Earthline Journal of Mathematical Sciences
E-ISSN: 2581-8147
Volume 14, Number 5, 2024, Pages 873-1029
https://doi.org/10.34198/ejms.14524.8731029

Mixed Variational Inequalities and Nonconvex Analysis

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Abstract

In this expository paper, we provide an account of fundamental aspects of mixed variational inequalities with major emphasis on the computational properties, various generalizations, dynamical systems, nonexpansive mappings, sensitivity analysis and their applications. Mixed variational inequalities can be viewed as novel extensions and generalizations of variational principles. A wide class of unrelated problems, which arise in various branches of pure and applied sciences are being investigated in the unified framework of mixed variational inequalities. It is well known that variational inequalities are equivalent to the fixed point problems. This equivalent fixed point formulation has played not only a crucial part in studying the qualitative behavior of complicated problems, but also provide us numerical techniques for finding the approximate solution of these problems. Our main focus is to suggest some new iterative methods for solving mixed variational inequalities and related optimization problems using resolvent methods, resolvent equations, splitting methods, auxiliary principle technique, self-adaptive method and dynamical systems coupled with finite difference technique. Convergence analysis of these methods is investigated under suitable conditions. Sensitivity analysis of the mixed variational inequalities is studied using the resolvent equations

Received: May 6, 2024; Accepted: June 2, 2024; Published: June 29, 2024

2020 Mathematics Subject Classification: 26A33, 26D15, 26A51, 47A12, 33B15, 33B10, 49J40, 65D15, 65D99, 91B50, 90C26, 90C33.

Keywords and phrases: variational inequalities, resolvent method, resolvent equations, auxiliary principle technique, dynamical system, iterative methods, convergence.

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method. Iterative methods for solving some new classes of mixed variational inequalities are proposed and investigated. Our methods of discussing the results are simple ones as compared with other methods and techniques. Results proved in this paper can be viewed as significant and innovative refinement of the known results.

1 Introduction

Variational inequality theory, which was introduced and considered in early sixties by Stampacchia [106], can be viewed as a natural extension and novel generalization of the variational principles. It is amazing that a wide class of unrelated problems, which arise in various different branches of pure and applied sciences, can be studied in the general and unified framework of variational inequalities. Variational inequality theory contains a wealth of new ideas and techniques. For the applications, motivation, numerical results and other aspects of variational inequalities, see [2–5, 17, 18, 20, 21, 25–34, 39, 41, 43, 48, 49, 52–55, 58, 59, 61, 63, 65, 66, 68, 69, 71, 73, 81–85, 88, 89, 92, 94, 96, 97, 100, 104–111, 113, 114] and the references therein.

In recent years, variational inequalities have been extended and generalized in various directions by using novel and innovative ideas and techniques, both for their own sake and for their applications. An important and useful generalization is called the mixed variational inequality or the variational inequality of the second kind. For the applications, formulations, motivation and numerical methods, see [27–30, 38, 39, 41, 48, 53, 54, 56, 57, 61, 83, 84, 96, 97, 100] and the references therein. It is well known the projection method and its variant forms including the Wiener-Hopf equations cannot be used to suggest numerical methods for solving the mixed variational inequalities. These facts motivated us to use the technique of the resolvent operator, the origin of which can be traced back to Martinet [42] and Brezis [21]. In this technique, the given operator is decomposed into the sum of two maximal monotone operators, whose resolvent are easier to evaluate than the resolvent of the original operator. Such a method is known as operator splitting method. This can lead to every efficient methods, since one can treat each part of the original operator independently. The operator splitting methods and related techniques have been analyzed and studied by many researchers including Glowinski and Le Tallec [30], and Tseng [108]. For an excellent account of the alternating direction implicit (splitting) methods, see Ames [6]. In the context of the mixed variational inequalities, Noor [53, 61] has used the resolvent operator technique to suggest several multi step splitting type methods. A useful feature of the forward-backward splitting method is that the resolvent step involves the subdifferential of the proper, convex and lower-semicontinuous only and the other part facilitates the problem decomposition. In passing, we point out that the tree-step iterative methods are also known Noor iterations, which contain Mann (one step)iteration, Ishikawa (two-step) iterations as special cases. It have shown the Noor orbit demonstrates that the boundary of the fixed point region is similar to natural features such as bird nests and certain types of peacock wing structures. This is demonstrated by geometrical and numerical analysis of composite Julia sets and composite Mandelbrot sets for the Noor iteration, see Chugh et al. [24] and Negi et al. [45]. In recent years, Noor iterations [58, 63] have been generalized and extended in various directions using innovative ideas to study various complicated and complex problems arising in fractal geometry, information sciences, data analysis, solar energy optimizations. For more details, see [12–14, 22–24, 44–46, 95, 101, 103, 112] and the references therein.

Equally important is the area of mathematical sciences known as the resolvent equations, which was introduced by Noor [53]. Noor [53] has established the equivalence between the mixed variational inequalities and the resolvent equations using essentially the resolvent operator technique. The resolvent equations are being used to develop powerful and efficient numerical methods for solving the mixed variational inequalities and related optimization problems.

In particular, if the nonlinear term in the mixed variational inequality is the indicator function of a closed convex set in the Hilbert space, then these splitting (forward-backward) methods reduce to the projection and extragradient methods for solving the standard variational inequalities. If the nonlinear term $\phi(.)$ in the mixed variational inequality is nonlinear, then the resolvent method, resolvent equations technique and splitting method can not be used to propose and suggest iterative methods for solving mixed variational inequalities. It is well known that to implement such type of the methods, one has to evaluate the projection, which is itself a difficult problems. Secondly, one can't extend the technique of resolvent method for solving some classes of mixed variational inequalities. These facts motivated to consider other methods. One of these techniques is known as the auxiliary principle. This technique is basically due to Lions and Stampacchia [41] and Noor [48] used this technique to discuss the existence of the solution of mildly nonlinear variational inequalities. See also Glowinski, Lions and Tremolieres [29]. Noor [49, 52–55, 58, 61, 63, 65, 66, 71] has used this technique to develop some predictor-corrector methods for solving variational inequalities. It can be shown [17,21,27,29,30,49,52–55,58,61,63,65,66,71,83,84,90–92,94,108,114] that various classes of methods including resolvent methods, resolvent equations, decomposition and descent can be obtained from this technique as special cases.

We consider some forward-backward splitting method for solving the mixed variational inequalities by modifying the extraresolvent method. Our suggested methods are in the spirit of the extraresolvent method performing an additional step forward and resolvent step at each iteration. The new method is easy to implement and versatile. As special cases, we obtain new methods for solving monotone mixed variational inequalities. The convergence criteria of the proposed implicit method is discussed under some mild conditions. Some numerical examples are given to illustrate the efficiency of the proposed methods.

Related to the mixed variational inequalities, is the problem of finding the fixed points of the nonexpansive mappings, which is the subject of current interest in functional analysis. It is natural to study these different problems in a unified framework. Motivated by the research going on these fields, we suggest and analyze three-step iterative methods for finding the common solution of these problems. The convergence criteria of these new iterative schemes under some mild conditions is considered.

The alternative equivalent fixed point technique is used to consider dynamical systems, in which the right-hand side of the ordinary differential equation is a resolvent operator. The novel feature of the resolvent dynamical systems is that the set of the stationary points of the dynamical system correspond to the set of the solution of the variational inequalities. Consequently, equilibrium problems which can be formulated in the setting of mixed variational inequalities can now be studied in the frame work of the dynamical systems. We prove that these resolvent dynamical systems have the global asymptotic stability properties for the pseudomonotone operator, which is a weaker condition than monotonicity. Using the forward-backward finite difference technique is used to suggest several new iterative methods for solving mixed variational inequalities along with convergence criteria.

In Section 9, we suggest and analyze a new self-adaptive three step iterative methods for solving mixed variational inequalities. The proposed method consists of three steps and the new iterate is obtained by using a descent direction. We prove that the new method is globally convergent under suitable mild conditions. An example is given to illustrate the efficiency and the implementation of the proposed method. Results are very encouraging and further efforts are required to improve these methods.

In Section 10, we study the sensitivity analysis of the mixed variational inequalities. It is worth mentioning that sensitivity analysis is important for several reasons. First, estimating problem data often introduces measurement errors, sensitivity analysis helps in identifying sensitive parameters that should be obtained with relatively high accuracy. Secondly, sensitivity analysis provides useful information for designing or planning various equilibrium systems. Furthermore, from mathematical and engineering point of view, sensitivity analysis can provide new insight regarding problems being studied.

Convexity theory contains a wealth of novel ideas and techniques, which have played the significant role in the development of almost all the branches of pure and applied sciences. Several new generalizations and extensions of the convex functions and convex sets have been introduced and studied to tackle unrelated complicated and complex problems in a unified manner. In several situations, the underlying the sets and functions may not be convex sets and convex functions. Theses facts and observations, motivated to introduce and consider several kind of nonconvex sets and nonconvex function with respect to arbitrary functions and bifunctions. Noor et al. [85, 88, 89, 92] introduced and studied the some new concepts of biconvex sets and biconvex functions involving an arbitrary bifunction. Optimality conditions of a sum of differentiable biconvex and non differentiable biconvex functions are characterized by a class of variational inequalities, which are called the mixed bivariational inequalities. Some new iterative methods for solving the mixed bivariational inequalities are proposed and analyzed using the auxiliary techniques. Several special cases are discussed as applications of the main results. These results are discussed in Section 11.

Harmonic functions and harmonic convex sets are important generalizations of the convex functions and convex sets. The harmonic means have novel applications in electrical circuits theory. It is known that the total resistance of a set of parallel resistors is obtained by adding up the reciprocals of the individual resistance values, and then taking the reciprocal of their total. More precisely, if u and v are the resistances of two parallel resistors, then the total resistance is computed by the formula:

$$\frac{1}{u} + \frac{1}{v} = \frac{uv}{u+v}$$

which is half the harmonic means. Al-Azemi et al. [2] studied the Asian options with harmonic average, which is a new direction in the study of the risk analysis and financial mathematics. Noor and Noor [74] have proved that the minimum of the differentiable harmonic convex functions on the harmonic convex can also be characterized via the harmonic variational inequalities. For the applications, motivations and numerical results for harmonic variational inequalities, see [3, 4, 74, 75]. In Section 12, mixed harmonic inequalities involving two two operators are investigated. Several important special cases are obtained as applications. Due to nature of the harmonic variational inequalities, we apply the auxiliary principle approach. Hybrid inertial iterative methods are proposed and analyzed for solving the mixed harmonic variational inequalities.

As indicated in Section 11 and Section 12, the concepts of biconvex sets and harmonic convex sets are distinctly two different classes of convex sets. It is natural to unify these concepts. Motivated by these facts and observations, we introduce the concepts of biharmonic convex sets and biharmonic functions with respect to an arbitrary bifunction , which is the subject of Section 13. Mixed biharmonic variational inequalities are analyzed applying the auxiliary principle technique. A wide class of iterative methods are suggested and investigated for solving the mixed biharmonic variational inequalities.

Exponentially mixed variational inequalities are introduced and studied in Section 14 associated with exponentially convex functions, which has important and interesting applications in artificial intelligence, data analysis, risk analysis, machine learning and medical images. For the recent developments in the exponentially convex functions and related optimizations, see [1, 7, 9, 15, 86–88, 98] and the references therein. If the nonlinear in the exponentially mixed variational inequalities is just continuous, then, we use the auxiliary principle technique for solving exponentially mixed variational inequalities. For the lower semi-continuous nonlinear, the exponentially mixed variational inequalities to the fixed point problems. This equivalent formulation is used to study the existence of the solution as well as to propose some multi step hybrid iterative methods. Several special cases of the obtained results are pointed out.

in Section 15, we have also discussed the change of variable technique for solving the variational inequalities, which is mainly due to Noor [51]. Applying this technique, one can establish the equivalence between the variational inequalities and fixed point method. This technique can be extended for solving the mixed variational inequalities, which need further research efforts. This technique have been used to develop modulus based methods for solving the system of absolute value equations.

Mixed variational inequalities theory is quite broad, so we shall content ourselves here to give the flavour of the ideas and techniques involved. The techniques used to analysis the iterative methods and other results for general variational inequalities are a beautiful blend of ideas of pure and applied mathematical sciences. In this paper, we have presented the main results regarding the development of various algorithms, their convergence analysis, dynamical systems and the sensitivity analysis of the mixed variational inequalities. Although this paper is expository in nature, our choice has been rather to consider a number of familiar and to us some interesting aspects of mixed variational inequalities. We also include some new results which we and our coworkers have recently obtained. The framework chosen should be seen as a model setting for more general results for other classes of variational inequalities and variational inclusions. It is true that each of these areas of applications require special consideration of peculiarities of the physical problem at hand and the inequalities that model it. However, many of the concepts and techniques, we have discussed are fundamental to all of these applications. One of the main purposes of this expository paper is to demonstrate the close connection among various classes of algorithms for the solution of the mixed variational inequalities. We would like to emphasize that the results obtained and discussed in this report may motivate and bring a large number of novel, innovate and potential applications, extensions and interesting topics in these areas. We have given a brief introduction of this fast growing field only. The interested reader is advised to explore this field further and discover novel and fascinating applications of this theory in other mathematical and engineering sciences.

2 Formulations and Basic Facts

Let H be a real Hilbert space, whose norm and inner product are denoted by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$, respectively.

Let $T : H \to H$ be a nonlinear operator and let $\phi : H \to H$ be a continuous function. We consider the problem of finding $u \in H$, such that

$$\langle Tu, v-u\rangle + \phi(v) - \phi(u) \geq 0, \quad \forall v \in H, \tag{2.1}$$

which is called the mixed variational inequalities, introduced and studied by Lions and Stampacchia [41]. A wide class of problems arising in pure and applied sciences can be studied via variational inequalities (2.1), see [27–30, 38, 39, 41, 48, 53, 54, 61, 69, 83, 84, 96, 97, 100, 114] and the references therein.

It has been shown that a large class of obstacle, unilateral, contact, free, moving, and equilibrium problems arising in regional, physical, mathematical, engineering and applied sciences can be studied in the unifying and general framework of (2.1). For example, the mixed variational inequality (2.1) characterizes the Signorini problem with non-local friction. If S is an open bounded domain in R^n with regular boundary ∂S, representing the interior of an elastic body subject to external forces and if a part of the boundary may come into contact with a rigid foundation, then (2.1) is simply a statement of the virtual work for an elastic body restrained by friction forces, assuming that a non-local law of friction holds. The strain energy of the body corresponding to an admissible displacement v is $\langle Tv, v\rangle$. Thus $\langle Tu, v-u\rangle, \forall u, v \in H$ is the work produced by the stresses through strains caused by the virtual displacement $v-u$. The friction forces are represented by the function $\phi(.)$. Similar problems arise in the study of the fluid flow through porous media. For the physical and mathematical formulation of the mixed variational inequalities of type (2.1), see [38].

We now discuss some important special cases of the mixed variational inequalities 2.1

Special Cases

(I). Let $F : H \longrightarrow R$ be a differentiable convex function and ϕ be a lower semi-continuous convex function. If $T = \nabla F$, then problem (2.1) is equivalent to finding $u \in H$ such that

$$0 \in \nabla F(u) + \partial\phi(u). \tag{2.2}$$

Problem (2.2) is nothing else than the convex optimization optimization problem:

$$\min_{u\in H}\{J(u)+\phi(u)\},$$

which were studied in [21, 26, 29].

(II). If the function $\phi(.)$ is the indicator function of a closed convex set K, then (2.1) reduces to the problem of finding $u \in K$ such that

$$\langle Tu, v-u\rangle \geq 0, \quad \forall v \in K, \tag{2.3}$$

which is called the variational inequality, introduced and studied by Stampacchia [106] and Lions and Stampacchia [41].

(III). If $K^* = \{u \in H : \langle u, v\rangle \geq 0, \forall v \in K\}$ is a polar(dual) cone, then problem(2.3) is equivalent to finding

$$u \in K, \quad Tu \in K^*, \quad \langle Tu, u\rangle = 0, \tag{2.4}$$

which is known as the nonlinear complementarity problem, introduced by Karamaridain [35]. For the applications and other aspects of the complementarity problems in engineering and applied sciences, see [31, 35, 63, 82–84, 100] and the references therein.

(IV). If $K = H$, then problem (2.3) is equivalent to finding $u \in H$, such that

$$\langle Tu, v\rangle = 0, \quad \forall v \in H, \tag{2.5}$$

which is known as weak formulation of the boundary value problems, see [**?**, 27, 38, 48].

(V). If the operator T is linear, positive, symmetric and the function $\phi(.)$ is a convex function, then minimum of the energy function Iv defined as

$$I[v] = \langle Tv, v\rangle + 2\phi(v), \quad \forall v \in H$$

can be characterized by the mixed variational inequality (2.1).

Definition 2.1. *An operator $T : H \to H$ is said to be:*

1. *Strongly monotone, if there exist a constant $\alpha > 0$, such that*

$$\langle Tu - Tv, u - v\rangle \geq \alpha \|u - v\|^2, \quad \forall u, v \in H.$$

2. *Lipschitz continuous, if there exist a constant $\beta > 0$, such that*

$$\|Tu - Tv\| \leq \beta \|u - v\|, \quad \forall u, v \in H.$$

3. *Monotone, if*

$$\langle Tu - Tv, u - v\rangle \geq 0, \quad \forall u, v \in H.$$

4. *Pseudo monotone, if*

$$\langle Tu, v - u\rangle \geq 0 \quad \Rightarrow \quad \langle Tv, v - u\rangle \geq 0, \quad \forall u, v \in H.$$

Remark 2.1. *Every strongly monotone operator is a monotone and monotone operator is a pseudo monotone, but the converse is not true.*

Definition 2.2. *If T is a maximal monotone operator on H, then, for a constant $\rho > 0$, the resolvent operator associated with T is defined by*

$$J_T(u) = (I + \rho T)^{-1}(u), \quad \forall u \in H,$$

where I is the identity operator. It is known that a monotone operator T is maximal monotone, if and only if, its resolvent operator J_T is defined everywhere. Furthermore, the resolvent operator J_T is nonexpansive, that is,

$$\|J_T(u) - J_T(v)\| \leq \|u - v\|, \quad \forall u, v \in H.$$

Remark 2.2. *Since the subdifferential $\partial\phi$ of a proper, convex and lower-semicontinuous $\phi : H \longrightarrow R \cup \{+\infty\}$ is a maximal monotone operator, we define by*

$$J_\varphi \equiv (I + \rho\partial\phi)^{-1},$$

the resolvent operator associated with $\partial\phi$ and $\rho > 0$ is a constant.

We also need the following result, known as the resolvent Lemma(best approximation) Lemma, which plays a crucial part in establishing the equivalence between the mixed variational inequalities and the fixed point problem. This result can be used in the analysing the convergence analysis of the resolvent implicit and explicit methods for solving the mixed variational inequalities and related optimization problems..

Lemma 2.1. *[21] For a given $z \in H$, $u \in H$ satisfies the inequality*

$$\langle u - z, v - u \rangle + \rho\phi(v) - \rho\phi(u) \geq 0, \quad \forall v \in H, \tag{2.6}$$

if and only if

$$u = J_\phi(z),$$

where J_ϕ is the resolvent operator.

It is well known that the resolvent operator J_ϕ is nonexpansive, that is,

$$\|J_\phi(u) - J_\phi(v)\| \leq \|u - v\|, \forall u, v \in H.$$

This property of the resolvent operator plays an important part in the derivation of our main results.

3 Resolvent Method

In this section, we use the fixed point formulation to suggest and analyze some new implicit methods for solving the mixed variational inequalities.

Using Lemma 2.1, one can show that the mixed variational inequalities are equivalent to the fixed point problems.

Lemma 3.1. *The function $u \in H$ is a solution of the mixed variational inequalities (2.1), if and only if, $u \in H$ satisfies the relation*

$$u = J_\phi[u - \rho Tu], \tag{3.1}$$

where J_ϕ is the resolvent operator and $\rho > 0$ is a constant.

Lemma 3.1 implies that the mixed variational inequality (2.1) is equivalent to the fixed point problem (3.1). This equivalent fixed point formulation was used to suggest some implicit iterative methods for solving the mixed variational inequalities. One uses (3.1) to suggest the following iterative methods for solving the mixed variational inequalities.

Algorithm 3.1. *For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme*

$$u_{n+1} = J_\phi[u_n - \rho T u_n], \quad n = 0, 1, 2, ... \tag{3.2}$$

which is known as the resolvent method and has been studied extensively.

Algorithm 3.2. *For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme*

$$u_{n+1} = J_\phi[u_n - \rho T u_{n+1}], \quad n = 0, 1, 2, ... \tag{3.3}$$

which is known as the extra-resolvent method.

Noor [61] has proved that the convergence of the extra-resolvent method for monotone operators.

Using the alternative equivalent fixed point formation (3.1), one can suggest and analyse the following method for solving mixed variational inequalities (2.1).

Algorithm 3.3. *For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme*

$$u_{n+1} = J_\phi[u_{n+1} - \rho T u_{n+1}], \quad n = 0, 1, 2, ... \tag{3.4}$$

which is known as the modified implicit double resolvent method.

We can rewrite the equation (3.1) as:

$$u = J_\phi[\frac{u+u}{2} - \rho T u]. \tag{3.5}$$

This fixed point formulation was used to suggest the following implicit method.

Algorithm 3.4. *For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme*

$$u_{n+1} = J_\phi[\frac{u_n + u_{n+1}}{2} - \rho T u_{n+1}], \quad n = 0, 1, 2, ... \tag{3.6}$$

For the implementation and numerical performance of Algorithm 14.18, one can use the predictor-corrector technique to suggest the following two-step iterative method for solving mixed variational inequalities.

Algorithm 3.5. *For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme*

$$\begin{aligned} y_n &= J_\phi[u_n - \rho T u_n] \\ u_{n+1} &= J_\phi[\frac{y_n + u_n}{2} - \rho T y_n], \qquad n = 0, 1, 2, ... \end{aligned}$$

which is an implicit method:

From equation (3.1), we have

$$u = J_\phi[u - \rho T(\frac{u+u}{2})]. \tag{3.7}$$

This fixed point formulation is used to suggest the implicit method for solving the mixed variational inequalities as

Algorithm 3.6. *For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme*

$$u_{n+1} = J_\phi[u_n - \rho T(\frac{u_n + u_{n+1}}{2})], \quad n = 0, 1, 2, ... \tag{3.8}$$

which is another implicit method, see Noor et al. [28].

To implement this implicit method, one can use the predictor-corrector technique to rewrite Algorithm 3.6 as equivalent two-step iterative method:

Algorithm 3.7. *For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme*

$$\begin{aligned} y_n &= J_\phi[u_n - \rho T u_n], \\ u_{n+1} &= J_\phi[u_n - \rho T(\frac{u_n + y_n}{2})], \quad n = 0, 1, 2, ... \end{aligned}$$

which is known as the mid-point implicit method for solving mixed variational inequalities.

It is obvious that Algorithm 3.6 and Algorithm 14.21 have been suggested using different variant of the fixed point formulations of the equation (3.1). It is natural to combine these fixed point formulation to suggest a hybrid implicit method for solving the mixed variational inequalities and related optimization problems, which is the main motivation of this paper.

One can rewrite the (3.1) as

$$u = J_{\phi}[\frac{u+u}{2} - \rho T(\frac{u+u}{2})]. \tag{3.9}$$

This equivalent fixed point formulation enables to suggest the following method for solving the mixed variational inequalities.

Algorithm 3.8. *For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme*

$$u_{n+1} = J_{\phi}[\frac{u_n + u_{n+1}}{2} - \rho T(\frac{u_n + u_{n+1}}{2})], \quad n = 0, 1, 2, ... \tag{3.10}$$

which is an implicit method.

We would like to emphasize that Algorithm 3.8 is an implicit method.

To implement the implicit method, one uses the predictor-corrector technique. We use Algorithm 14.14 as the predictor and Algorithm 3.8 as corrector. Thus, we obtain a new two-step method for solving the mixed variational inequalities.

Algorithm 3.9. *For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme*

$$\begin{aligned} y_n &= J_{\phi}[u_n - \rho T u_n] \\ u_{n+1} &= J_{\phi}\left[\left(\frac{y_n + u_n}{2}\right) - \rho T\left(\frac{y_n + u_n}{2}\right)\right], \quad n = 0, 1, 2, ... \end{aligned}$$

which is two step method ad appears to be new one.

From the above discussion, it is clear that Algorithm 3.8 and Algorithm 14.22 are equivalent. It is enough to prove the convergence of Algorithm 3.8, which is the main motivation of our next result.

Theorem 3.1. *Let the operator T be strongly monotone with constant $\alpha > 0$ and Lipschitz continuous with constant $\beta > 0$, respectively. Let $u \in H$ be solution of (2.1) and u_{n+1} be an approximate solution obtained from Algorithm 3.8. If there exists a constant $\rho > 0$, such that*

$$0 < \rho < \frac{2\alpha}{\beta^2}, \tag{3.11}$$

then the approximate solution u_{n+1} converge to the exact solution $u \in H$.

Proof. Let $u \in H$ be a solution of (2.1) and u_{n+1} be the approximate solution obtained from Algorithm 3.8. Then, from (14.49) and (14.30), we have

$$\begin{aligned}
\|u_{n+1} - u\|^2 &= \|J_\phi[(\frac{u_n + u_{n+1}}{2}) - \rho T(\frac{u_n + u_{n+1}}{2})] \\
&\quad - J_\phi[\frac{u+u}{2}) - \rho T(\frac{u+u}{2})]\|^2 \\
&\leq \|\frac{u_{n+1} + u_n}{2} - \frac{u+u}{2}) \\
&\quad - \rho(T(\frac{u_{n+1} + u_n}{2}) - T(\frac{u+u}{2}))\|^2 \\
&\leq (1 - 2\rho\alpha + \rho^2\beta^2)\|\frac{u_n - u}{2} + \frac{u_{n+1} - u}{2}\|^2,
\end{aligned} \tag{3.12}$$

where we have used the fact that the operator T is the strongly monotone with constant $\alpha > 0$ and Lipschitz continuous constant $\beta > 0$, respectively.

