math., 191*, Paper No. 103381, 19 pp. https://doi.org/10.1016/j.bulsci.2023.103381

[23] Tang, H., Srivastava, H. M., Li, S. H., & Deng, G. T. (2020). Majorization results for subclasses of starlike functions based on the sine and cosine functions. *Bull. Iranian Math. Soc., 46*(2), 381-388. https://doi.org/10.1007/s41980-019-00262-y

[24] Thomas, D. K. (2016). On the logarithmic coefficients of close to convex functions. *Proc. Amer. Math. Soc., 144*(4), 1681-1687. https://doi.org/10.1090/proc/12921

[25] Wanas, A. K., & Majeed, A. M. (2021). Second Hankel determinant for a certain subclass of λ-pseudo-starlike bi-univalent functions. *Iran. J. Math. Sci. Inform., 16*(2), 49-59. https://doi.org/10.52547/ijmsi.16.2.49

[26] Wanas, A. K., & Sokół, J. (2021). Second Hankel determinant for analytic and bi-univalent functions with respect to symmetric conjugate. *An. Univ. Oradea Fasc. Mat., 28*(1), 125-134.

[27] Wang, D. R., Huang, H. Y., & Long, B. Y. (2021). Coefficient problems for subclasses of close-to-star functions. *Iran. J. Sci. Technol. Trans. A Sci., 45*(3), 1071-1077. https://doi.org/10.1007/s40995-021-01115-2

[28] Wang, Z. G., Hussain, M., & Wang, X. Y. (2023). On sharp solutions to majorization and Fekete-Szego problems for starlike functions. *Miskolc Math. Notes, 24*(2), 1003-1019. https://doi.org/10.18514/MMN.2023.3986

[29] Wang, Z. G., Srivastava, H. M., Arif, M., Liu, Z. H., & Ullah, K. (2024). Sharp bounds on Hankel determinants of bounded turning functions involving the hyperbolic tangent function. *Appl. Anal. Discrete Math.* https://doi.org/10.2298/AADM221203013W