Thus, from (3.12), we have

$$\begin{aligned}
\|u_{n+1} - u\| &\leq \sqrt{1 - 2\rho\alpha + \rho^2\beta^2}\{\|\frac{u_n - u}{2}\| + \|\frac{u_{n+1} - u}{2}\|\} \\
&= \frac{1}{2}\sqrt{1 - 2\rho\alpha + \rho^2\beta^2}\|u_n - u\| \\
&\quad + \frac{1}{2}\sqrt{1 - 2\rho\alpha + \rho^2\beta^2}\|u_{n+1} - u\|,
\end{aligned} \tag{3.13}$$

which implies that

$$\begin{aligned}
\|u_{n+1} - u\| &\leq \frac{\frac{1}{2}\sqrt{1 - 2\rho\alpha + \rho^2\beta^2}}{1 - \frac{1}{2}\sqrt{1 - 2\rho\alpha + \rho^2\beta^2}}\|u_n - u\| \\
&= \theta\|u_n - u\|,
\end{aligned} \tag{3.14}$$

where

$$\theta = \frac{\frac{1}{2}\sqrt{1-2\rho\alpha+\rho^2\beta^2}}{1-\frac{1}{2}\sqrt{1-2\rho\alpha+\rho^2\beta^2}}.$$

From (8.32), it follows that $\theta < 1$. This shows that the approximate solution u_{n+1} obtained from Algorithm 3.8 converges to the exact solution $u \in H$ satisfying the mixed variational inequality (2.1).

Using (3.1), for a constant ξ, we have

$$u = J_\phi[u - \xi(u-u) - \rho Tu].$$

This fixed point formulation is used to suggest the following iterative method

Algorithm 3.10. *For given $u_0, u_1 \in H$, compute u_{n+1} by the iterative scheme*

$$u_{n+1} = J_\phi[u_n - \xi(u_n - u_{n-1}) - \rho Tu_n], \quad n = 0, 1, 2, ...$$

which can be written in the following equivalent form

Algorithm 3.11. *For given $u_0, u_1 \in H$, compute u_{n+1} by the iterative scheme*

$$\begin{aligned} y_n &= J_\phi[u_n - \rho Tu_n] \\ u_{n+1} &= J_\phi[y_n - \rho Tu_n], \quad n = 0, 1, 2, ... \end{aligned}$$

which is the inertial iterative method for solving the mixed variational inequalities, studied and analyzed by Noor et al. [81].

It is worth mentioning that Polyak [102] introduced the idea of inertial type method for speeding up the convergence of iterative methods.

For the applications of inertial methods, see [63, 81, 90, 91] and the references therein.

From equation (3.1), for a constant ξ, we have

$$u = J_\phi[u - \xi(u-u) - \rho T(u - \xi(u-u))].$$

This fixed point equivalent formulation is used to suggest iterative method for solving the variational inequalities.

Algorithm 3.12. *For given $u_0, u_1 \in H$, compute u_{n+1} by the iterative scheme*

$$u_{n+1} = J_\phi[u_n - \xi(u_n - u_{n-1}) - \rho T(u_n - \xi(u_n - u_{n-1}))], \quad n = 0, 1, 2, ...$$

Algorithm 14.25 is known as the inertial projection iterative method. For different and suitable choice of the parameter ξ, one can obtain various known and new known inertial resolvent type methods for solving variational inequalities and related optimization problems, see [81].
Algorithm 14.25 can be written in the following two step method:

Algorithm 3.13. *For a given $u_0, u_1 \in H$, compute u_{n+1} by the iterative schemes*

$$\begin{aligned} y_n &= u_n - \xi(u_n - u_{n-1}) \\ u_{n+1} &= J_\phi[y_n - \rho T y_n], \quad n = 0, 1, 2, ..., \end{aligned}$$

which is the subject of recent investigation and have been extended for other classes of mixed variational inequalities. It is worth mentioning that to implement the inertial-type methods, one has to choose two initial values, which is the main draw back of these inertial methods. □

4 Resolvent Equations Technique

We now consider the problem of solving the resolvent equations related to the mixed variational inequalities (2.1). Let T be an operator and $R_\phi = I - J_\phi$, where I is the identity operator and J_ϕ is the resolvent operator. We consider the problem of finding $z \in H$ such that

$$TJ_\phi z + \rho^{-1} R_\phi z = 0. \tag{4.1}$$

The equations of the type (4.1) are called the resolvent equations. It have been shown that the resolvent equations play an important part in the developments of iterative methods, sensitivity analysis and other aspects of the mixed variational inequalities, see [21, 53, 54] and references therein.

Lemma 4.1. *The element $u \in H$ is a solution of the mixed variational inequality (2.1), if and only if, $z \in H$ satisfies the resolvent equation (4.1), where*

$$u = J_\phi z, \tag{4.2}$$

$$z = u - \rho T u, \tag{4.3}$$

where $\rho > 0$ is a constant.

From Lemma 4.1, it follows that the mixed variational inequalities (2.1) and the resolvent equations (4.1) are equivalent. This alternative equivalent formulation has been used to suggest and analyze a wide class of efficient and robust iterative methods for solving the mixed variational inequalities and related optimization problems, see [21, 53, 54, 61, 89, 114] and the references therein.

We use the resolvent equations (4.1) to suggest some new iterative methods for solving the mixed variational inequalities. From (4.2) and (4.3),

$$\begin{aligned} z &= J_\phi z - \rho T J_\phi z \\ &= J_\phi[u - \rho T u] - \rho T J_\phi[u - \rho T u]. \end{aligned}$$

Thus, we have

$$u = \rho T u + \big[J_\phi[u - \rho T u] - \rho T J_\phi[u - \rho T u].$$

Consequently, for a constant $\alpha_n > 0$, we have

$$\begin{aligned} u &= (1-\alpha_n)u + \alpha_n\{J_\phi[u - \rho T u] + \rho T u - \rho T J_\phi[u - \rho T u]\} \\ &= (1-\alpha_n)u + \alpha_n\{y - \rho T y + \rho T u\}, \end{aligned} \tag{4.4}$$

where

$$y = J_\phi[u - \rho T u]. \tag{4.5}$$

Using (14.49) and (14.30), we can suggest the following new predictor-corrector method for solving the mixed variational inequalities.

Algorithm 4.1. *For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme*

$$y_n = J_\phi[u_n - \rho T u_n] \tag{4.6}$$

$$u_{n+1} = (1-\alpha_n)u_n + \alpha_n\Big\{y_n - \rho T y_n + \rho T u_n\Big\}. \tag{4.7}$$

Algorithm 4.1 can be rewritten in the following equivalent form:

Algorithm 4.2. *For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme*

$$\begin{aligned} u_{n+1} &= (1-\alpha_n)u_n \\ &+ \alpha_n\{J_\phi[u_n - \rho T u_n] - \rho T J_\phi[u_n - \rho T u_n] + \rho T u_n]\}, \end{aligned}$$

which is an explicit iterative method and appears to be a new one.

If $\alpha_n = 1$, then Algorithm 4.1 reduces to

Algorithm 4.3. *For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme*

$$\begin{aligned} y_n &= J_\phi[u_n - \rho T u_n] \\ u_{n+1} &= y_n - \rho T y_n + \rho T u_n, \end{aligned}$$

which appears to be a new one.

In a similar way, one can suggest the following inertial type iterative method for solving the mixed variational inequalities (2.1).

Algorithm 4.4. *For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme*

$$\begin{aligned} w_n &= u_n - \xi(u_n - u_{n-1}) \\ y_n &= J_\phi[w_n - \rho T w_n] \\ u_{n+1} &= y_n - \rho T y_n + \rho T u_n, \end{aligned}$$

which is three-step inertial iterative method.

Remark 4.1. *One can suggest several new iterative methods for solving mixed variational inequalities for appropriate suitable choice of operators and spaces. The convergence criteria and implementation of the these methods require further efforts.*

5 Splitting Methods

In this Section, we use the technique of updating the solution to suggest a class of three-step forward-backward projection-splitting methods for solving the mixed variational inequalities (2.1).

Using this technique, we again rewrite the equation (3.1) in the form:

$$\begin{aligned} u &= J_\phi[J_\phi[u-\rho Tu]-\rho TJ_\phi[u-\rho Tu]] \\ &= J_\phi[I-\rho T]J_\phi[I-\rho T]u \\ &= (I+\rho T)^{-1}\{J_\phi[I-\rho T]J_\phi[I-\rho T]+\rho T\}u \end{aligned} \tag{5.1}$$

or

$$y = J_\phi[u-\rho Tu] \tag{5.2}$$

$$u = J_\phi[y-\rho Ty]. \tag{5.3}$$

Using this fixed-point formulation, one can suggest and analyze the following iterative methods.

Algorithm 5.1. *For a given $u_0 \in H$, calculate the approximate solution u_{n+1} by the iterative schemes*

$$u_{n+1} = J_\phi[J_\phi[u_n-\rho Tu_n]-\rho TJ_\phi[u_n-\rho Tu_n]]$$

or

$$\begin{aligned} y_n &= J_\phi[u_n-\rho Tu_n] \\ u_{n+1} &= J_\phi[y_n-\rho Ty_n], \quad n=0,1,2,\ldots \end{aligned}$$

or

$$\begin{aligned} u_{n+1} &= J_\phi[I-\rho T]J_\phi[I-\rho T]u_n, \\ &= (I+\rho T)^{-1}\{J_\phi[I-\rho T]J_\phi[I-\rho T]+\rho T\}u_n, \quad n=0,1,2,\ldots \end{aligned}$$

which are known as the two-step forward-backward splitting methods and are different from the forward-backward splitting methods of Tseng [108].

Using the technique of Tseng [108], one can derive a number of iterative methods for solving mathematical programming problems. For the convergence analysis of Algorithm 5.1, see Section 6.

For a positive constant α, one can rewrite the equation (3.1) as:

$$\begin{aligned} u &= J_{\phi}[u - \alpha\{\eta(u - J_{\phi}[u - \rho Tu]) + \rho T(u - \eta R(u))\}] \\ &= P_K[u - \alpha\{\eta R(u) + \rho T(u - \eta R(u))\}] \\ &= P_K[u - \alpha d_2(u)] \end{aligned} \tag{5.4}$$

where

$$d_2(u) = \eta R(u) + \rho T(u - \eta R(u)).$$

Note that for $\alpha = 1$ and $\eta = 1$ equation (5.4) is equivalent to equations (5.1). This equivalent formulation is flexible and is used to suggest and analyze the following iterative method for solving the mixed variational inequalities (2.1).

Algorithm 5.2. *For a given $u_0 \in H$, compute u_{n+1} by the following iterative schemes*

Predictor step.

$$g(w_n) = (1 - \eta_n)g(u_n) + \eta_n J_{\phi}[g(u_n) - \rho_n Tu_n] = g(u_n) - \eta_n R(u_n),$$

where η_n satisfies

$$\eta_n \rho_n \langle Tu_n - Tw_n, R(u_n)\rangle \leq \sigma \|R(u_n)\|^2, \quad \sigma \in (0, 1).$$

Corrector step.

$$g(u_{n+1}) = J_{\phi}[g(u_n) - \alpha_n d_2(u_n)], \quad n = 0, 1, 2, \ldots$$

where

$$\begin{aligned} d_2(u_n) &= \eta_n R(u_n) + \rho_n Tw_n \\ \alpha_n &= \frac{\eta_n \langle R(u_n), R(u_n) - \rho_n Tu_n + \rho Tw_n\rangle}{\|d_2(u_n)\|^2}, \end{aligned}$$

which is called the self-adaptive projection method.

Some variant forms of Algorithm 5.2 have been studied by Noor [63] and Noor et al. [61]. It is interesting to note that for $\eta_n = 1$ and $\alpha_n = 1$, Algorithm 5.2 is exactly Algorithm 5.1, which was suggested by Noor [63]. One can study the convergence analysis of Algorithm 5.2 using the technique of Noor [63].

In a similar way, one can rewrite equation (3.1) as:

$$\begin{aligned} u &= J_\phi[J_\phi[J_\phi[u - \rho Tu] - \rho TJ_\phi[u - \rho Tu]] \\ &\quad -\rho TJ_\phi[J_\phi[u - \rho Tu] - \rho TJ_\phi[u - \rho Tu]]] \\ &= J_\phi[J_\phi[y - \rho Ty] - \rho TJ_\phi[y - \rho Ty]] \\ &= J_\phi[w - \rho Tw] \end{aligned} \tag{5.5}$$

where

$$y = J_\phi[u - \rho Tu] \tag{5.6}$$

$$w = J_\phi[y - \rho Ty] \tag{5.7}$$

We use the fixed-point formulation (5.5) to suggest the following three-step forward-backward splitting method:

Algorithm 5.3. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative schemes*

$$\begin{aligned} y_n &= J_\phi[u_n - \rho Tu_n] \\ w_n &= J_\phi[y_n - \rho Ty_n] \\ u_{n+1} &= J_\phi[u_n - \rho Tu_n], \quad n = 0, 1, 2, \ldots \end{aligned}$$

We now write the equation (5.5) in the form:

$$\begin{aligned} u &= J_\phi[I - \rho T]J_\phi[I - \rho T]J_\phi[I - \rho T]u \\ &= (I + \rho T)^{-1}\{J_\phi[I - \rho T]P_K[I - \rho T]J_\phi[I - \rho T] + \rho T\}u. \end{aligned}$$

This fixed-point formulation can be used to suggest and analyze the following iterative method.

Algorithm 5.4. *For a given $u_0 \in H$, compute the approximate solution by the iterative schemes:*

$$\begin{aligned} u_{n+1} &= J_\phi[I-\rho T]J_\phi[I-\rho T]J_\phi[I-\rho T]u_n \\ &= (I+\rho T)^{-1}\{J_\phi[I-\rho T]J_\phi[I-\rho T]J_\phi[I-\rho T] \\ &\quad +\rho T\}u_n, \quad n=0,1,2,\ldots \end{aligned}$$

Algorithm 5.4 is a three-step forward-backward projection splitting methods, which is different from the splitting method of Glowinski and Le Tallec [30] for solving the variational inequalities and can be viewed as a generalization of the modified forward-backward splitting method of Tseng [108]. Using the technique of Tseng [108] and Noor [63], one can develop various splitting-type methods for solving optimization and mathematical programming problems.

We now consider a self-adaptive projection-splitting method using the fixed-point formulation (5.1). For this purpose, we define the modified residue vector $R_1(u)$ by

$$\begin{aligned} R_1(u) &= u - w = u - J_\phi[y-\rho Ty] \\ &= u - J_\phi[J_\phi[u-\rho Tu] - \rho T J_\phi[u-\rho Tu]] \end{aligned}$$

From Lemma 3.1, it follows that $u \in H$ is a solution of (2.1), if and only if, $u \in H$ is a zero of the equation

$$R_1(u) = 0.$$

We remark that

$$x = (1-\eta)u + \eta J_\phi[y-\rho Ty] = u - R_1(u) \in H.$$

Based on the above discussions and observations, we can rewrite equation (3.1) in the form:

$$u = J_\phi[u - \alpha d_3(u)],$$

where

$$\begin{aligned} d_3(u) &= \eta R_1(u) + \rho T x \\ &= \eta R_1(u) + \rho T(u - \eta R_1(u)) \end{aligned}$$

and α is a positive constant. This fixed-point has been used to suggest and analyze the following iterative method.

Algorithm 5.5. *For a given $u_0 \in H$, compute u_{n+1} by the iterative schemes:*
Predictor step.

$$\begin{aligned} y_n &= J_\phi[u_n - \rho_n T u_n] \\ w_n &= J_\phi[y_n - \rho_n T y_n] \\ x_n &= u_n - \eta_n R_1(u_n), \end{aligned}$$

where η_n satisfies

$$\eta_n \rho_n \langle T u_n - T(u_n - \eta_n R_1(u_n)), R_1(u_n)\rangle \leq \sigma \|R_1(u_n)\|^2, \quad \sigma \in (0,1).$$

Corrector step.

$$u_{n+1} = J_\phi[u_n - \alpha_n d_3(u_n)], \quad n = 0, 1, 2, \ldots$$

$$\begin{aligned} d_3(u_n) &= \eta_n R_1(u_n) + \rho_n T x_n \\ \alpha_n &= \frac{\eta_n \langle R_1(u_n), D_1(u_n)\rangle}{\|d_3(u_n)\|^2} \\ D_1(u_n) &= R_1(u_n) - \rho_n T u_n + \rho T x_n. \end{aligned}$$

Here α_n is called the corrector step size which depends upon the modified resolvent equation. For $\alpha_n = 1$ and $\eta_n = 1$, Algorithm 5.5 coincides with the projection-splitting Algorithm 5.3 and Algorithm 5.4. Note that Algorithm 5.5 is quite different from the Algorithm 5.2 and other methods. For the convergence analysis of Algorithm 5.5, see Noor [63], where it has been shown that the convergence of Algorithm 5.5 requires only monotonicity. Using essentially the technique of updating the solution, one can develop several one-step, two-step, three-step and four-step forward-backward projection splitting methods for solving the mixed variational inequalities and related optimization problems..

6 Auxiliary Principle Technique

In the previous sections, we have considered and analyzed several resolvent-type methods for solving mixed variational inequalities. If the function $\phi(.)$ in the mixed variational inequality (2.1) is not lower-semicontinuous, then one cannot show that the mixed variational inequality (2.1) is not equivalent the fixed point problem. Consequently, one can't extend the technique of resolvent methods for solving mixed variational inequality. It is well known that to implement such type of the methods, one has to evaluate the resolvent, which is itself a difficult problems. These facts motivated us to consider other methods. One of these techniques is known as the auxiliary principle. This technique is basically due to Lions and Stampacchia [41]. See also Noor [63, 65]. Glowinski, Lions and Tremolieres [29] used this technique to study the existence of a solution of mixed variational inequalities. Noor [63, 65] has used this technique to develop some predictor-corrector methods for solving mixed variational inequalities. It can be shown [63, 83, 84, 84, 114] that various classes of methods including resolvent, resolvent ,equations, decomposition and descent can be obtained from this technique as special cases.

For a given $u \in H$ satisfying (2.1), consider the problem of finding $w \in H$, such that

$$\langle \rho Tu + w - u, v - w \rangle + \rho\phi(v) - \rho\phi(w) \geq 0, \quad \forall \quad v \in H, \tag{6.1}$$

where $\rho > 0$ is a constant.

Note that, if $w = u$, then w is clearly a solution of the mixed variational inequality (2.1). This simple observation enables us to suggest and analyze the following predictor-corrector method.

Algorithm 6.1. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by* *the iterative schemes*

$$\langle \mu T u_n + y_n - u_n, v - y_n \rangle + \mu\phi(v) - \mu\phi(y_n) \geq 0, \quad \forall v \in H, \tag{6.2}$$

$$\langle \beta T y_n + w_n - y_n, v - w_n \rangle + \beta\phi(v) - \beta\phi(w_n) \geq 0, \quad \forall v \in H, \tag{6.3}$$

$$\langle \rho T w_n + u_{n+1} - w_n, v - u_{n+1} \rangle + \rho\phi(v) - \rho\phi(u_{n+1}) \geq 0, \quad \forall v \in H, \tag{6.4}$$

where $\rho > 0, \beta > 0$ *and* $\mu > 0$ *are constants*

Algorithm 6.1 can be considered as a three-step predictor-corrector method, which was suggested and studied by Noor [104]. If $\mu = 0$, then Algorithm 6.1 reduces to:

Algorithm 6.2. *For a given* $u_0 \in H$, *compute the approximate solution* u_{n+1} *by the iterative schemes;*

$$\langle \beta T u_n + w_n - u_n, v - w_n \rangle + \beta\phi(v) - \beta\phi(w_n) \geq 0, \quad \forall v \in H,$$

$$\langle \rho T w_n + u_{n+1} - w_n, v - u_{n+1} \rangle + \rho\phi(v) - \rho\phi(u_{n+1}) \geq 0, \quad \forall \in H,$$

which is known as the two-step predictor-corrector method, see [104].

If $\mu = 0, \beta = 0$, then Algorithm 6.1 becomes:

Algorithm 6.3. *For a given* $u_0 \in H$, *compute* u_{n+1} *by the iterative scheme*

$$\langle \rho T u_n + u_{n+1} - u_n, v - u_{n+1} \rangle + \rho\phi(v) - \rho\phi(u_{n+1}) \geq 0, \quad \forall v \in H.$$

We note that, if the function $\phi(.)$ is proper, lower-semicontinuous and convex function, then Algorithm 6.1 can be written as

Algorithm 6.4. *For a given* $u_0 \in H$, *compute* u_{n+1} *by the iterative schemes*

$$\begin{aligned} y_n &= J_\phi[u_n - \mu T u_n] \\ w_n &= J_\phi[y_n - \beta T y_n] \\ u_{n+1} &= J_\phi[w_n - \rho T w_n], \quad n = 0, 1, 2, \ldots \end{aligned}$$

or

$$u_{n+1} = J_\phi[I-\mu T]J_\phi[I-\beta T]J_\phi[I-\rho T]u_n, \quad n=0,1,2,\ldots$$

or

$$\begin{aligned} u_{n+1} &= (I+\rho T)^{-1}\{J_\phi[I-\rho T]J_\phi[I-\rho T]J_\phi[I-\rho T] \\ &\quad +\rho T\}u_n, \quad n=0,1,2,\ldots, \end{aligned}$$

which is three-step forward-backward resolvent method and coincides with Algorithm 5.3 and Algorithm 5.4 for $\rho=\beta=\mu$.

Algorithm 6.1 is compatible with three-step splitting method of Glowinski and Le Tallec [39] and also can be considered as a generalization of a two-step forward-backward splitting method of Tseng [108].

For the analysis of Algorithm 6.1, we need the following concepts.

Lemma 6.1. *For all* $u,v\in H$, *we have*

$$\begin{aligned} 2\langle u,v\rangle &= \|u+v\|^2-\|u\|^2-\|v\|^2 \qquad (6.5)\\ \{\frac{-1}{4}\}\|v\|^2 &\le \langle u,v\rangle+\|u\|^2. \end{aligned}$$

We now study the convergence criteria of Algorithm 6.1.

Theorem 6.1. *Let* $\bar{u}\in K$ *be a solution of (2.1) and* $T: H\longrightarrow H$ *be a partially relaxed strongly monotone operator. If* u_{n+1} *is the approximate solution obtained from Algorithm 6.1, then*

$$\|u_{n+1}-\bar{u}\|^2 \le \|w_n-\bar{u}\|^2-(1-2\alpha\rho)\|u_{n+1}-w_n\|^2 \qquad (6.6)$$

$$\|w_n-\bar{u}\|^2 \le \|u_n-\bar{u}\|^2-(1-2\alpha\beta)\|w_n-y_n\|^2 \qquad (6.7)$$

$$\|y_n-\bar{u}\|^2 \le \|u_n-\bar{u}\|^2-(1-2\alpha\mu)\|y_n-u_n\|^2 \qquad (6.8)$$

Proof. Let $\bar{u}\in H$ be a solution of (2.1). Then

$$\rho\langle T\bar{u},v-\bar{u}\rangle+\rho\phi(v)-\phi(\bar{u})\ge 0, \quad \forall v\in H \qquad (6.9)$$

$$\beta\langle T\bar{u},v-\bar{u}\rangle+\beta\phi(v)-\beta\phi(\bar{u})\ge 0, \quad \forall v\in H \qquad (6.10)$$

$$\mu\langle T\bar{u},v-\bar{u}\rangle+\mu\phi(v)-\mu\phi(\bar{u})\ge 0, \quad \forall v\in H. \qquad (6.11)$$

Taking $v = \bar{u}$ in (6.2), $v = u_{n+1}$ in (6.9) and adding the resultant, we have

$$\begin{aligned} \langle u_{n+1} - w_n, \bar{u} - u_{n+1} \rangle\rangle &\geq \rho\langle Tw_n - T\bar{u}, u_{n+1} - \bar{u}\rangle \\ &\geq -\rho\alpha\|u_{n+1} - w_n\|^2, \end{aligned} \tag{6.12}$$

since T is partially relaxed strongly monotone with constant α.

Taking $v = \bar{u} - u_{n+1}$ and $u = u_{n+1} - w_n$ in (6.5), we have

$$\begin{aligned} 2\langle u_{n+1} - w_n, \bar{u} - u_{n+1} &= \|w_n - \bar{u}\|^2 - \|\bar{u} - u_{n+1}\|^2 \\ &\quad -\|u_{n+1} - w_n\|^2. \end{aligned} \tag{6.13}$$

From (6.12) and (6.13), we have

$$\|u_{n+1}g - \bar{u}\|^2 \leq \|w_n - \bar{u}\|^2 - (1 - 2\rho\alpha)\|u_{n+1} - w_n\|^2,$$

the required (6.6).

Now taking $v = \bar{u}$ in (6.3), $v = y_n$ in (6.22), adding the resultant and using the partially relaxed strongly monotonicity of T, we have

$$\langle w_n - y_n, \bar{u} - w_n \geq -\beta\rho\|y_n - w_n\|^2,$$

which implies, using Lemma 6.1,

$$\|w_n - \bar{u}\|^2 \leq \|y_n - \bar{u}\|^2 - (1 - 2\alpha\beta)\|y_n - w_n\|^2,$$

the required (6.7).

In a similar way, by taking $v = \bar{u}$ in (6.2), $v = y_n$ in (6.11), adding the resultant, using the partially relaxed strongly monotonicity and invoking Lemma 6.1, we obtain

$$\|y_n - \bar{u}\|^2 \leq \|u_n - \bar{u}\|^2 - (1 - 2\alpha\mu)\|y_n - u_n)\|^2,$$

the required (6.8). □

Theorem 6.2. *Let $\bar{u} \in H$ be a solution of (2.1) and u_{n+1} be the approximate solution obtained from Algorithm 6.1. If H is a finite dimensional space and $0 < \rho < 1/2\alpha, 0 < \beta < 1/2\alpha, 0 < \mu < 1/2\alpha$, then*

$$\lim_{n\to\infty} u_n = \bar{u}.$$

Proof. Let $\bar{u} \in H$ be a solution of (2.1). Then, from (6.6), (6.7) and (6.8), it follows that the sequences $\{\|u_n - \bar{u}\|\}, \quad \{\|y_n - \bar{u}\|\}$ and $\quad \{\|w_n - \bar{u}\|\}$ are nonincreasing and consequently the sequences $\{w_n\}, \{y_n\}$ and $\{u_n\}$ are bounded and

$$\begin{aligned}
\sum_{n=0}^{\infty}(1-2\alpha\rho)\|u_{n+1}-u_n\|^2 &\leq \|w_0-\bar{u}\|^2 \\
\sum_{n=0}^{\infty}(1-2\alpha\beta)\|w_n-u_n\|^2 &\leq \|y_0-\bar{u}\|^2 \\
\sum_{n=0}^{\infty}(1-2\alpha\mu)\|y_n-u_n\|^2 &\leq \|u_0-\bar{u}\|^2,
\end{aligned}$$

which implies that

$$\begin{aligned}
\lim_{n\to\infty}\|u_{n+1}-w_n\| &= 0 \\
\lim_{n\to\infty}\|w_n-y_n\| &= 0 \\
\lim_{n\to\infty}\|y_n-u_n\| &= 0.
\end{aligned}$$

Thus

$$\begin{aligned}
\lim_{n\longrightarrow\infty}\|u_{n+1}-u_n\| &= \lim_{n\longrightarrow\infty}\|u_{n+1}-w_n\| + \lim_{n\longrightarrow\infty}\|w_n-y_n\| \\
&\quad + \lim_{n\longrightarrow\infty}\|y_n-u_n\| = 0. \qquad (6.14)
\end{aligned}$$

Let $\hat{u}$ be a cluster point of $\{u_n\}$; there exists a subsequence $\{u_{n_i}\}$ such that $\{u_{n_i}\}$ converges to $\hat{u}$. Replacing w_n and y_n by u_{n_i} in (6.2), (6.3) and (6.4); and taking the limits and using (6.14), we have

$$\langle T\hat{u}, v-\hat{u}\rangle + \phi(v) - \phi(\hat{u}) \geq 0, \quad \forall v \in H.$$

This shows that $\hat{u} \in H$ solves the mixed variational inequality problem (2.1) and

$$\|u_{n+1} - \hat{u})\|^2 \leq \|u_n - \hat{u}\|^2,$$

which implies that the sequence $\{u_n\}$ has a unique cluster point and

$$\lim_{n\to\infty} u_n = \hat{u},$$

is the solution of (2.1), the required result. □

Remark 6.1. *It is worth mentioning that the one-step scheme, that is, Algorithm 14.14 convergence under the assumptions of Theorem 6.1 and Theorem 6.2. This clearly improves the convergence results for the one-step scheme. Our method can be considered as a new approach to consider the convergence analysis of three-step schemes. In the implementation of this scheme, one does not have to evaluate the projection, which is itself a problem. Our method of convergence is very simple as compared with other methods. Following the technique of Tseng [108], one can obtained new parallel and decomposition algorithms for solving a number of problems arising in optimization and mathematical programming.*

Remark 6.2. *We note that the auxiliary problem (6.4) is equivalent to finding the minimum of the functional $I[w]$ on the Hilbert space H, where*

$$I[w] = \frac{1}{2}\langle w - u, w - u\rangle + \langle \rho Tu, w - u\rangle + \rho\phi(w), \tag{6.15}$$

which is the quadratic optimization problem. This implies that the optimization programming techniques can be used to solve the mixed variational inequalities of type (2.1).

We again use the auxiliary principle technique to suggest some inertial proximal point methods for solving the mixed variational inequalities (2.1).

For a given $u \in H$ satisfying (2.1), consider the problem of finding a unique $w \in H$ such that

$$\langle \rho Tw + w - u, v - w\rangle + \rho\phi(v) - \rho\phi(w) \geq 0, \quad \text{for all} \quad v \in H, \tag{6.16}$$

where $\rho > 0$ is a constant.

Note that if $w = u$, then w is clearly a solution of the variational inequality (2.1). This simple observation enables us to suggest and analyze the following implicit proximal method.

Algorithm 6.5. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative schemes*

$$\langle \rho Tu_{n+1} + u_{n+1} - u_n, v - u_{n+1}\rangle + \rho\phi(v) - \rho\phi(u_{n+1}) \geq 0, \quad \forall v \in H, \tag{6.17}$$

where $\rho > 0$ is a constant.

Algorithm 6.5 is an implicit method. Using the technique of Noor [63], one can study the convergence criteria of Algorithm 6.5.
If the function $\phi(.)$ is a proper, lower-semicontinuous and convex function, then Algorithm 6.5 can be written in the equivalent form:

Algorithm 6.6. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative schemes*

$$u_{n+1} = J_\phi[u_n - \rho Tu_{n+1}], \quad n = 0, 1, 2, ... \tag{6.18}$$

which is known as the extraresolvent iterative method for solving the mixed variational inequality (2.1) in the sense of Kopervelich [39] extragradient method for variational inequalities.

We again use the auxiliary principle technique to suggest some inertial proximal point methods for solving the mixed variational inequalities (2.1).

For a given $u \in H$ satisfying (2.1), consider the problem of finding a unique $w \in H$ such that

$$\langle \rho T((1-\alpha)w + \alpha u) + w - ((1-\nu)w + \nu u), v - w\rangle + \rho\phi(v) - \rho\phi(w) \geq 0, \quad \forall \quad v \in H, \tag{6.19}$$

where $\rho > 0, \alpha > 0, \nu > 0$ are constants.

Note that, if $w = u$, then w is clearly a solution of the variational inequality (2.1). This simple observation enables us to suggest and analyze the following implicit proximal method.

Algorithm 6.7. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative schemes*

$$\begin{aligned}&\langle \rho T((1-\alpha)u_{n+1} + \alpha u_n) + u_{n+1} - ((1-\nu)u_{n+1} + \nu u_n), v - u_{n+1}\rangle \\ &+\rho\phi(v) - \rho\phi(u_{n+1}) \geq 0, \quad \forall v \in H,\end{aligned} \tag{6.20}$$

which can be rewritten in the following form

Algorithm 6.8. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative schemes*

$$u_{n+1} = J_\phi[((1-\nu)u_{n+1} + \nu u_n) - \rho T((1-\alpha)u_{n+1} + \alpha u_n)], \quad n = 0, 1, 2, ... \tag{6.21}$$

We again use the auxiliary principle technique to suggest some inertial proximal point methods for solving the mixed variational inequalities (2.1).

For a given $u \in H$ satisfying (2.1), consider the problem of finding a unique $w \in H$ such that

$$\langle \rho T((1-\alpha)u + \alpha u) + w - ((1-\nu)u + \nu u), v - w\rangle + \rho\phi(v) - \rho\phi(w) \geq 0, \quad \text{for all} \quad v \in H, \tag{6.22}$$

where $\rho > 0, \alpha > 0, \nu > 0$ are constants.

Note that if $w = u$, then w is clearly a solution of the mixed variational inequality (2.1). This simple observation enables us to suggest and analyze the following implicit proximal method.

Algorithm 6.9. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative schemes*

$$\begin{aligned}&\langle \rho T((1-\alpha)u_n + \alpha u_{n-1}) + u_{n+1} - ((1-\nu)u_n + \nu u_{n-1}), v - u_{n+1}\rangle \\ &+\rho\phi(v) - \rho\phi(u_{n+1}) \geq 0, \quad \forall v \in H. \end{aligned} \tag{6.23}$$

If the function $\phi(.)$ is a proper lower-semicontinuous and convex function, then Algorithm 6.9 reduces to

Algorithm 6.10. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative schemes*

$$u_{n+1} = J_\phi[(1-\nu)u_n + \nu u_{n-1} - \rho T((1-\alpha)u_n + \alpha u_{n-1})], \quad n = 0, 1, 2, \ldots \tag{6.24}$$

Or equivalently

Algorithm 6.11. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative schemes*

$$\begin{aligned} w_n &= (1-\nu)u_n + \nu u_{n-1} = u_n + \nu(u_n - u_{n-1}) \\ u_{n+1} &= J_\phi[w_n - \rho T w_n], \quad n = 0, 1, 2, \ldots \end{aligned}$$

which is called the inertial proximal iterative method for solving the mixed variational inequalities (2.1) and appears to be a new one.

We now suggest and analyze some iterative methods for mixed variational inequalities (2.1) using the auxiliary principle technique involving the Bregman distance function, which is mainly due to Zu and Marcotte [114] as developed by Noor [63, 65, 66].

For a given $u \in H$ satisfying the mixed variational inequality (2.1), we consider the auxiliary problem of finding a $w \in H$ such that

$$\langle \rho Tw, v - w\rangle + \langle E'(w) - E'(u), v - w\rangle + \rho\phi(v) - \phi(w) \geq 0, \quad \forall v \in H, \quad (6.25)$$

where $\rho > 0$ is a constant and $E'(u)$ is the differential of a strongly convex function $E(u)$ at $u \in H$. Since $E(u)$ is a strongly convex function, this implies that its differential E' is strongly monotone. Consequently it follows that the problem (6.25) has an unique solution.

Remark 6.3. *The function*

$$B(w, u) = E(w) - E(u) - \langle E'(u), w - u\rangle$$

associated with the convex function $E(u)$ is called the generalized Bregman distance function. By the strongly convexity of the function $E(u)$, the Bregman function $B(.,.)$ is nonnegative and $B(w, u) = 0$, if and only if $u = w, \forall u, w \in K$.

We note that, if $w = u$, then clearly w is solution of the mixed variational inequalities (2.1). This observation enables us to suggest and analyze the following iterative method for solving (2.1).

Algorithm 6.12. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\begin{aligned}\langle \rho Tu_{n+1}, v - u_{n+1}\rangle \quad + \quad & \langle E'(u_{n+1}) - E'(u_n), v - u_{n+1}\rangle \\ & + \rho\phi(v) - \rho\phi(u_{n+1}) \geq 0, \quad \forall v \in H, \quad (6.26)\end{aligned}$$

where $\rho > 0$ is a constant.

Algorithm 6.12 is called the proximal method for solving mixed variational inequalities (2.1). In passing we remark that the proximal point method was suggested by Martinet [42] in the context of convex programming problems as a regularization technique.

For suitable and appropriate choice of the operators and the spaces, one can obtain a number of known and new algorithms for solving mixed variational inequalities and related problems.

Theorem 6.3. *Let the operator T be monotone. If E be differentiable strongly convex function with module $\mu > 0$, then the approximate solution u_{n+1} obtained from Algorithm 6.12 converges to a solution $u \in H$ satisfying the mixed variational inequality (2.1).*

Proof. Let $u \in H$ be a solution of (2.1). Then

$$\langle Tu, v-u\rangle + \phi(v) - \phi(u) \geq 0, \quad \forall v \in H,$$

implies that

$$T(v, u-v) + \phi(v) - \phi(u) \geq 0, \quad \forall v \in H, \tag{6.27}$$

since T is a monotone operator.

Taking $v = u$ in (6.26) and $v = u_{n+1}$ in (6.27), we have

$$\rho T(u_{n+1}, u-u_{n+1}) + \langle E'(u_{n+1}) - E'(u_n), u - u_{n+1}\rangle \geq -\rho\phi(u) - \rho\phi(u_{n+1}). \tag{6.28}$$

and

$$\langle Tu_{n+1}, u-u_{n+1}\rangle - \phi(u) - \phi(u_{n+1}) \geq 0. \tag{6.29}$$

We now consider the Bregman function

$$B(u, w) = E(u) - E(w) - \langle E'(w), u-w\rangle \geq \beta\|u-w\|^2, \tag{6.30}$$

using strongly convexity of the convex function E.

Now combining (6.28),(6.29) and (6.30), we have

$$\begin{aligned} B(u,u_n)-B(u,u_{n+1}) &= E(u_{n+1})-E(u_n)-\langle E'(u_n),u-u_n\rangle \\ &\quad +\langle E'(u_{n+1}),u-u_{n+1}\rangle \\ &= E(u_{n+1})-E(u_n)-\langle E'(u_n)-E'(u_{n+1}),u-u_{n+1}\rangle \\ &\quad -\langle E'(u_n),u_{n+1}-u_n\rangle \\ &\geq \beta\|u_{n+1}-u_n\|^2+\langle E'(u_{n+1})-E'(u_n),u-u_{n+1}\rangle \\ &\geq \beta\|u_{n+1}-u_n\|^2. \end{aligned}$$

If $u_{n+1}=u_n$, then clearly u_n is a solution of the mixed variational inequality (2.1). Otherwise, it follows that $B(u,u_n)-B(u,u_{n+1})$ is nonnegative and we must have

$$\lim_{n\to\infty}\|u_{n+1}-u_n\|=0.$$

It follows that the the sequence $\{u_n\}$ is bounded. Let $\bar{u}$ be a cluster point of the subsequence $\{u_{n_i}\}$, and let $\{u_{n_i}\}$ be a subsequence converging toward $\bar{u}$. Now using the technique of Zhu and Marcotte [114], it can be shown that the entire sequence $\{u_n\}$ converges to the cluster point $\bar{u}$ satisfying the mixed variational inequalities (2.1). □

It is well-known that to implement the proximal point methods, one has to find the approximate solution implicitly, which is itself a difficult problem. To overcome this drawback, we now consider another method for solving the mixed variational inequality (2.1) using the auxiliary principle technique.

For a given $u\in H$ satisfying (2.1), find $w\in H$ such that

$$\langle\rho Tu,v-w\rangle+\langle E'(w)-E'(u),v-w\rangle+\rho\phi(v)-\rho\phi(w)\geq 0,\quad \forall v\in H, \tag{6.31}$$

where $E'(u)$ is the differential of a strongly convex function $E(u)$ at $u\in K$. Problem (6.31) has a unique solution, since E is strongly convex function. Note that problems (6.31) and (6.25) are quite different problems. It is clear that, for

$w = u$, w is a solution of (2.1). This fact allows us to suggest and analyze another iterative method for solving the mixed variational inequalities (2.1)).

Algorithm 6.13. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\begin{aligned} \langle \rho T u_n, v - u_{n+1} \rangle \quad + \quad & \langle E'(u_{n+1}) - E'(u_n), v - u_{n+1} \rangle \\ & \geq -\rho\phi(v) - \rho\phi(u_{n+1}), \quad \forall v \in H, \end{aligned} \tag{6.32}$$

for solving the mixed variational inequalities (2.1).

Remark 6.4. *For suitable and appropriate choice of the operators and the spaces, one can obtain various known and new algorithms for solving mixed variational inequalities and related optimization problems. One can consider the convergence analysis of Algorithm 14.26 using essentially the technique of Theorem 6.3.*

7 Nonexpansive Mappings

In recent years, Noor [58, 63] suggested and analyzed several three-step iterative methods for solving different classes of mixed variational inequalities. It has been shown that three-step schemes are numerically better than two-step and one-step methods. Related to the mixed variational inequalities, is the problem of finding the fixed points of the nonexpansive mappings, which is the subject of current interest in functional analysis. It is natural to study these different problems in a unified framework. Motivated by the research going on these fields, we suggest and analyze three-step iterative methods for finding the common solution of these problems. The convergence criteria of these new iterative schemes under some mild conditions is considered. For variational inclusions and nonexpansive mappings, see Noor and Huang [71] and the references therein.

We now recall some well known concepts and results.

Remark 7.1. *Let S be a nonexpansive mapping. We denote the set of the fixed points of S by $F(S)$ and the set of the solutions of the mixed variational inequalities (2.1) by $MVI(H,T)$.*

We can characterize the problem. If $x^* \in F(S) \cap MVI(H,t)$, then $x^* \in F(S)$ and $x^* \in MVI(H,T)$. Thus from Lemma 2.1, it follows that

$$x^* = Sx^* = J_\varphi[[x^* - \rho Tx^*] = SJ_\varphi[[x^* - \rho Tx^*]. \tag{7.1}$$

We can rewrite (7.1) in the following equivalent form using the technique of updating the solution as:

$$\begin{aligned} x^* &= SJ_\varphi[y^* - \rho Ty^*] \\ y^* &= SJ_\varphi[z^* - \rho Tz^*] \\ z^* &= SJ_\varphi[x^* - \rho Tx^*]. \end{aligned}$$

This alternative equivalent form plays a crucial role in suggesting three-step iterative schemes for solving variational inequalities. We here use this fixed point formulation to suggest the following multi-step iterative methods for finding a common element of two different sets of solutions of the fixed points of the nonexpansive mappings and the mixed variational inequalities.

Algorithm 7.1. *For a given $x_0 \in K$, compute the approximate solution x_n by the iterative schemes*

$$z_n = (1 - c_n)x_n + c_n SJ_\varphi[[x_n - \rho Tx_n], \tag{7.2}$$

$$y_n = (1 - b_n)x_n + b_n SJ_\varphi[[z_n - \rho Tz_n], \tag{7.3}$$

$$x_{n+1} = (1 - a_n)x_n + a_n SJ_\varphi[[y_n - \rho Ty_n], \tag{7.4}$$

where $a_n, b_n, c_n \in [0,1]$ for all $n \geq 0$ and S is the nonexpansive operator.

Algorithm 7.1 is a three-step predictor-corrector method.

Note that for $c_n \equiv 0$, Algorithm 7.1 reduces to:

Algorithm 7.2. *For an arbitrarily chosen initial point $x_0 \in K$, compute the sequence the approximate solution$\{x_n\}$ by the iterative schemes*

$$\begin{aligned} y_n &= (1 - b_n)x_n + b_n SJ_\varphi[[x_n - \rho Tx_n], \\ x_{n+1} &= (1 - a_n)x_n + a_n SJ_\varphi[[y_n - \rho Ty_n], \end{aligned}$$

where $a_n, b_n \in [0,1]$ *for all* $n \geq 0$ *and* S *is the nonexpansive operator.*

Algorithm 7.2 is called the two-step (Ishikawa iterations) iterative method.

For $b_n \equiv 1, a_n \equiv 1$, Algorithm 7.2 reduces to:

Algorithm 7.3. *For an arbitrarily chosen initial point* $x_0 \in K$, *compute the sequence* $\{x_n\}$ *by the iterative schemes*

$$\begin{aligned} y_n &= SJ_\varphi[[x_n - \rho T x_n], \\ x_{n+1} &= SJ_\varphi[[y_n - \rho T y_n]. \end{aligned}$$

or

$$x_{n+1} = SJ_\varphi[[SJ_\varphi[[x_n - \rho T x_n] - \rho T SJ_\varphi[[x_n - \rho T x_n]],$$

which is called extraresolvent algorithm.

For $b_n \equiv 0, c_n \equiv 0$, Algorithm 7.1 collapses to the following iterative method.

Algorithm 7.4. *For a given* $x_0 \in H$, *compute the approximate solution* x_{n+1} *by the iterative schemes:*

$$x_{n+1} = (1 - a_n)x_n + a_n SJ_\varphi[[x_n - \rho T x_n],$$

which is known as a Mann iteration.

If φ is the indicator function of a closed convex set K in H, then $J_\varphi = P_K$, the projection of H onto the closed convex set K. Consequently, Algorithms 7.1-7.4 collapse to the following iterative projection method for solving the classical variational inequalities.

Algorithm 7.5. *For a given* $x_0 \in K$, *compute the approximate solution* x_n *by the iterative schemes*

$$\begin{aligned} z_n &= (1 - c_n)x_n + c_n SP_K[x_n - \rho T x_n], \\ y_n &= (1 - b_n)x_n + b_n SP_K[z_n - \rho T z_n], \\ x_{n+1} &= (1 - a_n)x_n + a_n SP_K[y_n - \rho T y_n], \end{aligned}$$

where $a_n, b_n, c_n \in [0,1]$ for all $n \geq 0$ and S is the nonexpansive operator.

Algorithm 7.5 is a three-step predictor-corrector method.

Note that for $c_n \equiv 0$, Algorithm 7.5 reduces to:

Algorithm 7.6. *For an arbitrarily chosen initial point $x_0 \in K$, compute the sequence the approximate solution$\{x_n\}$ by the iterative schemes*

$$\begin{aligned} y_n &= (1-b_n)x_n + b_n SP_K[x_n - \rho T x_n], \\ x_{n+1} &= (1-a_n)x_n + a_n SP_K[y_n - \rho T y_n], \end{aligned}$$

where $a_n, b_n \in [0,1]$ for all $n \geq 0$ and S is the nonexpansive operator.

Algorithm 7.6 is called the two-step (Ishikawa iterations) iterative method.

For $b_n \equiv 1, a_n \equiv 1$, Algorithm 7.6 reduces to:

Algorithm 7.7. *For an arbitrarily chosen initial point $x_0 \in K$, compute the sequence $\{x_n\}$ by the iterative schemes*

$$\begin{aligned} y_n &= SP_K[x_n - \rho T x_n], \\ x_{n+1} &= SP_K[y_n - \rho T y_n]. \end{aligned}$$

or

$$x_{n+1} = SP_K[SP_K[x_n - \rho T x_n] - \rho T SP_K[x_n - \rho T x_n]],$$

which is called extragradient Algorithm.

For $b_n \equiv 0, c_n \equiv 0$, Algorithm 7.5 collapses to the following iterative method.

Algorithm 7.8. *For a given $x_0 \in K$, compute the approximate solution x_{n+1} by the iterative schemes:*

$$x_{n+1} = (1-a_n)x_n + a_n SP_K[x_n - \rho T x_n],$$

which is known as a Mann iteration.

This shows that three-step method suggested in this section is quite general and it includes several new and previously known algorithms for solving variational inequalities and nonexpansive mappings.

Definition 7.1. *A mapping $T : H \to H$ is called r-strongly monotone, if there exists a constant $r > 0$, such that*

$$\langle Tx - Ty, x - y\rangle \geq r\|x - y\|^2, \quad \forall x, y \in H.$$

Definition 7.2. *An operator $TH \to H$ is called cocoercive (inverse) strongly monotone, if there exists a constant $\alpha > 0$ such that*

$$\langle Tu - Tv, u - v\rangle \geq \alpha\|Tu - Tv\|^2, \quad \forall u, v \in H.$$

Definition 7.3. *A mapping $T : H \to H$ is called relaxed γ-cocoercive, if there exists a constant $\gamma > 0$, such that*

$$\langle Tx - Ty, x - y\rangle \geq -\gamma\|Tx - Ty\|^2, \quad \forall x, y \in H.$$

Definition 7.4. *A mapping $T : H \to H$ is called relaxed (γ, r)-cocoercive, if there exists constants $\gamma > 0, r > 0$, such that*

$$\langle Tx - Ty, x - y\rangle \geq -\gamma\|Tx - Ty\|^2 + r\|x - y\|^2 \quad \forall x, y \in H.$$

Definition 7.5. *A mapping $T : H \to H$ is called μ-Lipschitzian, if there exists a constant $\mu > 0$, such that*

$$\|Tx - Ty\| \leq \mu\|x - y\|, \quad \forall x, y \in H.$$

Remark 7.2. *Clearly a r-strongly monotonic mapping or a γ-inverse strongly monotonic mapping must be a relaxed (γ, r)-cocoercive mapping, but the converse is not true. Therefore the class of the relaxed (γ, r)-cocoercive mappings is the most general class.*

Remark 7.3. *From definition 7.4, it follows that, if the operator T is inverse strongly monotone with a constant $a\alpha > 0$, then it is Lipschitz continuous with a constant $\frac{1}{\alpha}$.*

Lemma 7.1. *[68] Suppose $\{\delta_k\}_{k=0}^{\infty}$ is a nonnegative sequence satisfying the following inequality:*

$$\delta_{k+1} \leq (1-\lambda_k)\delta_k + \sigma_k, \ k \geq 0$$

with $\lambda_k \in [0,1]$, $\sum_{k=0}^{\infty} \lambda_k = \infty$, and $\sigma_k = o(\lambda_k)$. Then $\lim_{k\to\infty} \delta_k = 0$.

We now investigate the strong convergence of Algorithms 7.1, 7.2 and 7.4 in finding the common element of two sets of solutions of the mixed variational inequalities $MVI(H,T)$and $F(S)$ and this is the main motivation of the next result.

Theorem 7.1. *Let T be a relaxed (γ, r)-cocoercive and μ-Lipschitzian mappingand S be a nonexpansive mapping of such that $F(S) \cap MVI(H,T) \neq \emptyset$. If*

$$0 < \rho < 2(r - \gamma\mu^2)/\mu^2, \quad \gamma\mu^2 < r, \tag{7.5}$$

$a_n, b_n, c_n \in [0,1]$ and $\sum_{n=0}^{\infty} a_n = \infty$, then x_n obtained from Algorithm 2.1 converges strongly to $x^* \in F(S) \cap MVI(H,T)$.

Proof. Let $x^* \in H$ be the solution of $F(S) \cap MVI(H,T)$. Then

$$\begin{aligned} x^* &= (1-c_n)x^* + c_n SJ_{\varphi}[x^* - \rho Tx^*] && (7.6)\\ &= (1-b_n)x^* + b_n SJ_{\varphi}x^* - \rho Tx^*] && (7.7)\\ &= (1-a_n)x^* + a_n SJ_{\varphi}[x^* - \rho Tx^*], && (7.8) \end{aligned}$$

where $a_n, b_n, c_n \in [0,1]$ are some constants. To prove the result, we need first to evaluate $||x_{n+1} - x^*||$ for all $n \geq 0$. From (7.4),(7.8), and the nonexpansive property of the resolvent operator J_{φ} and the nonexpansive mapping S, we have

$$\begin{aligned} & ||x_{n+1} - x^*|| \\ = \ & ||(1-a_n)x_n + a_n SJ_{\varphi}[y_n - \rho Ty_n] - (1-a_n)x^* - a_n SJ_{\varphi}[x^* - \rho Tx^*]|| \\ \leq \ & (1-a_n)\|x_n - x^*\| + a_n\|SJ_{\varphi}[y_n - \rho Ty_n] - SJ_{\varphi}[x^* - \rho Tx^*]\| \\ \leq \ & (1-a_n)\|x_n - x^*\| + a_n\|y_n - x^* - \rho(Ty_n - Tx^*)\|. \qquad (7.9) \end{aligned}$$

From the relaxed (γ, r)-cocoercive and μ-Lipschitzian definition on T,

$$\begin{aligned}
& \|y_n - x^* - \rho(Ty_n - Tx^*)\|^2 \\
= \ & \|y_n - x^*\|^2 - 2\rho\langle Ty_n - Tx^*, y_n - x^*\rangle + \rho^2\|Ty_n - Tx^*\|^2 \\
\leq \ & \|y_n - x^*\|^2 - 2\rho[-\gamma\|Ty_n - Tx^*\|^2 + r\|y_n - x^*\|^2] \\
& + \rho^2\|Ty_n - Tx^*\|^2 \\
\leq \ & \|y_n - y^*\|^2 + 2\rho\gamma\mu^2\|y_n - x^*\|^2 - 2\rho r\|y_n - x^*\|^2 + \rho^2\mu^2\|y_n - x^*\|^2 \\
= \ & [1 + 2\rho\gamma\mu^2 - 2\rho r + \rho^2\mu^2]\|y_n - x^*\|^2 \\
= \ & \theta^2\|y_n - x^*\|^2, \qquad (7.10)
\end{aligned}$$

where

$$\theta = \sqrt{1 + 2\rho\gamma\mu^2 - 2\rho r + \rho^2\mu^2}. \qquad (7.11)$$

From (7.5), we have $\theta < 1$.

Combining (7.9), (7.10) and (7.11), we have

$$\|x_{n+1} - x^*\| \leq (1 - a_n)\|x_n - x^*\| + a_n\theta\|y_n - x^*\|. \qquad (7.12)$$

From (7.3), (7.7) and the nonexpansivity of the operators S and J_φ, we have

$$\begin{aligned}
\|y_n - x^*\| & \leq (1 - b_n)\|x_n - x^*\| + b_n\|SJ_\varphi[z_n - \rho Tz_n] - SJ_\varphi[x^* - \rho Tx^*]\| \\
& \leq (1 - b_n)\|x_n - x^*\| + b_n\|[z_n - \rho Tz_n] - [x^* - \rho Tx^*]\|. \qquad (7.13)
\end{aligned}$$

Now from the relaxed (γ, r)-cocoercive and μ-Lipschitzian definition on T, it yields that

$$\begin{aligned}
& \|z_n - x^* - \rho[Tz_n - Tx^*]\|^2 \\
= \ & \|z_n - x^*\|^2 - 2\rho\langle Tz_n - Tx^*, z_n - x^*\rangle + \rho^2\|Tz_n - Tx^*\|^2 \\
\leq \ & \|z_n - x^*\|^2 - 2\rho[-\gamma\|Tz_n - Tx^*\|^2 + r\|z_n - x^*\|^2] \\
& + \rho^2\|Tz_n - Tx^*\|^2 \\
\leq \ & \|z_n - x^*\|^2 + 2\rho\gamma\mu^2\|z_n - x^*\|^2 - 2\rho r\|z_n - x^*\|^2 \\
& + \rho^2\mu^2\|z_n - x^*\|^2 \\
= \ & [1 + 2\rho\gamma\mu^2 - 2\rho r + \rho^2\mu^2]\|z_n - x^*\|^2 = \theta^2\|z_n - x^*\|^2, \qquad (7.14)
\end{aligned}$$

From (7.13) and (7.14), we have

$$\|y_n - x^*\| \leq (1-b_n)\|x_n - x^*\| + b_n\theta\|z_n - x^*\|. \tag{7.15}$$

In a similar way, we have

$$\begin{aligned} \|z_n - x^*\| &\leq (1-c_n)\|x_n - x^*\| + c_n\theta\|x_n - x^*\|, \\ &= \{(1-c_n(1-\theta))\}\|x_n - x^*\| \\ &\leq \|x_n - x^*\|. \end{aligned} \tag{7.16}$$

From (7.15) and (7.16, we obtain that

$$\begin{aligned} \|x_{n+1} - x^*\| &\leq (1-a_n)\|x_n - x^*\| \\ &\quad + a_n\|y_n - x^* - \rho(Ty_n - Tx^*)\| \\ &\leq (1-a_n)||x_n - x^*|| + a_n\theta\|y_n - x^*\| \\ &\leq (1-a_n)||x_n - x^*\| + a_n\theta\|z_n - x^*\| \\ &\leq (1-a_n)\|x_n - x^*\| + a_n\theta\|x_n - x^*\| \\ &= [1 - a_n(1-\theta)]\|x_n - x^*\|, \end{aligned}$$

and hence by Lemma 7.1,

$$\lim_{n\to\infty} \|x_n - x^*\| = 0,$$

completing the proof. □

Next we prove the strong convergence theorem of Algorithm 8.4 under the α-inverse strongly monotonicity.

Theorem 7.2. *Let T be an α-inverse strongly monotone mapping with a constant $\alpha > 0$ and S be a nonexpansive mapping such that $F(S) \cap MVI(H,T) \neq \emptyset$. If $\rho \subset [a,b] \subset (0, 2\alpha)$ and $a_n \subset [c,d]$ for some constants $c,d \in (0,1)$, then the sequence $\{x_n\}$ obtained from Algorithm 7.4 converges strongly to $x^* \in F(S) \cap MVI(H,T)$.*

Proof. It is well known that T is α-inverse strongly monotone with the constant $\alpha > 0$, then T is $\frac{1}{\alpha}$–Lipschitzian continuous.

Consider

$$\begin{aligned}
& \|x_n - x^* - \rho[Tx_n - Tx^*]\|^2 \\
= & \|x_n - x^*\|^2 + \rho^2\|Tx_n - Tx^*\|^2 - 2\rho\langle Tx_n - Tx^*, x_n - x^*\rangle \\
\leq & \|x_n - x^*\|^2 + \rho^2\|Tx_n - Tx^*\|^2 - 2\rho\alpha\|Tx_n - Tx^*\|^2 \\
= & \|x_n - x^*\|^2 + (\rho^2 - 2\rho\alpha)\|Tx_n - Tx^*\|^2 \\
\leq & \|x_n - x^*\|^2 + (\rho^2 - 2\rho\alpha) \cdot \frac{1}{\alpha^2}\|x_n - x^*\|^2 \\
= & (1 + \frac{(\rho^2 - 2\rho\alpha)}{\alpha^2})\|x_n - x^*\|^2. \qquad (7.17)
\end{aligned}$$

Set $\theta_1 = (1 + \frac{(\rho^2 - 2\rho\alpha)}{\alpha^2})^{1/2}$. Then from the condition $\rho \subset [a, b] \subset (0, 2\alpha)$, it follows that $\theta_1 \in (0, 1)$.

From (7.17) and the nonexpansive property of the operators of S and J_φ, we have

$$\begin{aligned}
& \|x_{n+1} - x^*\| \\
= & \|(1 - a_n)x_n + a_n SJ_\varphi[x_n - \rho Tx_n] - (1 - a_n)x^* - a_n SJ_\varphi[x^* - \rho Tx^*]\| \\
\leq & (1 - a_n)\|x_n - x^*\| + a_n\|SJ_\varphi[x_n - \rho Tx_n] - SJ_\varphi[x^* - \rho Tx^*]\| \\
\leq & (1 - a_n)\|x_n - x^*\| + a_n\|x_n - x^* - \rho(Tx_n - Tx^*)\| \\
\leq & (1 - a_n)\|x_n - x^*\| + a_n\theta_1\|x_n - x^*\| \\
= & [1 - a_n(1 - \theta_1)]\|x_n - x^*\|.
\end{aligned}$$

Therefore, it follows that

$$\lim_{n\to\infty} \|x_n - x^*\| = 0,$$

using Lemma 7.1, completing the proof. □

8 Dynamical Systems Technique

In this section, we consider the resolvent dynamical systems associated with the mixed variational inequalities. We investigate the convergence analysis of these new methods involving only the monotonicity of the operator. It is well known that the variational inequalities are equivalent to the fixed-point problems. This alternative formulation has played an important and fundamental part in developing a wide class of projection type methods for solving variational inequalities and complementarity problems. This equivalence has been used and analyzed to projected dynamical systems, in which the right-hand side of the ordinary differential equation is a projection operator. The novel feature of the projected dynamical systems is that the set of the stationary points of the dynamical system correspond to the set of the solution of the variational inequalities. Consequently, equilibrium problems which can be formulated in the setting of variational inequalities can now be studied in the frame work of the dynamical systems. Xia and Wang [110, 111] have shown that the projected dynamical systems can be used effectively in designing neural network for solving variational inequalities and related optimization problems. It is well known the projection method and its variant forms including the Wiener-Hopf equations cannot be used to suggest projected type dynamical systems for solving the mixed variational inequalities. These facts motivated us to use the technique of the resolvent operator. In this technique, the given operator is decomposed into the sum of two maximal monotone operators, whose resolvent are easier to evaluate than the resolvent of the original operator. Such a method is known as operator splitting method. This can lead to develop very efficient methods, since one can treat each part of the original operator independently. Using the resolvent operator technique, one can show that the mixed variational inequalities are equivalent to the fixed point-problems. We use this alternative equivalent formulation to suggest and analyze some resolvent dynamical systems associated with the mixed variational inequalities. We prove that these resolvent dynamical systems have the global asymptotic stability properties for the pseudomonotone operator, which is a weaker condition than monotonicity. It is worth mentioning that if the nonlinear term in the mixed variational inequalities is the indicator function of a closed convex set in a Hilbert space, then the resolvent operator is exactly the projection operator. Thus the resolvent dynamical systems include the projected dynamical systems studied in [28, 43, 83] as special case.

We use the equivalent fixed-point formulation to suggest and analyze the resolvent dynamical system:

$$\frac{du}{dt} = \lambda\{J_\phi[u - \rho\mathcal{T}(u)] - u\}, \quad u(t_0) = u_0 \in H, \tag{8.1}$$

associated with mixed variational inequality (2.1), where λ is a parameter. Such type of the dynamical systems are called the resolvent dynamical system. From the definition, it is clear that the solution of the dynamical system always stays in H. This implies that the qualitative results such as the existence, uniqueness and continuous dependence of the solution of (2.1) can be studied .

Using the updating technique of the solution, equation (8.1) can be written as

$$u = J_\phi[J_\phi[u - \rho T(u)] - \rho T J_\phi[u - \rho T(u)]] = J_\phi[I - \rho T]J_\phi[I - \rho T](u). \tag{8.2}$$

This fixed-point formulation has been used to suggest and analyze some two step forward-backward splitting algorithms for solving mixed variational inequalities (2.1). These splitting type methods can be parallelized and have potential applications in optimization and differential equations. We use this fixed-point formulation to suggest the following splitting type dynamical system associated with the mixed variational inequalities (2.1):

$$\frac{du}{dt} = \lambda\{J_\phi[J_\phi[u - \rho T(u)] - \rho T J_\phi[u - \rho T(u)]] - u\}, \quad u(t_0) = u_0 \in H. \tag{8.3}$$

Note this dynamical system is different from the resolvent dynamical system (8.2). These resolvent dynamical systems describe the disequilibrium adjustment processes, which may produce important transient phenomena prior to the achievement of a steady state. If the function ϕ is an indicator function of a closed convex set K in H, then the resolvent operator $J_\phi \equiv P_K$, the projection of H onto the convex set K. Consequently, the resolvent dynamical systems are exactly the projected dynamical systems associated with the variational inequalities considered. It has been shown that the projected dynamical systems are useful for computational schemes. From the view point of neural computation, the structure of these dynamical systems are simple and can be easily implemented in a parallel circuit network, see [28,43,63,83,109–111] and the references therein.

We now define the residue vector $R(u)$ by the relation

$$R(u) \quad = \quad u - J_\phi[u - \rho T(u)]. \tag{8.4}$$

It is clear that $u \in H$ is a solution of the mixed variational inequality (2.1), if and only if, $u \in H$ is a zero of the equation

$$R(u) = 0. \tag{8.5}$$

Definition 8.1. *The dynamical system is said to converge to the solution set K^* of (2.1), if, irrespective of the initial point, the trajectory of the dynamical system satisfies*

$$\lim_{t \to \infty} dist(u(t), K^*) = 0, \tag{8.6}$$

where

$$dist(u, K^*) = inf_{v \in K} \|u - v\|.$$

It is easy to see that, if the set K^* has a unique point u^*, then (8.6) implies that

$$\lim_{t \to \infty} u(t) = u^*.$$

If the dynamical system is stable at u^* in the Lyapunov sense, then the dynamical system is globally asymptotically stable at u^*.

Definition 8.2. *The dynamical system is said to be globally exponentially stable with degree η at u^* if, irrespective of the initial point, the trajectory of the dynamical system $u(t)$ satisfies*

$$\|u(t) - u^*\| \leq \mu_1 \|u(t_0) - u^*\| exp(-\eta(t - t_0)), \quad \textit{for all } t \geq t_0,$$

where η and μ_1 are positive constants independent of the initial point.

It is clear that globally exponentially stability is necessarily globally stability and the dynamical system converges arbitrarily fast.

Lemma 8.1. *Let $\hat{u}$ and $\hat{v}$ be real-valued nonnegative functions with domain $\{t : t \geq t_0\}$ and let $\alpha(t) = \alpha_0(|t - t_0|)$, where α_0 is a monotone increasing function. If, for $t \geq t_0$,*

$$u(t) \leq \alpha(t) + \int_{t_0}^{t} \hat{u}(s)\hat{v}(s)ds,$$

then

$$\hat{u}(t) \leq \alpha(t) exp\{\int_{t_0}^{t} \hat{v}(s)ds\}.$$

From now onward, we assume that the solution set K^* of the mixed variational inequalities (2.1) is nonempty and is bounded, unless otherwise specified.

We now study the main properties of the resolvent dynamical systems and analyze the global stability of the systems by using the technique of Xia and Wang [110, 111]. First of all, we discuss the existence and uniqueness of the resolvent dynamical system (8.1) and this is the main motivation of our next result.

Theorem 8.1. *Let the operator T be a Lipschitz continuous operator. Then, for each $u_0 \in H$, there exists a unique continuous solution $u(t)$ of the resolvent dynamical system (8.1) with $u(t_0) = u_0$ over $[t_0, \infty)$.*

Proof. Let

$$G(u) = \lambda\{J_{\varphi}[u - \rho T(u)] - u\},$$

where $\lambda > 0$ is a constant. For all $u, v \in R^n$, we have

$$\begin{aligned} \|G(u) - G(v)\| &\leq \{\|J_{\phi}[u - \rho T(u)] - J_{\phi}[v - \rho T(v)]\| + \|u - v\|\} \\ &\leq \lambda\|u - v\| + \lambda\|u - v - \rho(T(u) - T(v))\| \\ &\leq \lambda(2 + \rho\beta)\|u - v\|, \end{aligned}$$

where $\beta > 0$ is a Lipschitz constant of the operator T. This implies that the operator $G(u)$ is a Lipschitz continuous on the space H. So, for each $u_0 \in H$, there exists a unique and continuous solution $u(t)$ of the resolvent dynamical system (8.1), defined in the interval $t_0 \leq t < T$ with the initial condition $u(t_0) = u_0$. Let $[t_0, T)$ be its maximal interval of existence. We have to show that $T = \infty$.

Consider

$$\begin{aligned} \|G(u)\| &\leq \lambda\|J_{\phi}[u - \rho T(u)] - u\| \\ &\leq \lambda\{\|J_{\phi}[u - \rho T(u)] - J_{\phi}[u]\| + \|J_{\phi}[u] - J_{\phi}[u^*]\| + \|J_{\phi}[u^*] - u\|\} \\ &\leq \lambda\{2 + \rho\beta_1\}\|u\|| + \lambda\|u^*\| + \lambda\|J_{\phi}[u^*]\|, \end{aligned}$$

for any $u \in R^*$, then

$$\begin{aligned} \|u(t)\| &\leq \|u_0\| + \int_{t_0}^{t} \|Tu(s)\|ds \\ &\leq (\|u_0\| + k_1(t - T_0)) + k_2 \int_{t_0}^{t} \|u(s)\|ds, \end{aligned}$$

where $k_1 = \lambda(\|u^*\| + \|J_{\phi}[u^*]\|)$ and $k_2 = \lambda(2 + \rho\beta)$.

Hence by invoking Lemma 8.1, we have

$$\|u(t)\| \leq \{\|u_0\| + k_1(t - t_0)\}e^{k_2(t - t_0)}, \quad t \in [t, T).$$

This shows that the solution $u(t)$ is bounded on $[t, T)$. So, $T = \infty$. □

Theorem 8.2. *Let the operator T be locally Lipschitz continuous in a domain H. Then the resolvent dynamical system (8.1) is stable in the sense of Lyapunov and globally converges to the solution of the mixed variational inequality (2.1).*

Proof. Since the operator is Lipschitz continuous, it follows from Theorem that the resolvent dynamical system (8.1) has a unique continuous solution $u(t)$ over $[t, T)$ for any fixed $u_0 \in H$. Let $u(t, t_0; u_0)$ be he solution of the initial value problem (8.1). For a given $u^* \in H$, consider the following Lyapunov function

$$L(u) = \|u - u^*\|^2, \quad u \in H. \tag{8.7}$$

It is clear that $\lim_{n\to\infty} L(u_n) = +\infty$, whenever the sequence $\{u_n\} \subset H$ and $\lim_{n\to\infty} u_n = \infty$. Consequently, we conclude that the level sets of L are bounded. Let $u^* \in H$ be a solution of the mixed variational inequality (2.1). Then

$$\langle T(u), v - u^*\rangle + \phi(v) - \phi(u^*) \geq 0, \quad \text{for all } v \in H,$$

implies

$$\langle T(v), v - u^*\rangle + \phi(v) - \phi(u^*) \geq 0, \tag{8.8}$$

since the operator T is monotone.

Taking $v = J_\phi[u - \rho T(u)]$ in (8.8), we have

$$\langle T J_\phi[u - \rho T(u)], J_\phi[u - \rho T(u)]\rangle + \phi(J_\phi[u - \rho(u)]) - \phi(u^*) \geq 0. \tag{8.9}$$

Setting $v = u^*, u = J_\phi[u - \rho T(u)]$, and $z = u - \rho T J_\phi[u - \rho T(u)]$ in (2.6), we have

$$\begin{aligned} &\langle J_\phi[u - \rho T(u)] - u + \rho T J_\phi[u - \rho T(u)], u^* - J_\phi[u - \rho T(u)]\rangle \\ &+\rho\phi(u^*) - \rho\varphi(J_\phi[u - \rho T(u)]) \geq 0. \end{aligned} \tag{8.10}$$

Adding (8.9), (8.10) and using (8.7) we obtain

$$\langle u - u^*, R(u)\rangle \geq \|R(u)\|^2. \tag{8.11}$$

Thus we have

$$\begin{aligned}\frac{d}{dt}L(u) &= \frac{dL}{du}\frac{du}{dt} \\ &= 2\lambda\langle u-u^*, J_\phi[u-\rho T(u)]\rangle = 2\lambda\langle u-u^*, -R(u)\rangle \\ &\leq -2\lambda\|R(u)\|^2 \leq 0.\end{aligned}$$

This implies that $L(u)$ is a global Lyapunov function for the resolvent dynamical system (8.1) and the system is stable in the sense of Lyapunov. Since $\{u(t) : t \geq t_0\} \subset K_0$, where $K_0 = \{u \in H : L(u) \leq L(u_0)\}$ and the function is continuous differentiable on H., it follows from LaSalle's invariance principle that the trajectory will converge to Ω, the largest invariant subset of the following subset:

$$E = \{u \in R^n : \frac{dL}{dt} = 0\}.$$

Note that, if $\frac{dL}{dt} = 0$, then

$$\|u - J_\phi[u - \rho T(u)]\| = 0,$$

and hence u is the equilibrium point of the resolvent dynamical system (8.1, that is,

$$\frac{du}{dt} = 0.$$

Conversely, if $\frac{du}{dt} = 0$, then it follows that $\frac{dL}{dt} = 0$. Thus, we conclude that

$$E = \{u \in R^n : \frac{du}{dt} = 0\} = K \cap K^*,$$

which is nonempty, convex and invariant set containing the solution set K^*. So

$$\lim_{t\to\infty} \text{dis}(u(t), E) = 0.$$

Therefore the resolvent dynamical system (8.1) converges globally to the solution set of the mixed variational inequalities (2.1). In particular, if the set $E = \{u^*\}$, then

$$\lim_{t\to\infty} u(t) = u^*.$$

Hence the resolvent system (8.1) is globally asymptotically stable. □

Theorem 8.3. *Let the operator T be a Lipschitz continuous with constant $\beta > 0$. If $\lambda < 0$, then the resolvent dynamical system (8.1) converges globally exponentially to the unique solution of the mixed variational inequality (2.1).*

Proof. From Theorem 8.3, we see that there exits a unique continuously differentiable solution of the resolvent dynamical system over $[t_0, \infty)$. Then, from (8.1) and (3.1), we have

$$\begin{aligned} \frac{dL}{dt} &= 2\lambda\langle u(t) - u^*, J_\varphi[u(t) - \rho T(u(t))]\rangle \\ &= -2\lambda\|u(t) - u^*\|^2 + 2\lambda\langle u(t) - u^*, J_\phi[u(t) - \rho T(u(t))] - u^*\rangle, \quad (8.12) \end{aligned}$$

where $u^* \in H$ is the solution of the mixed variational inequality (2.1), that is

$$u^* = J_\phi[u^* - \rho T(u^*)].$$

Now using the nonexpansivity of the resolvent operator J_ϕ and the Lipschitz continuity of the operator $\mathcal{T}$, we have

$$\begin{aligned} \|J_\phi[u - \rho T(u)] - J_\phi[u^* - \rho T(u^*)]\| &\leq \|u - u^*\| + \rho\|T(u) - T(u^*)\| \\ &\leq (1 + \beta\rho)\|u - u^*\|. \quad (8.13) \end{aligned}$$

From (8.12) and (8.13), we have

$$\frac{d}{dt}\|u(t) - u^*\|^2 \leq 2\alpha\lambda\|u(t) - u^*\|,$$

where

$$\alpha = \rho\beta.$$

Thus, for $\lambda = -\lambda_1$, where λ_1 is a positive constant, we have

$$\|u(t) - u^*\| \leq \|u(t) - u^*\|e^{-\alpha\lambda_1(t-t_0)},$$

which shows that the trajectory of the solution of the resolvent dynamical system (8.1) will converge to the unique solution of the mixed variational inequality. □

We use the resolvent dynamical system (8.1) to suggest some iterative for solving the mixed variational inequalities (2.1). These methods can be viewed in the sense of Koperlevich [39] and Noor [63] involving the double resolvent operator.

For simplicity, we take $\lambda = 1$. Thus the dynamical system(8.1) becomes

$$\frac{du}{dt} + u = J_\phi[u - \rho Tu], \quad u(t_0) = \alpha. \tag{8.14}$$

We construct the implicit iterative method using the forward difference scheme. Discretizing (8.14), we have

$$\frac{u_{n+1} - u_n}{h} + u_{n+1} = J_\phi[u_n - \rho Tu_{n+1}], \tag{8.15}$$

where h is the step size. Now, we can suggest the following implicit iterative method for solving the mixed variational inequality (2.1).

Algorithm 8.1. *For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme*

$$u_{n+1} = J_\phi\left[u_n - \rho Tu_{n+1} - \frac{u_{n+1} - u_n}{h}\right], \quad n = 0, 1, 2, \ldots.$$

This is an implicit method and is quite different from the implicit method. Using the resolvent Lemma, Algorithm 8.1 can be rewritten in the equivalent form as:

Algorithm 8.2. *For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme*

$$\begin{aligned} \langle \rho Tu_{n+1} \quad + \quad & \{\frac{(1+h)u_{n+1} - (1+h)u_n}{h}\}, v - u_{n+1}\rangle \\ & + \rho\phi(v) - \rho\phi(u_{n+1}) \geq 0, \forall v \in H. \end{aligned} \tag{8.16}$$

We now study the convergence analysis of algorithm 8.2 under some mild conditions.

Theorem 8.4. *Let $u \in H$ be a solution of the mixed variational inequality (2.1). Let u_{n+1} be the approximate solution obtained from (8.16). If T is monotone, then*

$$\|u - u_{n+1}\|^2 \leq \|u - u_n\|^2 - \|u_n - u_{n+1}\|^2. \tag{8.17}$$

Proof. Let $u \in H$ be a solution of (2.1). Then

$$\langle Tv, v - u\rangle + \phi(v) - \phi(u) \geq 0, \quad \forall v \in H, \tag{8.18}$$

since T is a monotone operator.

Set $v = u_{n+1}$ in (8.18), to have

$$\langle Tu_{n+1}, u_{n+1} - u\rangle + \phi(u_{n+1}) - \phi(u) \geq 0. \tag{8.19}$$

Take $v = u$ in equation (8.16), we have

$$\langle \rho Tu_{n+1} + \{\frac{(1+h)u_{n+1} - (1+h)u_n}{h}\}, u - u_{n+1}\rangle + \rho\phi(u) - \rho\phi(u_{n+1}) \geq 0. \tag{8.20}$$

From (8.19) and (8.20), we have

$$\langle (1+h)u_{n+1} - (1+h)u_n, u - u_{n+1}\rangle \geq 0. \tag{8.21}$$

From (8.21) and using $2\langle a, b\rangle = \|a+b\|^2 - \|a\|^2 - \|b\|^2, \quad \forall a, b \in H$, we obtain

$$\|u_{n+1} - u\|^2 \leq \|u - u_n\|^2 - \|u_{n+1} - u_n\|^2.$$

the required result (8.17). □

Theorem 8.5. *Let $u \in H$ be the solution of mixed variational inequality (2.1). Let u_{n+1} be the approximate solution obtained from (8.16). If T is a monotone operator, then u_{n+1} converges to $u \in H$ satisfying (2.1).*

Proof. Let T be a monotone operator. Then, from (8.17), it follows the sequence $\{u_i\}_{i=1}^{\infty}$ is a bounded sequence and

$$\sum_{i=1}^{\infty} \|u_n - u_{n+1}\|^2 \leq \|u - u_0\|^2,$$

which implies that

$$\lim_{n\to\infty} \|u_{n+1} - u_n\|^2 = 0. \tag{8.22}$$

Since sequence $\{u_i\}_{i=1}^{\infty}$ is bounded, so there exists a cluster point $\hat{u}$ to which the subsequence $\{u_{i_k}\}_{k=1}^{\infty}$ converges. Taking limit in (8.16)and using (8.22), it follows that $\hat{u} \in H$ satisfies

$$\langle T\hat{u}, v - \hat{u}\rangle + \phi(v) - \phi(\hat{u}) \geq 0, \quad \forall v \in H,$$

and

$$\|u_{n+1} - u\|^2 \leq \|u - u_n\|^2.$$

Using this inequality, one can show that the cluster point $\hat{u}$ is unique and

$$\lim_{n\to\infty} u_{n+1} = \hat{u}.$$

□

We now suggest an other implicit iterative method for solving (2.1). Discretizing (8.14), we have

$$\frac{u_{n+1} - u_n}{h} + u_{n+1} = J_\phi[u_{n+1} - \rho T u_{n+1}], \tag{8.23}$$

where h is the step size.

This formulation enable us to suggest the following iterative method.

Algorithm 8.3. *For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme*

$$u_{n+1} = J_\phi\left[u_{n+1} - \rho T u_{n+1} - \frac{u_{n+1} - u_n}{h}\right], \quad n = 0, 1, 2, \ldots.$$

Using resolvent Lemma, Algorithm 8.3 can be rewritten in the equivalent form as:

Algorithm 8.4. *For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme*

$$\langle \rho T u_{n+1} + \{\frac{u_{n+1} - u_n}{h}\}, v - u_{n+1} + \phi(v) - \phi)u_{n+1})\rangle \geq 0, \quad \forall v \in H. \tag{8.24}$$

Again using the dynamical systems, we can suggested some iterative methods for solving the variational inequalities and related optimization problems.

Algorithm 8.5. *For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme*

$$u_{n+1} = J_\phi\Big[\frac{(h+1)u_n - u_{n+1}}{h} - \rho T u_n\Big], \quad n = 0, 1, 2, \ldots,$$

which can be written in the equivalent form as

Algorithm 8.6. *For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme*

$$\langle \rho T u_n + \{\frac{h+1}{h}(u_{n+1} - u_n)\}, v - u_{n+1}\rangle + \phi(v) - \phi(u_{n+1}) \geq 0, \quad \forall v \in H. \quad (8.25)$$

Using the dynamical system associated with the mixed variational inequalities (2.1), one can suggest and analyze a wide class of iterative methods for solving the mixed variational inequalities. The comparison and implementations of these methods with other methods is an interesting problem and needs further efforts.

We now introduce the second order dynamical system associated with the variational inequality (2.1), which is the main aim of this paper. To be more precise, we consider the problem of finding $\mu \in \mathtt{H}$ such that

$$\gamma\ddot{\mu} + \dot{\mu} = \lambda\{J_\phi[\mu - \rho T\mu] - \mu\}, \quad \mu(a) = \alpha, \quad \mu(b) = \beta, \quad (8.26)$$

where $\gamma > 0, \lambda > 0$ and $\rho > 0$ are constants. We would like to emphasize that the problem (8.26) is indeed a second order boundary vale problem.

The equilibrium point of the dynamical system (8.26) is naturally defined as follows.

Definition 8.3. *An element $\mu \in \mathcal{H}$, is an equilibrium point of the dynamical system (8.26), if,*

$$\gamma\frac{d^2\mu}{dx^2} + \frac{d\mu}{dx} = 0.$$

Thus it is clear that $\mu \in H$ is a solution of the variational inequality (2.1), if and only if, $\mu \in H$ is an equilibrium point.

From (8.26), we have

$$\mu = J_\phi[\mu - \rho T\mu].$$

Thus, we can rewrite (8.26) as follows:

$$\mu = J_\phi\big[\mu - \rho T\mu + \gamma \frac{d^2\mu}{dx^2} + \frac{d\mu}{dx}\big]. \tag{8.27}$$

For $\lambda = 1$, the problem (8.26) is equivalent to finding $\mu \in \Omega$ such that

$$\gamma\ddot{\mu} + \dot{\mu} + \mu = P_\Omega\big[\mu - \rho T\mu\big], \quad \mu(a) = \alpha, \quad \mu(b) = \beta. \tag{8.28}$$

The problem (8.28) is called the second dynamical system, which is in fact a second order boundary value problem. This interlink among various areas is fruitful from numerical analysis in developing implementable numerical methods for finding the approximate solutions of the variational inequalities. Consequently, we can explore the ideas and techniques of the differential equations to suggest and propose hybrid proximal point methods for solving the variational inequalities and related optimization problems.

We discretize the second-order dynamical systems (8.28) using central finite difference and backward difference schemes to have

$$\gamma\frac{\mu_{n+1} - 2\mu_n + \mu_{n-1}}{h^2} + \frac{\mu_n - \mu_{n-1}}{h} + \mu_n = J_\phi[\mu_n - \rho T\mu_{n+1}], \tag{8.29}$$

where h is the step size.

If $\gamma = 1, h = 1$, then, from equation (8.29) we have

Algorithm 8.7. *For a given $\mu_0 \in H$, compute μ_{n+1} by the iterative scheme*

$$\mu_{n+1} = J_\phi[\mu_n - \rho T\mu_{n+1}],$$

which is the the extragradient method of Korpelevich [35] for solving the variational inequalities.

Algorithm 8.7 is an implicit method. To implement the implicit method, we use the predictor-corrector technique to suggest the method.

Algorithm 8.8. *For given $\mu_0, \mu_1 \in H$, compute μ_{n+1} by the iterative scheme*

$$\begin{aligned} y_n &= (1-\theta_n)\mu_n + \theta_n\mu_{n-1} \\ \mu_{n+1} &= J_\phi[\mu_n - \rho T y_n], \end{aligned}$$

is called the two-step inertial iterative method, where $\theta_n \in [0, 1]$ is a constant.

Problem (8.28) can be rewritten as

$$\gamma\ddot{\mu} + \dot{\mu} + \mu = J_\phi\big[(1-\theta_n)\mu + \theta_n\mu - \rho T((1-\theta_n)\mu + \theta_n\mu)\big], \tag{8.30}$$

where $\gamma > 0, \theta_n \in [0, 1]$ are constants.

Discretising the system (8.30), we have

$$\begin{aligned}&\gamma\frac{\mu_{n+1} - 2\mu_n + \mu_{n-1}}{h^2} + \frac{\mu_{n+1} - \mu_n}{h} + \mu_n \\ &= J_\phi[(1-\theta_n)\mu_n + \theta_n\mu_{n-1} - \rho T((1-\theta_n)\mu_n + \theta_n\mu_{n-1})]\end{aligned}$$

from which, for $\gamma = 0, \quad h = 1$, we have

Algorithm 8.9. *For a given $\mu_0, \mu_1 \in H$, compute μ_{n+1} by the iterative scheme*

$$\mu_{n+1} = J_\phi[(1-\theta_n)u_n + \theta_n\mu_{n-1} - \rho T((1-\theta_n)\mu_n + \theta_n\mu_{n-1})].$$

Or equivalently

Algorithm 8.10. *For a given $\mu_0, \mu_1 \in H$, compute μ_{n+1} by the iterative scheme*

$$\begin{aligned}y_n &= (1-\theta_n)\mu_n + \theta_n\mu_{n-1} \\ \mu_{n+1} &= J_\phi[y_n - \rho T y_n]\end{aligned}$$

which is called the new inertial iterative method for solving the variational inequality.

We discretize the second-order dynamical systems (8.28) using central finite difference and backward difference schemes to have

$$\gamma\frac{\mu_{n+1} - 2\mu_n + \mu_{n-1}}{h^2} + \frac{\mu_n - \mu_{n-1}}{h} + \mu_{n+1} = J_\phi\big[\mu_n - \rho T\mu_{n+1}\big],$$

where h is the step size.

Using this discrete form, we can suggest the following an iterative method for solving the variational inequalities (2.1).

Algorithm 8.11. *For given $\mu_0, \mu_1 \in H$, compute μ_{n+1} by the iterative scheme*

$$\mu_{n+1} = J_\phi\big[\mu_n - \rho T\mu_{n+1} - \frac{\gamma\mu_{n+1} - (2\gamma - h)\mu_n + (\gamma - h)\mu_{n-1}}{h^2}\big].$$

Algorithm 8.11 is called the inertial proximal method for solving the variational inequalities and related optimization problems. This is a new proposed method.

We note that, for $\gamma = 0, \quad h = 1$, Algorithm 8.11 reduces to the following iterative method for solving variational inequalities (2.1).

Algorithm 8.12. *For given $\mu_0, \mu_1 \in H$, compute μ_{n+1} by the iterative scheme*

$$\mu_{n+1} = J_\phi[\mu_{n-1} - \rho T\mu_{n+1)}].$$

We again discretize the second-order dynamical systems (8.28) using central difference scheme and forward difference scheme to suggest the following inertial proximal method.

Algorithm 8.13. *For a given $\mu_0, \mu_1 \in H$, compute μ_{n+1} by the iterative scheme*

$$\mu_{n+1} = J_\phi[\mu_{n+1} - \rho T\mu_{n+1} - \frac{(\gamma + h)\mu_{n+1} - (2\gamma + h)\mu_n + \gamma\mu_{n-1}}{h^2}].$$

Algorithm 8.13 is quite different from other inertial proximal methods.

If $\gamma = 0$, then Algorithm 8.13 collapses to:

Algorithm 8.14. *For a given $\mu_0 \in H$, compute μ_{n+1} by the iterative scheme*

$$\mu_{n+1} = J_\phi[\mu_{n+1} - \rho\mathcal{T}\mu_{n+1} - \frac{\mu_{n+1} - \mu_n}{h}].$$

Algorithm 8.13 is an proximal method. Such type of proximal methods were suggested by Noor [65] using the fixed point problems.

Rewriting the problem (8.28) in the following form

$$\gamma\ddot{\mu} + \dot{\mu} + \mu = J_{\phi}\big[(\frac{\mu+\mu}{2}) - \rho T(\frac{\mu+\mu}{2})\big], \tag{8.31}$$

and descretizing, we obtain

Algorithm 8.15. *For given $\mu_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme*

$$u_{n+1} = J_{\phi}\big[(\frac{\mu_n+\mu_{n+1}}{2}) - \rho T(\frac{\mu_n+\mu_{n+1}}{2})\big],$$

which is an implicit iterative method.

Using the predictor and corrector technique, we suggest the following two-step iterative method for solving the variational inequalities.

Algorithm 8.16. *For given $\mu_0 \in \mathcal{H}$, compute the approximate solution μ_{n+1} by the iterative scheme*

$$\begin{aligned} y_n &= J_{\phi}\big[\mu_n - \rho\mathcal{T}\mu_n\big] \\ \mu_{n+1} &= J_{\phi}\big[\frac{\mu_n+y_n}{2} - \rho T(\frac{\mu_n+y_n}{2})\big]. \end{aligned}$$

Algorithm 8.16 is a two step iterative method.

Clearly Algorithm 8.15 and Algorithm 14.24 are equivalent. It is enough to prove the convergence of Algorithm 8.15, which is the main motivation of our next result.

Theorem 8.6. *Let the operator T be Lipschitz continuous with constant $\beta > 0$. Let $u \in \mathcal{H}$ be a solution of (2.1) and μ_{n+1} be an approximate solution obtained from Algorithm 8.15. If there exists a constant $\rho > 0$, such that*

$$\rho < \frac{1-\sigma}{\beta}, \quad \sigma < 1, \tag{8.32}$$

then the approximate solution μ_{n+1} converge to the exact solution $\mu \in \Omega$.

Proof. Let $\mu \in \mathcal{H}$ be a solution of(2.1) and μ_{n+1} be the approximate solution obtained from Algorithm 14.23. Then, using the Lipchitz continuity of the operator T, we obtain

$$\begin{aligned}
\|\mu_{n+1}-\mu\| &= \|J_\phi[(\frac{\mu_n+\mu_{n+1}}{2})-\rho T(\frac{\mu_n+\mu_{n+1}}{2})]-J_\phi[\frac{\mu+\mu}{2}-\rho T(\frac{\mu+\mu}{2})]\| \\
&\leq \|(\frac{\mu_n+\mu_{n+1}}{2})-(\frac{\mu+\mu}{2})-\rho(T(\frac{\mu_{n+1}+\mu_n}{2})-T(\frac{\mu+\mu}{2}))\| \\
&\leq \|(\frac{\mu_n+\mu_{n+1}}{2})-(\frac{\mu+\mu}{2})\|+\rho(T(\frac{\mu_{n+1}+\mu_n}{2})-T(\frac{\mu+\mu}{2}))\| \\
&\leq (1+\rho\beta)\|(\frac{\mu_n+\mu_{n+1}}{2})-(\frac{\mu+\mu}{2})\| \\
&\leq \frac{(1+\rho\beta)}{2}\{\|\mu_{n+1}-\mu\|+\|\mu_n-\mu\|\},
\end{aligned}$$

which implies

$$\|\mu_{n+1}-\mu\| \leq \frac{\sigma+\rho\beta}{2-\sigma-\rho\beta}\|\mu_n-\mu\| = \theta\|(\frac{\mu_n+\mu_{n+1}}{2})-(\frac{\mu+\mu}{2})\|,$$

where

$$\theta = \frac{\sigma+\rho\beta}{2-\sigma-\rho\beta}.$$

□

Using the predictor and corrector technique, we suggest the following multi-step method for solving the variational inequalities.

Algorithm 8.17. *For given $\mu_0 \in H$, compute the approximate solution u_{n+1} by the iterative schemes*

$$\begin{aligned}
y_n &= (1-\alpha_n)\mu_n + \alpha_n J_\phi\big[\mu_n - \rho T\mu_n\big] \\
w_n &= (1-\beta_n)y_n + \beta_n J_\phi\big[(\frac{\mu_n+y_n}{2})-\rho T(\frac{\mu_n+y_n}{2})\big] \\
\mu_{n+1} &= (1-\beta_n)w_n + \beta_n J_\phi\big[(\frac{w_n+y_n}{2})-\rho T(\frac{w_n+y_n}{2})\big],
\end{aligned}$$

which is called three-step iterative method, where α_n, η_n, η_n are constants.

Algorithm 8.18. *For given $\mu_0 \in H$, compute the approximate solution u_{n+1} by the iterative schemes*

$$\begin{aligned} t_n &= (1-\theta_n)\mu_n + \theta_n \mu_{n-1} \\ y_n &= (1-\alpha_n)t_n + \alpha_n J_\phi\big[(\frac{\mu_n + t_n}{2}) - \rho T(\frac{\mu_n - t_n}{2})\big] \\ w_n &= (1-\beta_n)y_n + \beta_n J_\phi\big[(\frac{\mu_n + y_n}{2}) - \rho T(\frac{\mu_n + y_n}{2})\big] \\ \mu_{n+1} &= (1-\zeta_n)w_n + \zeta_n J_\phi\big[(\frac{w_n + y_n}{2}) - \rho T(\frac{w_n + y_n}{2})\big], \end{aligned}$$

is called four step inertial iterative method, where $\theta_n, \alpha_n, \beta_n, \zeta_n$ are constants.

Applying the technique and idea of the second order boundary value associated with variational inequalities, one can suggest and consider a wide class of hybrid multi step iterative methods for solving variational inequalities, complementarity and related optimization problems.

Zeng et al. [114] have investigated the fractional dynamical systems associated with variational inequalities. They have investigated the criteria for the asymptotically stability of the equilibrium points. We would like to point out that our results are more general than the results of Zeng et al. [114]. These ideas and techniques may inspire the interested readers for further research in this area. We now suggest a new fractional resolvent dynamical system associated with mixed variational inequalities.

$$D_t^\alpha u = \gamma\{-R(u) - \rho T J_\phi[u - \rho T u] + \rho T u\}, \quad u(0) = \alpha, \quad u \in H, \tag{8.33}$$

where $0 < \alpha < 1$ and γ is a constant, associated with problem mixed variational inequality. For more applications and motivation, see [36, 37, 47].

For $\alpha = 1$, problem (8.33) reduces to finding $u \in H$ such that

$$\frac{du}{dx} = \gamma\{-R(u) - \rho T J_\phi[u - \rho T u] + \rho T u\}, \quad u(0) = \alpha, \quad u \in H, \tag{8.34}$$

is called the resolvent dynamical system, which appear to be a new one. Using the technique of this section, one can investigate the asymptotically stability and other aspects.

9 Selfadaptive Iterative Methods

It is known that the convergence of the projection iterative requires that the underlying operator must be strongly monotone and Lipschitz continuous. These strict conditions rules out the applications of the projection iterative method in several important problems. This fact has motivated to modify the projection method in several directions. Korpelevich [39] proposed the extragradient method, the convergence of which needs only the Lipschitz continuity of the monotone operator. To implement this method, one has to calculate the Lipschitz continuity constant, which is itself a difficult problem. In order to overcome this advantage, Noor [63] suggested some two-step projection methods for solving the variational inequalities using the technique of updating the solution and proved that the convergence of the modified two-step method requires only the partially relaxed strongly monotonicity of the involved operator. Note that the partially relaxed strongly mononotonicity implies monotonicity, but the converse is not true. The modified two-step is also called the predictor-corrector operator. It has been shown that three-step iterative methods [30, 32] are more efficient than two-step and one-step iterative methods. For the applications of Noor iterations in solar panel optimization, see Natarajan et al. [46] and the references therein. Motivated by these results, we suggest and analyze a new self-adaptive three step iterative method for solving mixed variational inequalities. The proposed method consists of three steps and the new iterate is obtained by using a descent direction. We prove that the new method is globally convergent under suitable mild conditions. An example is given to illustrate the efficiency and the implementation of the proposed method. Results are very encouraging and further efforts are required to improve these methods

Lemma 9.1. *$u^* \in H$ is solution of the mixed variational inequality (2.1), if and only if, $u^* \in H$ satisfies the relation:*

$$u^* = J_\phi[u^* - \rho T(u^*)], \tag{9.1}$$

where $J_\phi = (I + \rho\partial\phi)^{-1}$ is the resolvent operator.

From Lemma 9.1, it is clear that u is solution of (2.1), if and only if, u is a zero point of the function

$$r(u,\rho) = u - J_{\phi}[u - \rho T(u)] = 0.$$

Lemma 9.2. *[20] $\forall u \in H$ and $\rho' \geq \rho > 0$, it holds that*

$$\|r(u,\rho')\| \geq \|r(u,\rho)\| \tag{9.2}$$

and

$$\frac{\|r(u,\rho')\|}{\rho'} \leq \frac{\|r(u,\rho)\|}{\rho}. \tag{9.3}$$

Lemma 9.3. *For all $v, w \in H$, we have*

$$\|J_{\phi}(w) - J_{\phi}(v)\|^2 \leq \langle w - v, J_{\phi}(w) - J_{\phi}(v)\rangle. \tag{9.4}$$

Proof. By using (2.6), we get

$$\langle w - J_{\phi}(w), J_{\phi}(w) - J_{\phi}(v)\rangle + \rho\varphi(J_{\phi}(v)) - \rho\phi(J_{\phi}(w)) \geq 0 \tag{9.5}$$

and

$$\langle v - J_{\phi}(v), J_{\phi}(v) - J_{\phi}(w)\rangle + \rho\phi(J_{\phi}(w)) - \rho\phi(J_{\phi}(v)) \geq 0. \tag{9.6}$$

Adding (9.5) and (9.6), we obtain

$$\langle v - w, J_{\phi}(v) - J_{\phi}(w)\rangle \geq \|J_{\phi}(v) - J_{\phi}(w)\|^2.$$

□

Throughout this paper, we make following assumptions.

Assumptions:

- H is a finite dimension space.

- T is continuous and monotone operator on H, that is,,

$$\langle T(u) - T(u'), u' - u\rangle \geq 0 \qquad \forall u', u \in H.$$

- The solution set of problem (2.1) denoted by Ω^* is nonempty.

We now suggest and analyze the new method for solving for solving mixed variational inequalities (2.1). For given $u^k \in H$ and $\rho_k > 0$, each iteration of the proposed method consists of three steps, the first step offers $\tilde{u}^k$, the second step makes $\bar{u}^k$ and the third step produces the new iterate u^{k+1}.

Algorithm 9.1. *Step 1. Given* $u^0 \in H$, $\epsilon > 0$, $\rho_0 = 1$, $\nu > 1$, $\mu \in (0, \sqrt{2})$, $\gamma \in (0, 2)$, $\tau \in (0, 1)$, $\eta_1 \in (0, \tau)$, $\eta_2 \in (\tau, \nu)$ *and let* $k = 0$.

Step 2. If $\|r(u^k, 1)\| \leq \epsilon$, *then stop. Otherwise, go to Step 3.*

Step 3. 1) For a given $u^k \in H$, *calculate the two predictors*

$$\tilde{u}^k = J_\phi[u^k - \rho_k T(u^k)], \tag{9.7a}$$

$$\bar{u}^k = J_\phi[\tilde{u}^k - \rho_k T(\tilde{u}^k)]. \tag{9.7b}$$

2) If $\|r(\tilde{u}^k, 1)\| \leq \epsilon$, *then stop . Otherwise, continue.*

3) If ρ_k *satisfies both*

$$r_1 := \frac{\|\rho_k[\langle \tilde{u}^k - \bar{u}^k, T(u^k) - T(\tilde{u}^k)\rangle - \langle u^k - \bar{u}^k, T(\tilde{u}^k) - T(\bar{u}^k)\rangle]\|}{\|\tilde{u}^k - \bar{u}^k\|^2} \leq \mu^2 \tag{9.8}$$

and

$$r_2 := \frac{\|\rho_k(T(\tilde{u}^k) - T(\bar{u}^k))\|}{\|\tilde{u}^k - \bar{u}^k\|} \leq \nu, \tag{9.9}$$

then go to Step 4; otherwise, continue.

4) Perform an Armijo-like line search via reducing ρ_k

$$\rho_k := \rho_k * \frac{0.8}{\max(r_1, 1)} \tag{9.10}$$

and go to Step 3.

Step 4. Take the new iteration u^{k+1}, by setting

$$u^{k+1}(\alpha_k) = J_\phi[u^k - \alpha_k d(\tilde{u}^k, \bar{u}^k))], \tag{9.11}$$

where

$$\alpha_k = \frac{\langle u^k - \bar{u}^k, d(\tilde{u}^k, \bar{u}^k)\rangle}{\|d(\tilde{u}^k, \bar{u}^k)\|^2} \tag{9.12}$$

and

$$d(\tilde{u}^k, \bar{u}^k) := (\tilde{u}^k - \bar{u}^k) - \rho_k(T(\tilde{u}^k) - T(\bar{u}^k)). \tag{9.13}$$

Step 5. Adaptive rule of choosing a suitable ρ_{k+1} as the start prediction step size for the next iteration

1) Prepare a proper ρ_{k+1},

$$\rho_{k+1} := \begin{cases} \rho_k * \tau/r_2 & \text{if } r_2 \leq \eta_1, \\ \rho_k * \tau/r_2 & \text{if } r_2 \geq \eta_2, \\ \rho_k & \text{otherwise.} \end{cases} \tag{9.14}$$

2) Return to Step 2, with k replaced by $k+1$.

We show that Algorithm 3.1 is well-defined. To see this, we need to show that the Armijo-like line search procedure is well defined.

Lemma 9.4. *In the kth iteration, if $\|r(u^k, 1)\| \geq \epsilon$, then the Armijo-like line search procedure with criteria (9.8) and (9.9) is finite.*

Proof. Assume for contradiction that ρ_k does not satisfy criterion (9.8) or (9.9) in finite Armijo-like line search procedure. Consequently, $\rho_k \to 0(see(9.10))$. Without losing generality, we can assume $\rho_k < 1$. Let us consider two possible cases.

Case 1. Criterion (9.8) fails to be satisfied. It follows that

$$\mu^2\|\tilde{u}^k - \bar{u}^k\|^2 < \|\rho_k[\langle \tilde{u}^k - \bar{u}^k, T(u^k) - T(\tilde{u}^k)\rangle - \langle u^k - \bar{u}^k, T(\tilde{u}^k) - T(\bar{u}^k)\rangle]\|.$$

This implies that either

$$\frac{1}{2}\mu^2\|\tilde{u}^k - \bar{u}^k\|^2 < \|\rho_k\langle\tilde{u}^k - \bar{u}^k, T(u^k) - T(\tilde{u}^k)\rangle\| \tag{9.15}$$

or

$$\frac{1}{2}\mu^2\|\tilde{u}^k - \bar{u}^k\|^2 < \|\rho_k\langle u^k - \bar{u}^k, T(\tilde{u}^k) - T(\bar{u}^k)\rangle\| \tag{9.16}$$

holds.

If (9.15) holds, by using the Cauchy-Schwarz inequality and dividing both sides of (9.15) by ρ_k, we have that

$$\frac{\mu^2\|\tilde{u}^k - \bar{u}^k\|}{2\rho_k} < \|T(u^k) - T(\tilde{u}^k)\|. \tag{9.17}$$

Note that

$$\|\tilde{u}^k - \bar{u}^k\| = \|\tilde{u}^k - J_\phi[\tilde{u}^k - \rho_k T(\tilde{u}^k)]\| = \|r(\tilde{u}^k, \rho_k)\|, \tag{9.18}$$

substituting above equality into (9.17) and using inequality (9.3), we find that

$$\frac{1}{2}\mu^2\|r(\tilde{u}^k, 1)\| \leq \frac{\mu^2\|r(\tilde{u}^k, \rho_k)\|}{2\rho_k} < \|T(\tilde{u}^k) - T(\bar{u}^k)\|. \tag{9.19}$$

It is easy to see that $\tilde{u}^k \to u^k, \bar{u}^k \to u^k$ (since $\rho_k \to 0$). Consequently, $T(\tilde{u}^k) \to T(u^k), T(\bar{u}^k) \to T(u^k)$ and $r(\tilde{u}^k, 1) \to r(u^k, 1)$ due to continuity of $T(u)$ and $r(u, 1)$, respectively. When we take $\rho_k \to 0$ in (9.19), we get $\|r(u^k, 1)\| \leq 0$. But this contradicts the assertion that $\epsilon \leq \|r(u^k, 1)\|$.

Let us turn to deal with (9.16). Since u^k is bounded, then we have $\|T(u^k)\| \leq M$. Note that

$$\|u^k - \tilde{u}^k\| = \|u^k - J_\phi[u^k - \rho_k T(u^k)]\| \leq \|\rho_k T(u^k)\| \leq \rho_k M,$$

then we have

$$\|u^k - \bar{u}^k\| \leq \|u^k - \tilde{u}^k\| + \|\tilde{u}^k - \bar{u}^k\| \leq \|\rho_k T(u^k)\| + \|\tilde{u}^k - \bar{u}^k\| \leq \rho_k M + \|\tilde{u}^k - \bar{u}^k\|. \tag{9.20}$$

In (9.16), using Cauchy-Schwarz inequality and (9.20), we get immediately,

$$\frac{1}{2}\mu^2\|\tilde{u}^k - \bar{u}^k\|^2 < \rho_k\|u^k - \bar{u}^k\|\|T(\tilde{u}^k) - T(\bar{u}^k)\| \leq \rho_k(\rho_k M + \|\tilde{u}^k - \bar{u}^k\|)\|T(\tilde{u}^k) - T(\bar{u}^k)\|. \tag{9.21}$$

Dividing both sides of (9.21) by ρ_k^2, using the equality (9.18) and inequality (9.3) again, we obtain

$$\frac{1}{2}\mu^2\|r(\tilde{u}^k,1)\|^2 \le \frac{\mu^2\|r(\tilde{u}^k,\rho_k)\|^2}{2\rho_k^2} < (M + \frac{\|r(\tilde{u}^k,\rho_k)\|}{\rho_k})\|T(\tilde{u}^k) - T(\bar{u}^k)\|. \quad (9.22)$$

By taking $\rho_k \to 0$ in above inequality, we obtain $\|r(u^k,1)\| \le 0$. Therefore, $\|r(u^k,1)\| = 0$, contradicting that u^k is not a solution.

Case 2. Condition (9.9) is violated. Then we must have

$$\nu\|\tilde{u}^k - \bar{u}^k\| < \|\rho_k(T(\tilde{u}^k) - T(\bar{u}^k))\|. \quad (9.23)$$

The proof is quite similar to the Case 1. Dividing both sides of (9.23) by ρ_k and taking $\rho_k \to 0$, we get the contradiction.

From the above observations, we assert that our proposed algorithm is well-defined. □

Lemma 9.5. *Let u^* be a solution of problem (2.1). For given $u^k \in H$, let $\tilde{u}^k, \bar{u}^k$ be the predictors produced by (9.7a) and (9.7b), then we have*

$$\langle u^k - \bar{u}^k, d(\tilde{u}^k,\bar{u}^k)\rangle \ge (2-\mu^2)\|\tilde{u}^k - \bar{u}^k\|^2. \quad (9.24)$$

Proof. Note that $\tilde{u}^k = J_\phi[u^k - \rho_k T(u^k)]$, $\bar{u}^k = J_\phi[\tilde{u}^k - \rho_k T(\tilde{u}^k)]$, we can apply (9.4) with $v = u^k - \rho_k T(u^k)$, $w = \tilde{u}^k - \rho_k T(\tilde{u}^k)$ to obtain

$$\langle u^k - \rho_k T(u^k) - (\tilde{u}^k - \rho_k T(\tilde{u}^k)), \tilde{u}^k - \bar{u}^k\rangle \ge \|\tilde{u}^k - \bar{u}^k\|^2.$$

By some manipulations, we have

$$\langle u^k - \tilde{u}^k, \tilde{u}^k - \bar{u}^k\rangle \ge \|\tilde{u}^k - \bar{u}^k\|^2 + \rho_k\langle \tilde{u}^k - \bar{u}^k, T(u^k) - T(\tilde{u}^k)\rangle.$$

Then, we obtain

$$\begin{aligned}\langle u^k - \tilde{u}^k, d(\tilde{u}^k,\bar{u}^k)\rangle &= \langle u^k - \tilde{u}^k, \tilde{u}^k - \bar{u}^k\rangle - \rho_k\langle u^k - \tilde{u}^k, T(\tilde{u}^k) - T(\bar{u}^k)\rangle \\ &\ge \|\tilde{u}^k - \bar{u}^k\|^2 + \rho_k\langle \tilde{u}^k - \bar{u}^k, T(u^k) - T(\tilde{u}^k)\rangle \\ &\quad -\rho_k\langle u^k - \tilde{u}^k, T(\tilde{u}^k) - T(\bar{u}^k)\rangle. \quad (9.25)\end{aligned}$$

Using (9.25), (9.8) and the definition of $d(\tilde{u}^k, \bar{u}^k)$, we get

$$\begin{aligned}\langle u^k - \bar{u}^k, d(\tilde{u}^k, \bar{u}^k)\rangle &= \langle u^k - \tilde{u}^k, d(\tilde{u}^k, \bar{u}^k)\rangle + \langle \tilde{u}^k - \bar{u}^k, d(\tilde{u}^k, \bar{u}^k)\rangle \\ &\geq \|\tilde{u}^k - \bar{u}^k\|^2 + \rho_k\langle \tilde{u}^k - \bar{u}^k, T(u^k) - T(\tilde{u}^k)\rangle \\ &\quad -\rho_k\langle u^k - \tilde{u}^k, T(\tilde{u}^k) - T(\bar{u}^k)\rangle \\ &\quad +\|\tilde{u}^k - \bar{u}^k\|^2 - \rho_k\langle \tilde{u}^k - \bar{u}^k, T(\tilde{u}^k) - T(\bar{u}^k)\rangle \\ &\geq (2-\mu^2)\|\tilde{u}^k - \bar{u}^k\|^2.\end{aligned}$$

Hence, (9.24) holds and the proof is completed. □

We now focus on investigating the convergence of the proposed method. The following theorem plays a crucial role in the convergence of the proposed

Theorem 9.1. *Let u^* be a solution of problem (2.1) and let $u^{k+1} = u^{k+1}(\gamma\alpha)$ be the sequence obtained from Algorithm 9.1. Then u^k is bounded and*

$$\|u^{k+1} - u^*\|^2 \leq \|u^k - u^*\|^2 - \frac{\gamma(2-\gamma)(2-\mu^2)^2}{(1+\nu)^2}\|\tilde{u}^k - \bar{u}^k\|^2. \tag{9.26}$$

Proof. For any $u^* \in \Omega^*$ solution of problem (2.1), we have

$$\langle \rho_k T(u^*), \bar{u}^k - u^*\rangle + \rho_k\phi(\bar{u}^k) - \rho_k\phi(u^*) \geq 0.$$

Using the monotonicity of T, we obtain

$$\langle \rho_k T(\bar{u}^k), \bar{u}^k - u^*\rangle + \rho_k\phi(\bar{u}^k) - \rho_k\phi(u^*) \geq 0. \tag{9.27}$$

Substituting $w = \tilde{u}^k - \rho_k T(\tilde{u}^k)$ and $v = u^*$ into (9.5), we get

$$\langle \tilde{u}^k - \rho_k T(\tilde{u}^k) - \bar{u}^k, \bar{u}^k - u^*\rangle + \rho_k\varphi(u^*) - \rho_k\phi(\bar{u}^k) \geq 0. \tag{9.28}$$

Adding (9.27) and (9.28), and using the definition of $d(\tilde{u}^k, \bar{u}^k)$, we have

$$\langle d(\tilde{u}^k, \bar{u}^k), \bar{u}^k - u^*\rangle \geq 0. \tag{9.29}$$

Since $u^* \in H$ be a solution of problem (2.1), then

$$\begin{aligned}\|u^{k+1}-u^*\|^2 &\leq \|u^k-u^*-\gamma\alpha_k d(\tilde{u}^k,\bar{u}^k)\|^2\\ &= \|u^k-u^*\|^2-2\gamma\alpha_k\langle u^k-u^*, d(\tilde{u}^k,\bar{u}^k)\rangle\\ &\quad +\gamma^2\alpha_k^2\|d(\tilde{u}^k,\bar{u}^k)\|^2 \end{aligned} \tag{9.30}$$

Adding (9.29) (multiplied by $2\gamma\alpha_k$) to (9.30) and using (9.12)

$$\begin{aligned}\|u^{k+1}-u^*\|^2 &\leq \|u^k-u^*\|^2-2\gamma\alpha_k\langle u^k-\bar{u}^k, d(\tilde{u}^k,\bar{u}^k)\rangle+\gamma^2\alpha_k^2\|d(\tilde{u}^k,\bar{u}^k)\|^2\\ &= \|u^k-u^*\|^2-\gamma(2-\gamma)\alpha_k\langle u^k-\bar{u}^k, d(\tilde{u}^k,\bar{u}^k)\rangle\\ &\leq \|u^k-u^*\|^2-\gamma(2-\gamma)\alpha_k(2-\mu^2)\|\tilde{u}^k-\bar{u}^k\|^2 \end{aligned} \tag{9.31}$$

where the last inequality follows from (9.24)

Recalling the definition of $d(\tilde{u}^k,\bar{u}^k)$ (see (9.13)) and applying Criterion (9.9), it is easy to see that

$$\|d(\tilde{u}^k,\bar{u}^k)\|^2 \leq (\|\tilde{u}^k-\bar{u}^k\|+\|\rho_k(T(\tilde{u}^k)-T(\bar{u}^k))\|)^2 \leq (1+\nu)^2\|\tilde{u}^k-\bar{u}^k\|^2. \tag{9.32}$$

Moreover, by using (9.24) together with (9.32), we get

$$\alpha_k = \frac{\langle u^k-\bar{u}^k, d(\tilde{u}^k,\bar{u}^k)\rangle}{\|d(\tilde{u}^k,\bar{u}^k)\|^2} \geq \frac{2-\mu^2}{(1+\nu)^2} > 0, \qquad \mu \in (0,\sqrt{2}). \tag{9.33}$$

Substituting (9.33) in (9.31), we get the assertion of this theorem. Since $\gamma \in [1,2)$ and $\mu \in (0,\sqrt{2})$ we have

$$\|u^{k+1}-u^*\| \leq \|u^k-u^*\| \leq \ldots \leq \|u^0-u^*\|.$$

Then the sequence u^k is bounded. □

We now present the convergence result of the proposed method.

Theorem 9.2. *If $\inf_{k=0}^{\infty}\rho_k := \rho > 0$, then any cluster point of the sequence $\{\tilde{u}^k\}$ generated by the proposed method is a solution of problem (2.1).*

Proof. It follows from (9.26) that

$$\lim_{k\to\infty} \|\tilde{u}^k - \bar{u}^k\| = 0.$$

Since the sequence u^k is bounded, $\{\tilde{u}^k\}$ is also bounded, it has at least a cluster point. Let u^∞ be a cluster point of $\{\tilde{u}^k\}$ and the subsequence $\{\tilde{u}^{k_j}\}$ converges to u^∞. Using the continuity of $r(u,\rho)$ and inequality (9.2), it follows that

$$\|r(u^\infty,\rho)\| = \lim_{k_j\to\infty} \|r(\tilde{u}^{k_j},\rho)\| \le \lim_{k_j\to\infty} \|r(\tilde{u}^{k_j},\rho_{k_j})\| = \lim_{k_j\to\infty} \|\tilde{u}^{k_j} - \bar{u}^{k_j}\| = 0.$$

This means that u^∞ is a solution of problem (2.1). □

Numerical Results

To illustrate the efficiency of the proposed method, we consider a spacial case of problem (2.1), by taking

$$\varphi(u) = \begin{cases} 0, & \text{if} \quad u \in R^n_+; \\ +\infty, & \text{otherwise} . \end{cases}$$

We consider the variational inequality problem of finding $u \in K$ such that

$$\langle Tu, v-u\rangle \ge 0, \quad v \in K. \tag{9.34}$$

where

$$T(u) = D(u) + Mu + q,$$

$D(u)$ and $Mu+q$ are the nonlinear part and linear part of $T(u)$ respectively. We form the linear part in the test problems as

$$M = \begin{pmatrix} 1 & 2 & 2 & \cdots & 2 \\ 0 & 1 & 2 & \cdots & 2 \\ 0 & 0 & 1 & \cdots & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}_{n\times n} \quad \text{and} \quad q = \begin{pmatrix} -1 \\ -1 \\ -1 \\ \vdots \\ -1 \end{pmatrix}_{n\times 1}.$$

In $D(u)$, the nonlinear part of $T(u)$, the components are chosen to be $D_j(u) = d_j * \arctan(u_j)$, where d_j is a random variable in $(0, 1)$. In all tests we take $\rho_0 = 1, \tau = 0.7, \eta_1 = 0.2, \eta_2 = 0.95, \mu = 0.95, \nu = 1.95, \gamma = 1.9$. We employ $\|r(u, 1)\| \leq 10^{-7}$ as the stopping criterion and choose $u^0 = 0$ as the initial iterative points. All codes were written in Matlab, we compare the proposed method with those in [17] and [18]. The test results for problems (9.34) with different dimensions are reported in Table 9, k is the number of iterations and l denotes the number of evaluations of mapping T.

Table 9 Numerical results for problem (9.34)

Dimension of the problem	Method [17]			Method [18]			New method		
	k	l	CPU(Sec.)	k	l	CPU(Sec.)	k	l	CPU(Sec.)
n=100	281	572	0.07	228	465	0.04	144	467	0.05
n=200	315	637	0.19	242	496	0.17	156	487	0.2
n=500	508	1090	4.77	531	1109	4.97	228	789	3.59
n=600	569	1243	7.59	520	1160	7.30	256	1023	6.18
n=800	657	1427	14.57	568	1236	13.11	297	1194	9.83
n=900	788	1621	21.22	635	1321	17.40	375	1178	15.58
n=1000	735	1567	21.57	574	1285	17.02	303	1000	13.14

Table 9 shows that the proposed method is very efficient algorithm even for large-scale classical nonlinear complementarity problems. Moreover, it demonstrates computationally that the new method is more effective than the methods presented in [17] and [18] in the sense that the new method needs fewer iteration and less evaluation numbers of T, which clearly illustrate its efficiency

10 Sensitivity Analysis

In recent years variational inequalities are being used as mathematical programming models to study a large number of equilibrium problems arising in finance, economics, transportation, operations research and engineering sciences. The behaviour of such equilibrium problems as a result of changes in the problem data is always of concern. In this section, we study the sensitivity analysis of the mixed variational inequalities (2.1), that is, examining how solutions of such problems change when the data of the problems are changed. We like to mention that sensitivity analysis is important for several reasons. First, estimating problem data often introduces measurement errors, sensitivity analysis helps in identifying sensitive parameters that should be obtained with relatively high accuracy. Second, sensitivity analysis may help to predict the future changes of the equilibrium as a result of changes in the governing system. Third, sensitivity analysis provides useful information for designing or planning various equilibrium systems. Furthermore, from mathematical and engineering point of view, sensitivity analysis can provide new insight regarding problems being studied can stimulate new ideas and techniques for problem solving. due to these and other reasons, there has been increasing interest in studying the sensitivity analysis of variational inequalities and related optimization problems. Sensitivity analysis for variational inequalities has been studied by many authors including Dafermos [26], Noor [55] and Noor et al. [83, 84] using quite different techniques. The techniques suggested so far vary with the problem being studied. Dafermos [26] used the equivalence between the variational inequalities and the fixed-point problem to study the sensitivity analysis of the classical variational inequalities. This technique has been modified and extended by many authors for studying the sensitivity analysis of various other classes of variational inequalities. This approach has strong geometrical flavor. It is well known that the mixed variational inequalities are equivalent to the resolvent equations, see Noor [53, 54]. This fixed-point equivalence is obtained by a suitable and appropriate rearrangement of the resolvent equations. The resolvent equation approach is quite general, flexible unified and provides us with a new technique to study the sensitivity analysis of variational inequalities without assuming the differentiability of the given data. Our analysis is in the spirit of Noor [55].

We now consider the parametric versions of the problem (2.1). To be more precise, let M be an open subset of H in which the parameter λ takes values. Let

$T(u, \lambda)$ be a given operator defined on $H \times M$ and takes values in H. From now onward, we denote $T_\lambda(.) := T(., \lambda)$ unless otherwise specified. The parametric mixed variational inequality problem is to find $(u, \lambda) \in H \times M$ such that

$$\langle T_\lambda u, v - u\rangle + \phi(v) - \phi(u) \geq 0, \quad v \in H. \tag{10.1}$$

We also assume that the parametric mixed variational inequality (10.1) has a unique solution $\bar{u}$ for some $\bar{\lambda} \in M$.

Related to the parametric mixed variational inequality (10.1), we consider the parametric resolvent equations. We consider the problem of finding $(z, \lambda) \in H \times M$ such that

$$T_\lambda g^{-1} J_{\phi_\lambda} z + \rho^{-1} R_{\phi_\lambda} z = 0, \tag{10.2}$$

where $\rho > 0$ ia constant and $R_{\phi_\lambda} \equiv I - J_{\phi_\lambda}$, is defined on the set of (z, λ) with $\lambda \in M$ and takes values in H. The equations of the type (10.2) are called the parametric resolvent equations.

Using Lemma 4.1, one can easily establish the equivalence between problems (10.1) and (10.2).

Lemma 10.1. *The parametric mixed variational inequality (10.1) has solution $(u, \lambda \in H \times M$, if and only if, the parametric resolvent equation (10.2) has a solution (z, λ), if*

$$u = J_{\phi_\lambda} z, \tag{10.3}$$

$$z = u - \rho T_\lambda(u). \tag{10.4}$$

From Lemma 10.2, we see that the problems (10.1) and (10.2) are equivalent. We use this equivalence to study the sensitivity analysis of the general variational inclusion (2.1). We assume that for some $\bar{\lambda} \in M$, problem (10.2) has a unique solution $\bar{z}$ and X is a closure of a ball in H centered at $\bar{z}$. We want to investigate those conditions under which for each λ in a neighborhood of $\bar{z}$, problem (10.2) has a unique solution $z(\lambda)$ near $\bar{z}$ and the function $z(\lambda)$ is Lipschitz continuous and differentiable.

First of all, we recall the following well known concepts.

Definition 10.1. *Let $T_\lambda(.)$ be an operator on $X \times M$. Then, $\lambda \in M, \quad \forall u, v \in X$, the operator T_λ is said to be:*
(a) locally strongly monotone, if there exists a constant $\alpha > 0$ such that

$$\langle T_\lambda(u) - T_\lambda(v), u - v\rangle \geq \alpha\|u - v\|^2.$$

(b) locally Lipschitz continuous, if there exists a constant $\beta > 0$ such that

$$\|T_\lambda(u) - T_\lambda(v)\| \leq \beta\|u - v\|.$$

From (a) and (b), it follows that $\alpha \leq \beta$.

We now consider the case, when the solutions of the parametric resolvent equations (10.2) lie in the interior of X. Following the ideas and technique of Dafermos [26] and Noor [55], we consider the map

$$\begin{aligned} F_\lambda(z) &= J_{\phi_\lambda} z - \rho T_\lambda(u), \quad \forall (z, \lambda) \in X \times M, \\ &= u - \rho T_\lambda(u), \end{aligned} \tag{10.5}$$

where

$$u = J_{\phi_\lambda} z. \tag{10.6}$$

We have to show that the map $F_\lambda(z)$ defined by (10.5) has a fixed point, which is solution of the resolvent equation (10.2). We have to show that the map $F_\lambda(z)$ defined by (10.5) is a contraction map with respect to z uniformly in $\lambda \in M$.

Lemma 10.2. *Let $T_\lambda(.)$ be a locally strongly monotone with constant $\alpha > 0$ and locally Lipschitz continuous with constant $\beta > 0$. Then*

$$\|F_\lambda(z_1) - F_\lambda(z_2)\| \leq \theta\|z_1 - z_2\|,$$

$$\theta = \sqrt{1 - 2\alpha\rho + \beta^2\rho^2}, \tag{10.7}$$

for

$$0 \leq \rho \leq \frac{2\alpha}{\beta^2}. \tag{10.8}$$

Proof. $\forall z_1, z_2 \in H$, and $\lambda \in M$, we have

$$\|F_\lambda(z_1) - F_\lambda(z_2)\| \leq +\|u_1 - u_2 - \rho(T_\lambda(u_1) - T_\lambda(u_2))\|. \tag{10.9}$$

Using the strongly monotonicity and Lipschitz continuity of the operator T_λ, we have

$$\begin{aligned} \|u_1 - u_2 - (T_\lambda(u_1) - T_\lambda(u_2))\|^2 &\leq \|u_1 - u_2\|^2 - 2\langle(u_1 - u_2, \rho(T_\lambda(u_1) - T_\lambda(u_2))\rangle \\ &\quad +\rho^2\|T_\lambda(u_1) - T_\lambda(u_2)\|^2 \\ &\leq (1 - 2\rho\alpha + \rho^2\beta^2)\|u_1 - u_2\|^2. \end{aligned} \tag{10.10}$$

$$\|u_1 - u_2\| \leq \|J_{\phi_\lambda} z_1 - J_{\phi_\lambda} z_2\| \leq \|z_1 - z_2\|. \tag{10.11}$$

From (10.6), (10.9) and (10.11), we obtain

$$\begin{aligned} \|F_\lambda(z_1) - F_\lambda(z_2)\| &\leq \{\sqrt{1 - 2\rho\alpha + \rho^2\beta^2}\}\|z_1 - z_2\| \\ &= \theta\|z_1 - z_2\|, \quad \text{using (10.7)}. \end{aligned}$$

From (10.8), it follows that $\theta < 1$ and consequently, the map $F_\lambda(z)$ defined by (10.5) is a contraction map and has a fixed point $z(\lambda)$, which is a solution of the resolvent equation (10.2). □

Remark 10.1. *From Lemma 10.2, we see that the map $F_\lambda(z)$ defined by (10.5) has a unique fixed point $z(\lambda)$, that is, $z(\lambda) = F_\lambda(z)$. Also, by assumption,, the function $\bar{z}$, for $\lambda = \bar{\lambda}$ is a solution of the parametric resolvent equation (10.2). Again using Lemma 10.2, we see that $\bar{z}$, for $\lambda = \bar{\lambda}$, is a fixed point of $F_\lambda(z)$ and it is also a fixed point of $F_{\bar{\lambda}}(z)$. Consequently, we conclude that*

$$z(\bar{\lambda}) = \bar{z} = F_{\bar{\lambda}}(z(\bar{\lambda})).$$

Using Lemma 10.2 and technique of Noor [55], we can prove the continuity of the solution $z(\lambda)$ of the parametric resolvent equation (10.2). We include its proof to convey an idea of the technique.

Lemma 10.3. *Assume that the operator $T_\lambda(.)$ is locally Lipschitz continuous with respect to the parameter λ. If the operators $T_\lambda(.)$ is locally Lipschitz continuous and the map $\lambda \longrightarrow J_{\phi_\lambda}$ is continuous (or Lipschitz continuous), then the function $z(\lambda)$ satisfying (10.2) is (Lipschitz) continuous at $\lambda = \bar{\lambda}$.*

Proof. $\forall \lambda \in M$, invoking Lemma 10.2. and the triangle inequality, we have

$$\begin{aligned} \|z(\lambda) - z(\bar{\lambda})\| &\leq \|F_\lambda(z(\lambda)) - F_{\bar{\lambda}}(z(\bar{\lambda}))\| + \|F_\lambda(z(\bar{\lambda}) - F_{\bar{\lambda}}(z(\bar{\lambda}))\| \\ &\leq \theta\|z(\lambda) - z(\bar{z})\| + \|F_\lambda(z(\bar{\lambda})) - F_{\bar{\lambda}}(z(\bar{\lambda}))\|. \end{aligned} \tag{10.12}$$

From (10.12) and the fact that the operator $T_\lambda(.)$ is a locally Lipschitz continuous with respect to the parameter λ, we have

$$\begin{aligned} \|F_\lambda(z(\bar{\lambda})) - F_{\bar{\lambda}}(z(\bar{\lambda}))\| &= \|u(\bar{\lambda}) - u(\bar{\lambda}) + \rho(T_\lambda(u(\bar{\lambda})) - T_{\bar{\lambda}}(u(\bar{\lambda})))\| \\ &\leq \rho\mu\|\lambda - \bar{\lambda}\|. \end{aligned} \tag{10.13}$$

Combining (10.12) and (10.13), we obtain

$$\|z(\lambda) - z(\bar{\lambda})\| \leq \frac{\rho\mu}{1-\theta}\|\lambda - \bar{\lambda}\|, \quad \forall \lambda, \bar{\lambda} \in M,$$

from which the required result follows. □

We now state and prove the main result of this section and is the motivation of our next result.

Theorem 10.1. *Let $\bar{u}$ be the solution of the parametric general variational inclusion (10.1) and $\bar{z}$ be the solution of the parametric resolvent equation (10.2) for $\lambda = \bar{\lambda}$. Let $T_\lambda(.)$ be the locally strongly monotone Lipschitz continuous operator. Let the operator g be also strongly monotone Lipschitz continuous operator. If the map $\lambda \longrightarrow J_{\phi_\lambda}$ is (Lipschitz) continuous at $\lambda = \bar{\lambda}$, then there exists a neighborhood*

$N \subset H$ of $\bar{\lambda}$ such that for $\lambda \in N$, the parametric resolvent equation (10.2) has a unique solution $z(\lambda)$ in the interior of X, $z(\bar{\lambda}) = \bar{z}$ and $z(\lambda)$ is (Lipschitz) continuous at $\lambda = \bar{\lambda}$.

Proof. Its proof follows from Lemma 10.2, Lemma 10.3 and Remark 10.1. □

Remark 10.2. *It is known that the mixed variational inequality (2.1) is equivalent to finding $u \in H$ such that*

$$0 \in Tu + \partial\phi(u). \tag{10.14}$$

This is known as finding zero of the the variational inclusion.

For given two monotone operators T, A , consider finding zero of sum of two monotone operator such that

$$0 \in Tu + A(u),$$

which is called the general variational inclusion and includes (10.14) as a special case. One can easily study the sensitivity analysis of the general variational inclusions using the above technique.

11 Mixed Bivariational Inequalities

In this section, we consider and study the mixed bivariational inequalities. For the sake of completeness and to convey the main ideas, we recall the some concepts and basic results, which are mainly due to Noor et al. [73, 76, 85, 88, 89, 92].

Definition 11.1. *A set $K_\beta \subseteq H$ is said to be a biconvex set with respect to an arbitrary bifunction $\beta(\cdot - \cdot)$, if*

$$u + \lambda\beta(v - u) \in K_\beta, \qquad \forall u, v \in K_\beta, \lambda \in [0, 1].$$

The biconvex set K_β is also called β-connected set. We would like to mention that, if $u+\beta(v-u) = v, \forall u, v \in K_\beta$, then $\beta(v-u) = v-u, \forall u, v \in K$. Consequently, the biconvex set K_β reduces to the convex set K. Thus, $K_\beta \subseteq K$. Thus, it shows that every convex set is a biconvex set, but the converse is not true.

For example, the set $K_\beta = R - (-\frac{1}{2}, \frac{1}{2})$ is an biconvex set with respect to η, where

$$\beta(v-u) = \begin{cases} v-u, & \text{for} \quad v>0, u>0 \quad \text{or} \quad v<0, u<0 \\ u-v, & \text{for} \quad v<0, u>0 \quad \text{or} \quad v<0, u<0. \end{cases}$$

It is clear that K_β is not a convex set.

From now onward K_β is a nonempty closed biconvex set in H with respect to the bifunction $\beta(\cdot - \cdot)$, unless otherwise specified.

We now introduce some new concepts of biconvex functions and their variants forms.

Definition 11.2. *A function F on the biconvex set K_β is said to be a strongly biconvex with respect to the bifunction $\beta(\cdot - \cdot)$, if there exists a constant $\mu > 0$ such that*

$$F(u + \lambda\beta(v-u)) \leq (1-\lambda)F(u) + \lambda F(v) - \mu\lambda(1-\lambda)\|\beta(v-u)\|^2, \forall u, v \in K_\beta, \lambda \in [0,1].$$

A function F is said to be strongly biconcave, if and only if, $-F$ is strongly biconvex function. Consequently, we have a new concept.

Definition 11.3. *A function F is said to be strongly affine biconvex involving an arbitrary bifunction $\beta(\cdot - \cdot)$, if*

$$F(u + \lambda\beta(v-u)) = (1-\lambda)F(u) + \lambda F(v) - \mu\lambda(1-\lambda)\|\beta(v-u)\|^2, \forall u, v \in K_\beta, \lambda \in [0,1].$$

Note that every strongly biconvex function is a strongly affine biconvex, but the converse is not true.

If $\beta(v-u) = v-u$, then the strongly biconvex function becomes a strongly convex function, that is,

$$F(u + \lambda(v-u)) \leq (1-\lambda)F(u) + \lambda F(v) - \mu\lambda(1-\lambda)\|\beta(v-u)\|^2, \quad \forall u, v \in K, \lambda \in [0,1].$$

Definition 11.4. *The function F on the biconvex set K_β is said to be strongly quasi biconvex with respect to the bifunction $\beta(\cdot - \cdot)$, if*

$$F(u+\lambda\beta(v-u)) \leq \max\{F(u), F(v)\} - \mu\lambda(1-\lambda)\|\beta(v-u)\|^2, \quad \forall u, v \in K_\beta, \lambda \in [0,1].$$

Definition 11.5. *The function F on the biconvex set K_β is said to be strongly* log*-biconvex with respect to the bifunction $\beta(\cdot - \cdot)$, if*

$$F(u+\lambda\beta(v-u)) \leq (F(u))^{1-\lambda}(F(v))^{\lambda} - \mu\lambda(1-\lambda)\|\beta(v-u)\|^2, \quad \forall u, v \in K_\beta, \lambda \in [0,1].$$

where $F(\cdot) > 0$.

We can rewrite the Definition 11.5 in the following form

Definition 11.6. *The function F on the biconvex set K_β is said to be strongly* log*-biconvex with respect to the bifunction $\beta(\cdot - \cdot)$, if*

$$\begin{aligned}\log F(u+\lambda\beta(v-u)) &\leq (1-\lambda)\log F(u) + \lambda \log F(v) \\ &\quad -\mu\lambda(1-\lambda)\|\beta(v-u)\|^2, \quad \forall u, v \in K_\beta, \lambda \in [0,1].\end{aligned}$$

where $F(\cdot) > 0$.

This definition can be used to discus the properties of the differentiable strongly log-biconvex functions.

From the above definitions, we have

$$\begin{aligned}F(u+\lambda\beta(v-u)) &\leq (F(u))^{1-\lambda}(F(v))^{\lambda} - \mu\lambda(1-\lambda)\|\beta(v-u)\|^2 \\ &\leq (1-\lambda)F(u) + \lambda F(v) - \mu\lambda(1-\lambda)\|\beta(v-u)\|^2 \\ &\leq \max\{F(u), F(v)\} - \mu\lambda(1-\lambda)\|\beta(v-u)\|^2.\end{aligned}$$

This shows that every strongly log-biconvex function is strongly biconvex function and every strongly biconvex function is a strongly quasi-biconvex function. However, the converse is not true.

For $\lambda = 1$, Definition 11.2 and 11.5 reduce to the following condition.

Assumption 11.1.

$$F(u+\beta(v-u)) \leq F(v), \quad \forall v,u \in K_\beta,$$

which is called the Condition A.

To derive the main results, we need the following assumptions regarding the bifunction $\beta(\cdot - \cdot)$.

Assumption 11.2. *The bifunction $\beta(,-,)$ said to satisfy the assumptions, if*

$$\begin{aligned}(i).\quad \beta(\gamma\beta(v-u)) &= \gamma\beta(v-u), \forall u,v \in K_\beta, \quad \gamma \in R^n.\\ (ii).\quad \beta(v-u-\gamma\beta(v-u)) &= (1-\gamma)\beta(v-u), \forall u,v \in K_\beta,\end{aligned}$$

which is called the Condition M.

Remark 11.1. *Let $\beta(\cdot - \cdot): K_\beta \times K_\beta \to H$ satisfy the assumption*

$$\beta(v-u) = \beta(v-z) + \beta(z-u), \forall u,v,z \in K_\beta.$$

One can easily show that $\beta(v-u) = 0 \quad \forall u,v \in K_\beta$. Consequently

$$\beta(v-u) = 0 \quad \Leftrightarrow \quad v = u, \forall u,v \in K_\beta.$$

Also

$$\beta(v-u) + \beta(u-v) = 0, \quad \forall u,v \in K_\beta.$$

This implies that the bifunction $\beta(.-.)$ is skew symmetric.

Theorem 11.1. *Let K_β be a biconvex function in H and the condition M hold. If the function F is a differentiable strongly biconvex function with constant $\mu > 0$, then the following are equivalent.*

(i). *The function F is a strongly biconvex function.*

(ii). $F(v) - F(u) \geq \langle F'(u), \beta(v-u)\rangle + \mu\|\beta(v-u)\|^2, \quad \forall u,v \in K_\beta.$

(iii). $\langle F'(u), \beta(v-u)\rangle + \langle F'(v), \beta(u-v)\rangle \leq -\mu\{\|\beta(v-u)\|^2 + \|\beta(u-v)\|^2\|\}, \quad \forall u,v \in K_\beta.$

We now discuss the optimality conditions for the differentiable biconvex functions.

Consider the energy functional $I[v]$ defined as:

$$I[v] = F(v) + \phi(v), \quad \forall v \in H, \tag{11.1}$$

where F and ϕ are two suitable biconvex functions.

Theorem 11.2. *Let F be a differentiable biconvex function and ϕ be a nondifferentiable biconvex function. If $u \in H$ is the minimum of the energy functional $I[v]$, if and only if, $u \in H$ satisfies the*

$$\langle F'(u), \beta(v-u)\rangle + \phi(v) - \phi(u) \geq 0, \quad \forall v \in H. \tag{11.2}$$

Proof. Let $u \in H$ be a minimum of the functional $I[v]$. Then

$$I(u) \leq I(v), \forall v \in H. \tag{11.3}$$

Note the the whole apace H is a biconvex set. This implies that, $\forall u, v \in H, \quad \lambda \in [0,1]$,

$$v_\lambda = u + \lambda\beta(v-u) \in H.$$

Taking $v = v_\lambda$ in (11.3), we have

$$\begin{aligned} F(u) + \phi(u) &\leq F(u + \lambda\beta(v-u)) + \phi(u + \lambda\beta(v-u)) \\ &\leq F(u + \lambda\beta(v-u)) + \phi(u) + \lambda(\phi(v) - \phi(u)), \end{aligned}$$

from which, we have

$$\begin{aligned} 0 &\leq \lim_{\lambda\to 0}\{\frac{F(u+\lambda\beta(v-u)) - F(u)}{\lambda}\} + \phi(v) - \phi(u) \\ &\leq \langle F'(u), \beta(v-u)\rangle + \phi(v) - \phi(u), \end{aligned} \tag{11.4}$$

which is the inequality (11.2).

Conversely, let $u \in H$ satisfy (11.2). We have to show that $u \in H$ is the minimum of the functional $I[v]$ defined by (11.1).

Since F is differentiable biconvex function, so

$$F(u+\lambda\beta(v-u)) \leq F(u)+\lambda(F(v)-F(u)), \forall u, v \in H,$$

which implies that

$$F(v)-F(u) \geq \lim_{\lambda \to 0}\{\frac{F(u+\lambda\beta(v-u))-F(u)}{\lambda}\} \tag{11.5}$$

From (11.1), (11.2) and (11.5), we obtain

$$\begin{aligned} I[u]-I[v] &= -\{F(v)-F(u)+\phi(v)-\phi(u)\} \\ &\leq -\{\langle F'(u), \beta(v-u)\rangle+\phi(v)-\phi(u)\} \\ &\leq 0. \end{aligned}$$

This implies that

$$I[u] \leq I[v], \quad \forall v \in H,$$

This shows that $u \in H$ is the minimum of the functional $I[v]$ defined by (11.2).

□

The inequality of the type (11.2) is called the mixed bivariational inequality and appears to new one.

It is worth mentioning that inequalities of the type (11.2) may not arise as a minimization of the biconvex functions. This motivated us to consider a more general mixed bivariational inequality of which (11.2) is a special case.

For a given operator T, bifunction $\beta(.-.)$ and continuous function ϕ, consider the problem of finding $u \in H$, such that

$$\langle Tu, \beta(v-u)\rangle+\phi(v)-\phi(u) \geq 0, \forall v \in H, \tag{11.6}$$

which is called mixed bivariational inequality.

It is worth mentioning that for suitable and appropriate choice of the operators, biconvex sets and spaces, one can obtain a wide class of variational inequalities and optimization problems. This shows that the mixed bivariational inequalities are quite flexible and unified ones.

Iterative methods and convergence analysis

Due to the inherent nonlinearity, the projection method and its variant form can not be used to suggest the iterative methods for solving these bivariational inequalities. To overcome these drawback, one may use the auxiliary principle technique of to suggest and analyze some iterative methods for solving the mixed bivariational inequalities (11.6). This technique does not involve the concept of the projection and resolvent operators , which is the main advantage of this technique. We again use the auxiliary principle technique coupled with Bergman functions. These applications are based on the type of convex functions associated with the Bregman distance. We now suggest and analyze some iterative methods for mixed bivariational inequalities (11.6) using the auxiliary principle technique coupled with Bregman distance functions.

For a given $u \in H$ satisfying the mixed bivariational inequality (11.6), we consider the auxiliary problem of finding a $w \in H$ such that

$$\langle \rho Tw, \beta(v-w) + \langle E'(w) - E'(u), \beta(v-w)\rangle + \rho(\phi(v) - \phi(u)) \geq 0, \quad \forall v \in H, \quad (11.7)$$

where $\rho > 0$ is a constant and $E'(u)$ is the differential of a strongly biconvex function $E(u)$ at $u \in K_\beta$. Since $E(u)$ is a strongly biconvex function, this implies that its differential E' is strongly β-monotone. Consequently it follows that the problem (11.6) has an unique solution.

Remark 3.1: The function $B(w,u) = E(w) - E(u) - \langle E'(u), \beta(w,u)\rangle$ associated with the biconvex function $E(u)$ is called the generalized Bregman function. By the strongly boiconvexity of the function $E(u)$, the Bregman function $B(.,.)$ is nonnegative and $B(w,u) = 0$, if and only if $u = w, \forall u, w \in H$.

We note that, if $w = u$, then clearly w is solution of the mixed bivariational inequality (11.7). This observation enables us to suggest and analyze the following iterative method for solving (11.7).

Algorithm 11.1. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\begin{aligned}\langle \rho T u_{n+1}, \beta(v - u_{n+1})\rangle \quad + \quad & \langle E'(u_{n+1}) - E'(u_n), \beta(v - u_{n+1})\rangle \\ & +\rho(\phi(v) - \phi(u_{n+1})) \geq 0, \quad \forall v \in H, \quad (11.8)\end{aligned}$$

where $\rho > 0$ is a constant.

Algorithm 11.1 is called the proximal method for solving mixed bivariational inequalities (11.6). In passing we remark that the proximal point method was suggested in the context of convex programming problems as a regularization technique, see Martinet [42].

If $\beta(v - u) = v - u$, then Algorithm 11.1 collapses to:

Algorithm 11.2. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\begin{aligned}\langle \rho T(u_{n+1}), v - u_{n+1}\rangle \quad + \quad & \langle E'(u_{n+1}) - E'(u), v - u_{n+1}\rangle \\ & +\rho(\phi(v) - \phi(u)) \geq 0, \quad \forall v \in H,\end{aligned}$$

for solving the mixed variational inequality.

For suitable and appropriate choice of the operators and the spaces, one can obtain a number of known and new algorithms for solving variational inequalities and related problems.

Theorem 11.3. *Let the operator T be pseudomonotone. Let E be differentiable higher order strongly biconvex function with module $\nu > 0$ and Condition M hold. If $\rho\mu \leq \nu$, then the approximate solution u_{n+1} obtained from Algorithm 11.1 converges to a solution $u \in H$ satisfying the mixed bivariational inequality (11.6).*

Proof. Let $u \in H$ be a solution of mixed bivariational inequality (11.6). Then

$$\langle Tu, \beta(v-u)\rangle + \phi(v) - \phi(u) \geq 0, \quad \forall v \in H,$$

implies that

$$-\langle Tv, \beta(u-v)) + \phi(v) - \phi(u)\rangle \geq 0, \quad \forall v \in H, \tag{11.9}$$

since T is β-pseudomonotone.

Taking $v = u$ in (11.8) and $v = u_{n+1}$ in (11.9), we have

$$\begin{aligned} \langle \rho T(u_{n+1}), \beta(u, u-n+1)\rangle \quad &+ \quad \langle E'(u_{n+1}) - E'_k(u_n, \beta(u-u_{n+1})\rangle \\ +\rho(\phi(u) - \phi(u_{n+1})) \geq 0. & \end{aligned} \tag{11.10}$$

and

$$-\langle Tu_{n+1}, \beta(u-u_{n+1})\rangle + \phi(u_{n+1}) - \phi(u) \geq 0. \tag{11.11}$$

We now consider the Bregman distance function

$$B(u,w) = E(u) - E(w) - \langle E'(w, \beta(u-w)\rangle \geq \nu\|\beta(v-u)\|^2, \tag{11.12}$$

using higher order strongly biconvexity of E.

Now combining (11.12), (11.10) and (11.11), we have

$$\begin{aligned} B(u,u_n) - B(u,u_{n+1}) &= E(u_{n+1}) - E(u_n) - \langle E'(u_n), \beta(u-u_n)\rangle \\ &\quad +\langle E'(u_{n+1}), \beta(u-u_{n+1})\rangle \\ &= E(u_{n+1}) - E(u_n) - \langle E'(u_n) - E'(u_{n+1}, \beta(u-u_{n+1})\rangle \\ &\quad -\langle E'(u_n, u_{n+1} - u_n\rangle \\ &\geq \nu\|\beta(u_{n+1}-u_n)\|^2 + \langle E'(u_{n+1}) - E'(u_n), \beta(u-u_{n+1})\rangle \\ &\geq \nu\|\beta(u_{n+1}-u_n)\|^2 - \rho\langle T(u_{n+1}), \beta(u-u_{n+1})\rangle \\ &\quad -\rho\mu\|\beta(u-u_{n+1})\|^2 \\ &\geq (\nu - \rho\mu)\|\beta(u_{n+1}-u_n)\|^2. \end{aligned}$$

If $u_{n+1} = u_n$, then clearly u_n is a solution of the problem(11.6). Otherwise, it follows that $B(u, u_n) - B(u, u_{n+1})$ is nonnegative and we must have

$$\lim_{n\to\infty} \|\beta(u_{n+1} - u_n)\| = 0.$$

from which, we have

$$\lim_{n\to\infty} \|u_{n+1} - u_n\| = 0.$$

It follows that the sequence $\{u_n\}$ is bounded. Let $\bar{u}$ be a cluster point of the subsequence $\{u_{n_i}\}$, and let $\{u_{n_i}\}$ be a subsequence converging toward $\bar{u}$. Now using the technique of Zhu and Marcotte [115], it can be shown that the entire sequence $\{u_n\}$ converges to the cluster point $\bar{u}$ satisfying the mixed bivariational inequality (11.6). □

It is well-known that to implement the proximal point methods, one has to find the approximate solution implicitly, which is itself a difficult problem. To overcome this drawback, we now consider another method for solving the mixed bivariational inequality(11.6) using the auxiliary principle technique.

For a given $u \in H$, find $w \in H$ such that

$$\langle \rho T(u, \beta(v - w)\rangle + \langle E'(w) - E', \beta(v - w)\rangle + \rho(\phi(v) - \phi(u)) \geq 0, \quad \forall v \in H, \quad (11.13)$$

where $E'(u)$ is the differential of a biconvex function $E(u)$ at $u \in H$. Problem (11.13) has a unique solution, since E is strongly biconvex function. Note that problems (11.13) and (11.7) are quite different problems.

It is clear that for $w = u$, w is a solution of (11.6). This fact allows us to suggest and analyze another iterative method for solving the mixed bivariational inequality (11.6).

Algorithm 11.3. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\begin{aligned}\langle \rho T u_n, \beta(v - u_{n+1})\rangle \quad + \quad & \langle E'(u_{n+1}) - E'(u_n), \beta(v - u_{n+1})\rangle \\ & + \rho(\phi(v) - \phi(u_{n+1})) \geq 0, \quad \forall v \in H, \quad (11.14)\end{aligned}$$

for solving the mixed bivariational inequality (11.6).

If $\beta(v,u)) = v - u$, Algorithm 11.3 collapses to:

Algorithm 11.4. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative schemes*

$$\rho\langle Tu_n, v - u_{n+1}\rangle + \langle E'(u_{n+1}) - E'(u_n), v - u_{n+1}\rangle + \rho(\phi(v) - \phi(u_{n+1})) \geq 0, \quad \forall v \in H,$$

for solving the mixed variational inequalities and appears to be a new one.

We now again use the auxiliary principle to suggest some more iterative methods for solving bivariational inequalities.

For a given $u \in H$ satisfying (11.6), find $w \in H$ such that

$$\langle \rho T(w, \beta(v - w)\rangle + \langle w - u + \alpha(u - u), v - w\rangle + \rho(\phi(v) - \phi(u)) \geq 0, \quad \forall v \in H, \tag{11.15}$$

which is the auxiliary mixed bivariational inequality. We note that, if $w = u$, w is a solution of (11.6). This fact allows us to suggest and analyze another iterative method for solving the mixed bivariational inequality (11.6).

Algorithm 11.5. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative schemes*

$$\rho\langle Tu_{n+1}, \beta(v - u_{n+1})\rangle + \langle u_{n+1} - u_n) + \alpha(u_n - u_{n-1}), v - u_{n+1}\rangle + \rho(\phi(v) - \phi(u_{n+1})) \geq 0, \quad \forall v \in H, \tag{11.16}$$

where α is a constant.

Algorithm 11.5 is called the inertial proximal method for solving the mixed bivariational inequalities (11.6).

For $\alpha = 0$, Algorithm 11.5 becomes:

Algorithm 11.6. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative schemes*

$$\begin{aligned}\rho\langle Tu_{n+1}, \beta(v - u_{n+1})\rangle \quad + \quad & \langle u_{n+1} - u_n), v - u_{n+1}\rangle \\ & +\rho(\phi(v) - \phi(u_{n+1})) \geq 0, \quad \forall v \in H,\end{aligned}$$

which is called the proximal method for solving the mixed bivariational inequalities (11.6).

If $\beta(. - .) = v - u$, then Algorithm 11.6 reduces to:

Algorithm 11.7. *For a given $u_0, u_1 \in H$, compute the approximate solution u_{n+1} by the iterative schemes*

$$\begin{aligned}\rho\langle Tu_{n+1}, v - u_{n+1}\rangle + \langle u_{n+1} - u_n) + \alpha(u_n - u_{n-1}), v - u_{n+1}\rangle \\ +\rho(\phi(v) - \phi(u_{n+1})) \geq 0, \quad \forall v \in H.\end{aligned}$$

Algorithm 11.7 is known as the inertial proximal method for solving mixed variational inequalities.

We now consider the convergence analysis of Algorithm 11.5.

Theorem 11.4. *Let $\bar{u} \in H$ be a solution of (11.6) and let u_{n+1} be the approximate solution obtained from Algorithm 11.5. If the $T : H \longrightarrow R$ is pseudo β-monotone, then*

$$\begin{aligned}\|u_{n+1} - \bar{u}\|^2 \quad \leq \quad & \|u_n - \bar{u}\|^2 - \|u_{n+1} - u_n - \alpha_n(u_n - u_{n-1})\|^2 \\ & +\alpha_n\{\|u_n - \bar{u}\|^2 - \|u_{n-1} - \bar{u}\|^2 + 2\|u_n - u_{n-1}\|^2\}. \quad (11.17)\end{aligned}$$

Proof. Let $\bar{u} \in H$ be a solution of (11.6). Then

$$\langle T\bar{u}, \beta(v - \bar{u})\rangle + \phi(v) - \phi(\bar{u}) \geq 0, \quad \forall v \in H,$$

implies that

$$-\langle Tv, \beta(\bar{u} - v)\rangle + \phi(v) - \phi(\bar{u}) \geq 0, \quad \forall v \in H, \quad (11.18)$$

since T is pseudo β-monotone.

Taking $v = u_{n+1}$ in (11.18), we have

$$\langle Tu_{n+1}, \beta(\bar{u} - u_{n+1})\rangle + \phi(u_{n+1}) - \phi(\bar{u}) \geq 0. \tag{11.19}$$

Now taking $v = \bar{u}$ in (11.16), we obtain

$$\begin{aligned}\langle \rho Tu_{n+1}, \beta(\bar{u} - u_{n+1})\rangle \quad + \quad & \langle u_{n+1} - u_n - \alpha_n(u_n - u_{n-1}), \bar{u} - u_{n+1}\rangle \\ & +\phi(\bar{u}) - \phi(u) \geq 0. \end{aligned} \tag{11.20}$$

From (11.19) and (11.20), we have

$$\langle u_{n+1} - u_n - \alpha_n(u_n - u_{n-1}), \bar{u} - u_{n+1}\rangle \geq -\langle \rho Tu_{n+1}, \beta(\bar{u} - u_{n+1})\rangle \geq 0, \tag{11.21}$$

One can write (11.21) in the form

$$\langle u_{n+1} - u_n, \bar{u} - u_{n+1}\rangle \geq \alpha_n\langle u_n - u_{n-1}, \bar{u} - u_n + u_n - u_{n+1}\rangle. \tag{11.22}$$

Using the inequality $2\langle u, v\rangle = \|u + v\|^2 - \|u\|^2 - \|v\|^2, \forall u, v \in H$ and rearranging the terms in (11.22), one can easily obtain (11.17), the required result. □

Theorem 11.5. *Let H be a finite dimensional space. Let u_{n+1} be the approximate solution obtained from Algorithm 11.5 and $\bar{u} \in H$ be a solution of (11.6). If there exists $\alpha \in (0, 1)$ such that $0 \leq \alpha_n \leq \alpha, \quad \forall n \in N$ and*

$$\sum_{n=1}^{\infty} \alpha_n \|u_n - u_{n-1}\|^2 \leq \infty,$$

then $\lim_{n \longrightarrow \infty} u_n = \bar{u}$.

Proof. Let $\bar{u} \in K_\beta$ be a solution of (11.6). First we consider the case $\alpha_n = 0$. In this case, we see from (11.17) that the sequence $\{\|\bar{u} - u_n\|\}$ is nonincreasing and consequently $\{u_n\}$ is bounded. Also from (11.17), we have

$$\sum_{n=0}^{\infty} \|u_{n+1} - u_n\|^2 \leq \|u_0 - \bar{u}\|^2,$$

which implies that

$$\lim_{n \longrightarrow \infty} \|u_{n+1} - u_n\| = 0. \tag{11.23}$$

Let $\hat{u}$ be the cluster point of $\{u_n\}$ and the subsequence $\{u_{n_j}\}$ of the sequence $\{u_n\}$ converge to $\hat{u} \in H$. Replacing u_n by u_{n_j} in (11.17) and taking the limit $n_j \longrightarrow \infty$ and using (11.23), we have

$$\langle T\hat{u}, \beta(v - \hat{u})\rangle + \phi(v) - \phi(\hat{u}) \geq 0, \quad \forall v \in H,$$

which implies that $\hat{u}$ solves the mixed bihemivariational inequality problem (11.6) and

$$\|u_{n+1} - u_n\|^2 \leq \|u_n - \bar{u}\|^2.$$

Thus it follows from the above inequality that the sequence $\{u_n\}$ has exactly one cluster point $\hat{u}$ and

$$\lim_{n \longrightarrow \infty} u_n = \hat{u}.$$

Now we consider the case $\alpha_n > 0$. From (11.17), we have

$$\begin{aligned}\sum_{n+1}^{\infty} \|u_{n+1} - u_n &- \alpha_n(u_n - u_{n-1})\|^2 \leq \|u_0 - \bar{u}\|^2 \\ &+ \sum_{n=1}^{\infty} \{\alpha\|u_n - \bar{u}\|^2 + 2\|u_n - u_{n-1}\|^2\} \leq \infty,\end{aligned}$$

which implies that

$$\lim_{n \longrightarrow \infty} \|u_{n+1} - u_n - \alpha_n(u_n - u_{n-1})\|^2 = 0.$$

Repeating the above arguments as in the case $\alpha_n = 0$, one can easily show that $\lim_{n \to \infty} u_n = \hat{u}$, the required result. □

For a given $u \in H$ satisfying the mixed bivariational inequality (11.6), consider the auxiliary problem of finding $w \in H$ such that

$$\langle \rho T u, \beta(v - w)\rangle + \langle w - u, v - w\rangle + \rho(\phi(v) - \phi(w)) \geq 0, \quad \forall v \in H, \tag{11.24}$$

where $\rho > 0$ is a constant. Problem (11.24) is known as the auxiliary bivariational inequality. We note that, if $w = u$, then clearly w is a solution of the problem (11.6). This observation enables us to suggest and analyze the following iterative method for solving the problem(11.6).

Algorithm 11.8. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\langle \rho T w_n, \beta(v - w_n)\rangle + \langle u_{n+1} - w_n, v - u_{n+1}\rangle + \phi(v) - \phi(u_{n+1}) \geq 0, \quad \forall v \in H$$
$$\langle \nu T(u_n, \beta(v - u_n))\rangle + \langle w_n - u_n, v - w_n\rangle + \rho(\phi(v) - \phi(w_n)) \geq 0, \quad \forall v \in H,$$

where $\rho > 0$ and $\nu > 0$ are constants.

Algorithm 11.8 is two-step predictor-corrector method for solving the mixed bivariational inequalities (11.6).

Remark 11.2. *For suitable and appropriate choice of the operators and the spaces, one can obtain various known and new algorithms for solving bivariational inequality (11.6) and related optimization problems. Convergence analysis of these new algorithms can be considered and investigated using the above techniques and ideas. It is an interesting problem from both analytically and numerically point of views.*

12 Mixed Harmonic Variational Inequalities

Convexity theory contains a wealth of novel ideas and techniques, which have played the significant role in the development of almost all the branches of pure and applied sciences. Several new generalizations and extensions of the convex functions and convex sets have been introduced and studied to tackle unrelated complicated and complex problems in a unified manner. Harmonic functions and harmonic convex sets are important generalizations of the convex functions and convex sets. The harmonic means have novel applications in electrical circuits theory. It is known that the total resistance of a set of parallel resistors is obtained by adding up the reciprocals of the individual resistance values, and then taking the reciprocal of their total. More precisely, if u and v are the resistances of two parallel resistors, then the total resistance is computed by the formula: $\frac{1}{u} + \frac{1}{u} = \frac{uv}{u+v}$, which is half the harmonic means. Al-Azemi et al. [11] studied the Asian options with harmonic average, which can be viewed a new direction in the study of the risk analysis and financial mathematics. Noor et al [74] have shown that the minimum of the differentiable harmonic convex function F on the harmonic convex set $\mathcal{C}_h$ can be characterized by a class of variational inequalities,

$$\langle F'(u), \frac{uv}{u-v}\rangle \geq 0, \quad \forall v \in \mathcal{C}_h. \tag{12.1}$$

Motivated and inspired by this result, Noor and Noor [74] considered the problem of finding $u \in \mathcal{C}_h$ such that

$$\langle Tu, \frac{uv}{u-v}\rangle \geq 0, \quad \forall v \in \mathcal{C}_h, \tag{12.2}$$

which is called the harmonic variational inequality. For the formulation, motivation, numerical methods, generalizations and other aspects of harmonic variational inequalities, see [3, 4, 74, 75].

It is natural to study these different problems in a unified framework. This motivated us to introduce and consider a class of mixed harmonic variational inequality. We now prove that the optimality conditions of the difference of two differentiable harmonic functions can be characterized by a class of nonlinear harmonic variational inequalities. This motivated us to introduce and consider some classes of nonlinear harmonic variational inequality involving two harmonic operators. Several special cases such as variational inequalities, absolute value harmonic variational inequalities, harmonic complementarity problems, harmonic Riesz-Frechet representation results and system of absolute value equations. The projection method, resolvent method, Wiene-Hopf equations technique and descent methods are not applicable to propose numerical methods for solving mixed harmonic variational inequalities. We apply the auxiliary principle technique to develop some iterative schemes for solving various classes of variational inequalities and equilibrium problems. As special cases, one obtain several known and new results for harmonic variational inequalities, variational inequalities and related optimization problems. Results obtained in this paper, represent an improvement and refinement of the known results for harmonic variational inequalities and their variant forms.

Definition 12.1. *[2] Let j be locally Lipschitz continuous at a given point $x \in \mathcal{H}$ and v be any other vector in $\mathcal{H}$. The Clarke's generalized directional derivative of j at x in the direction v, denoted by $j^0(x; v)$, is defined as*

$$f^0(x; v) = \lim_{t \to 0^+} \sup_{h \to 0} \frac{f(x + h + tv) - f(x + h)}{t}.$$

The generalized gradient of j at x, denoted $\partial j(x)$, is defined to be subdifferential of the function $j^0(x; v)$ at 0. That is

$$\partial j(x) = \{w \in \mathcal{H} : \langle w, v\rangle \leq j^0(x; v), \quad \forall v \in \mathcal{H}.\}$$

Lemma 12.1. *Let j be a locally Lipschitz continuous at a given point $x \in \mathcal{H}$ with a constant L. Then*

(i). $\partial j(x)$ *is a none-empty compact subset of $\mathcal{H}$ and $\|\xi\| \leq L$ for each $\xi \in \partial j(x)$.*

(ii). *For every $v \in \mathcal{H}$, $\quad j^0(x; v) = max\{\langle \xi, c\rangle : \xi \in \partial j(x)\}$.*

(iii). *The function $v \longrightarrow j^0(x; v)$ is finite, positively homogeneous, subadditive,*
convex and continuous.

(iv). $j^0(x; -v) = (-j)^0(x; v)$.

(v). *$j^0(x; v)$ is upper semicontinuous as a function of $(x; v)$.*

(vi). *$\forall x \in \mathcal{H}$, there exists a constant $\alpha > 0$ such that*

$$|j^0(x; v)| \leq \alpha \|v\|, \quad \forall v \in \mathcal{H}.$$

If j is convex on $\mathcal{C}$ and locally Lipschitz continuous at $x \in \mathcal{C}$, then $\partial j(x)$ coincides with the subdifferential $j'(x)$ of j at x in the sense of convex analysis , and $j^0(x;v)$ coincides with the directional derivative $j'(x;v)$ for each $v \in H$, that is, $j^0(x;v) = \langle j'(x), v\rangle$.

For the sake of completeness and to convey the main ideas, we include the relevant details.

Definition 12.2. *[8, 74] The set $\mathcal{C}_h$ is said to be a harmonic convex set, if*

$$\frac{uv}{v+\lambda(u-v)} \in \mathcal{C}_h, \quad \forall u, v \in \mathcal{C}_h, \quad \lambda \in [0,1].$$

Definition 12.3. *[8, 74] The function ϕ on the harmonic convex set $\mathcal{C}_h$ is said to be harmonic convex, if*

$$\phi(\frac{uv}{v+\lambda(u-v)}) \leq (1-\lambda)\phi(u) + \lambda\phi(v), \quad \forall u, v \in \mathcal{C}_h \quad \lambda \in [0,1].$$

The function ϕ is said to be harmonic concave function, if and only if, $-\phi$ is harmonic convex function.

We recall that the minimum of a differentiable harmonic convex function on the harmonic convex set $\mathcal{C}_h$ can be characterized by a class of variational inequities, which is called the harmonic variational inequality. This is result is due to Noor and Noor [74].

Theorem 12.1. *Let ϕ be a differentiable harmonic convex function on the harmonic convex set $\mathcal{C}_h$. Then $u \in \mathcal{C}_h$ is a minimum of ϕ, if and only if, $u \in \mathcal{C}_h$ satisfies the inequality*

$$\langle \phi'(u), \frac{uv}{u-v}\rangle \geq 0, \quad \forall v \in \mathcal{C}_h. \tag{12.3}$$

The inequality of type (12.3) is called the harmonic variational inequality.

Proof. Let $u \in \mathcal{C}_h$ is a minimum of differentiable harmonic convex function ϕ. Then

$$\phi(u) \leq \phi(v), \quad \forall v \in \mathcal{C}_h. \tag{12.4}$$

Since $\mathcal{C}_h$ is a harmonic convex set, so $\forall u, v \in \mathcal{C}_h$, $v_\lambda = \frac{uv}{u+\lambda(u-v)} \in \mathcal{C}_h$. Replacing v by v_λ) in (12.4) and diving by λ and taking limit as $\lambda \to 0$, we have

$$0 \leq \frac{\phi(\frac{uv}{u+\lambda(u-v)}) - \phi(u)}{\lambda} = \langle \phi'(u), \frac{uv}{u-v} \rangle$$

the required result (12.3). Conversely, let the function ϕ be exponentially harmonic convex function on the harmonic convex set $\mathcal{C}_h$. Then

$$\frac{uv}{v+\lambda(u-v)} \leq (1-\lambda)\phi(u) + \lambda\phi(v) = \phi(u) + \lambda(\phi(v) - \phi(u)),$$

which implies that

$$\phi(v) - \phi(u) \quad \geq \quad \lim_{\lambda \to 0} \frac{\phi(\frac{uv}{v+\lambda(u-v)}) - \phi(u)}{\lambda} = \langle \phi'(u), \frac{uv}{u-v} \rangle \geq 0, \quad \text{using} \quad (12.3).$$

Consequently, it follows that

$$\phi(u) \leq \phi(v), \quad \forall v \in \mathcal{C}_h.$$

This shows that $u \in \mathcal{C}_h$ is the minimum of the differentiability harmonic convex function. □

We would like to mention that Theorem 12.1 implies that harmonic optimization programming problem can be studied via the harmonic variational inequality (12.3).

Using the ideas and techniques of Theorem 12.1, we can derive the following result.

Theorem 12.2. *Let ϕ be a differentiable harmonic convex functions on the harmonic convex set $\mathcal{C}_h$. Then*

$$(i). \qquad \phi(v) - \phi(u) \geq \langle \phi'(u), \frac{uv}{u-v} \rangle, \quad \forall u, v \in \mathcal{C}_h.$$

$$(ii). \qquad \langle \phi'(u) - \phi'(v), \frac{uv}{v-u} \rangle \geq 0, \quad \forall u, v \in \mathcal{C}_h.$$

Motivated by Theorem 12.1 and Theorem 12.2, we introduce some new concepts.

Definition 12.4. *An operator* T *is said to be a harmonic monotone operator, if and only if,*

$$\langle \mathrm{T}u - \mathrm{T}v, \frac{uv}{u-v} \rangle \geq 0, \quad \forall u, v \in \mathrm{H}.$$

Definition 12.5. *An operator* T *is said to a harmonic pseudomonotone operator, if*

$$\langle Tu, \frac{uv}{u-v} \rangle \geq 0 \quad \Rightarrow \quad -\langle Tv, \frac{uv}{u-v} \rangle \geq 0, \quad \forall u, v \in H.$$

An harmonic monotone operator is a harmonic pseudomonotone operator, but the converse is not true.

Consider the energy (virtual) functional

$$I[v] = F(v) - \phi(v), \tag{12.5}$$

where $F(v)$ and $\phi(v)$ are two harmonic convex functions.

We now consider the optimality conditions of the energy function $I[v]$ defined by (12.6) under suitable conditions.

Theorem 12.3. *Let F and $\phi(v)$ be a differentiable harmonic convex functions on the convex set $\mathcal{C}_h$. If $u \in \mathcal{C}_h$ is the minimum of the functional $I[v]$ defined by (12.5), then*

$$\langle F'(u), \frac{uv}{u-v} \rangle - \langle \phi'(u), \frac{uv}{u-v} \rangle \geq 0, \quad \forall v, u \in \mathcal{C}_h. \tag{12.6}$$

Proof. Let $u \in \mathcal{C}_h$ be a minimum of the functional $I[v]$. Then

$$I[u] \leq I[v], \quad \forall v \in K.$$

which implies that

$$F(u) - \phi(u) \leq F(v) - \phi(v), \quad \forall v \in \mathcal{C}_h. \tag{12.7}$$

Since $\mathcal{C}_h$ is a convex set, so, $\forall u, v \in \mathcal{C}_h, \quad \lambda \in [0,1]$, $v_t = \frac{uv}{(1-\lambda)v + \lambda u} \in \mathcal{C}_h$.

Taking $v = v_t$ in (12.7), we have

$$F(u) - \phi(u) \leq F(v_t) - \phi(v_t), \quad \forall v \in \mathcal{C}_h. \tag{12.8}$$

This implies that

$$0 \leq F(\frac{uv}{(1-\lambda)v + \lambda u}) - F(u) - \phi(\frac{uv}{(1-\lambda)v + \lambda u}) - \phi(u), \quad \forall v \in \mathcal{C}_h. \tag{12.9}$$

Dividing the above inequality by λ and taking limit as $\lambda \to 0$, we have

$$\begin{aligned} 0 &\leq \frac{F(\frac{uv}{(1-\lambda)v+\lambda u}) - F(u)}{\lambda} - \frac{\phi(\frac{uv}{(1-\lambda)v+\lambda u}) - \phi(u)}{\lambda} \\ &= \langle F'(u), \frac{uv}{u-v} \rangle - \langle \phi'(u), \frac{uv}{u-v} \rangle, \end{aligned}$$

which is the required (12.6).

Since F is differentiable harmonic convex function, so

$$F(\frac{uv}{v + \lambda(u-v))}) \leq F(u) + \lambda(F(v) - F(u)), \quad \forall u, v \in \mathcal{C}_h$$

from which, we have

$$F(v) - F(u) \geq \lim_{\lambda \to 0} \{ \frac{F(\frac{uv}{(1-\lambda)v+\lambda u}) - F(u)}{\lambda} \} = \langle F'(u), \frac{uv}{u-v} \rangle \tag{12.10}$$

In a similar way,

$$\phi(v) - \phi(u) \geq \lim_{\lambda \to 0} \{ \frac{\phi(\frac{uv}{(1-\lambda)v+\lambda u}) - \phi(u)}{\lambda} \} = \langle \phi'(u), \frac{uv}{u-v} \rangle \tag{12.11}$$

From (12.11) and (12.10), we have

$$F(v) + \phi(v) - (F(u) + \phi(u)) \geq = \langle F'(u), \frac{uv}{u-v} \rangle - \langle \phi'(u), \frac{uv}{u-v} \rangle \geq 0.$$

Consequently, it follows that $u \in \mathcal{C}_h$ such that

$$F(u) - \phi(u) \leq (F(v) - \phi(v)), \quad \forall v \in \mathcal{C}_h,$$

which shows that $u \in \mathcal{C}_h$ is the minimum of the function $I[v]$ defined by (12.5). □

Remark 12.1. *The inequality of the type (12.6) is called the mildly nonlinear harmonic variational inequality.*

Essentially using the technique of Theorem 12.3, one can prove the following result.

Theorem 12.4. *Let F be a differentiable harmonic convex function and $\phi(v)$ be a non differentiable harmonic convex function. If $u \in \mathcal{H}$ is the minimum of the functional $I_1[v] = F(v) + \phi(v)$, then*

$$\langle F'(u), \frac{uv}{u-v}\rangle + \phi(v) - \phi(u) \geq 0, \quad \forall v, u \in H, \tag{12.12}$$

which is called the mixed harmonic variational inequality.

In many applications, the inequalities of the type (12.6) may not arise as the minimum of the sum of the differentiable harmonic convex functions. These facts motivated us to consider more general harmonic variational inequality, which contains the inequalities (12.6) and (12.12) as a special case.

For given nonlinear continuous operators $T, A : H \longrightarrow H$, we consider the problem of finding $u \in H$ such that

$$\langle Tu, \frac{uv}{u-v}\rangle + \langle A(u), \frac{uv}{u-v}\rangle + \phi(v) - \phi(u) \geq 0, \quad \forall v \in H, \tag{12.13}$$

which is called the *mixed nonlinear harmonic variational inequality.*

We note that, if $\langle A(u), \frac{uv}{u-v}\rangle = A(u : \frac{uv}{u-v})$ the problem (12.13) reduces to finding $u \in H$ such that

$$\langle Tu, \frac{uv}{u-v}\rangle + A(u; \frac{uv}{u-v}) + \phi(v) - \phi(u) \geq 0, \quad \forall v \in H, \tag{12.14}$$

which is called the mixed harmonic hemivariational inequality involving two harmonic operators, see Noor and Noor [75].

We now discuss some new and known classes of variational inequalities and related optimization problems.

(i). If $\langle A(u), \frac{uv}{u-v}\rangle = \varphi'(u; \frac{uv}{u-v})$, where $\phi'(u)$ denotes derivative of the harmonic convex function $\phi(u)$ in the direction $\frac{uv}{u-v}$, then problem (12.13) reduces to finding $u \in H$, such that

$$\langle Tu, \frac{uv}{u-v}\rangle + \varphi'(u; \frac{uv}{u-v}) + \phi(v) - \phi(u \quad 0, \quad \forall v \in H, \tag{12.15}$$

which is also called the mixed harmonic directional variational inequality.

(ii). For $\langle A(u), \frac{uv}{v-u}\rangle = J^0(u; \frac{uv}{u-v})$, the problem (12.13) reduces to finding $u \in H$, such that

$$\langle Tu, \frac{uv}{u-v} + \rangle + J^0(u; \frac{uv}{u-v}) + \phi(v) - \phi(u \geq 0, \quad \forall v \in H, \tag{12.16}$$

which is known as mixed harmonic hemivariational inequality. Hemivariational inequalities have important applications in superpotential analysis of elasticity and structural analysis.

(iii). If $\varphi(.)$ is a smooth and convex function, then $\phi'(u; \frac{uv}{u-v}) = \langle \phi'(u), \frac{uv}{u-v}\rangle$, and consequently problem (12.15) is equivalent to finding $u \in \mathcal{C}_h$ such that

$$\langle Tu, \frac{uv}{u-v}\rangle + \langle \varphi'(u), \frac{uv}{u-v}\rangle + \phi(v) - \phi(u \geq 0, \quad \forall v \in H, \tag{12.17}$$

which is called the nonlinear mixed harmonic variational inequality.

(iv). If $\langle A(u), \frac{uv}{u-v}) = -\langle Au, \frac{uv}{u-v}\rangle$ then the problem (12.13) reduces to finding $u \in H$ such that

$$\langle Tu, \frac{uv}{u-v}\rangle - \langle A(u), \frac{uv}{u-v}\rangle + \phi(v) - \phi(u \geq 0, \quad \forall v \in H, \tag{12.18}$$

which is called the mildly nonlinear harmonic variational inequality involving the difference of two monotone operators.

(v). If $(\mathcal{C}_h)^\star = \{u \in \mathcal{H} : \langle u, \frac{uv}{u-v}\rangle \geq 0, \quad \forall v \in \mathcal{C}_h\}$ is a polar harmonic convex cone of the harmonic convex $\mathcal{C}_h$, then problem (12.13) is equivalent to fining $u \in \mathcal{H}$, such that

$$\frac{uv}{u-v} \in \mathcal{C}_h, \quad Tu + A(u) \in (\mathcal{C}_h)^\star, \quad \langle Tu + A(u), \frac{uv}{u-v}\rangle = 0, \tag{12.19}$$

is called the general harmonic complementarity problem. For the applications, numerical methods and other aspects of complementarity problems, see [21, 28, 33, 34, 85, 92] and the references therein.

(vii). For $Au = A|u|$, the problem (12.13) reduces to finding $u \in H$ such that

$$\langle Tu + A|u|, \frac{uv}{u-v}\rangle + \phi(v) - \phi(u \geq 0, \quad \forall v \in H, \tag{12.20}$$

which is called the system of absolute value harmonic equations.

(viii). If $\langle A(u), \frac{uv}{u-v}\rangle = 0$, and ϕ is the indicator function of the harmonic convex set $\mathcal{C}_h$, then problem (12.13) reduces to finding $u \in \mathcal{C}_h$ such that

$$\langle Tu, \frac{uv}{v-u} \geq 0, \quad \forall v \in \mathcal{C}_h, \tag{12.21}$$

which is called the harmonic variational inequalities, introduced and studied by Noor and Noor [74, 75]. For the recent applications, motivation, numerical methods, sensitivity analysis and local uniqueness of solutions of harmonic variational inequalities and related optimization problems, see [3, 4, 74, 75] and the references therein.

This show that the problem (12.13) is quite and unified one. Due to the structure and nonlinearity involved, one has to consider its own. It is an open problem to develop unified numerical implementation numerical methods for solving the harmonic variational inequalities.

Iterative Methods and Convergence Analysis

In this section, we apply the auxiliary principle technique to suggest and analyze some inertial iterative methods for solving harmonic mixed variational inequalities (12.13).

For a given $u \in H$ satisfying (12.13), consider the problem of finding $w \in H$ such that

$$\langle \rho T(w+\eta(u-w)), \frac{uw}{u-w}\rangle \quad + \quad \langle w-u, v-w\rangle + \langle \rho A(w), \frac{uw}{u-w}\rangle) \\ +\rho\phi(v) - \rho\phi(u \geq 0, \quad \forall v \in H, \qquad (12.22)$$

where $\rho > 0, \eta \in [0,1]$ are constants.

Inequality of type (14.1) is called the auxiliary mixed harmonic variational inequality.

If $w = u$, then w is a solution of (12.13). This simple observation enables us to suggest the following iterative method for solving (12.13).

Algorithm 12.1. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\langle \rho T(u_{n+1}+\eta(u_n-u_{n+1})), \frac{vu_{n+1}}{v-u_{n+1}}\rangle + \langle u_{n+1}-u_n, v-u_{n+1}\rangle \\ \geq -\langle \rho A(u_{n+1}), \frac{vu_{n+1}}{v-u_{n+1}}\rangle - \rho\phi(v) - \rho\phi(u_{n+1}), \quad \forall v \in H.$$

Algorithm 12.1 is called the hybrid proximal point algorithm for solving harmonic hemivariational inequalities(12.13).

Special Cases

We now consider some cases of Algorithm 12.1.

(I). For $\eta = 0$, Algorithm 12.1 reduces to:

Algorithm 12.2. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\langle \rho T u_{n+1}, \frac{v u_{n+1}}{v - u_{n+1}} \rangle + \langle u_{n+1} - u_n, v - u_{n+1} \rangle$$
$$\geq -\langle \rho A(u_{n+1}), \frac{v u_{n+1}}{v - u_{n+1}} \rangle - \rho\phi(v) - \rho\phi(u_{n+1}), \quad \forall v \in H. \quad (12.23)$$

(II). If $\eta = 1$, then Algorithm 12.1 reduces to:

Algorithm 12.3. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\langle \rho T u_n, \frac{v u_{n+1}}{v - u_{n+1}} \rangle + \langle u_{n+1} - u_n, v - u_{n+1} \rangle$$
$$\geq -\langle \rho A(u_{n+1}), \frac{v u_{n+1}}{v - u_{n+1}} \rangle - \rho\phi(v) - \rho\phi(u_{n+1}), \quad \forall v \in H.$$

(III). If $\eta = \frac{1}{2}$, then Algorithm 12.1 collapses to:

Algorithm 12.4. *For a given $u_0 \in \mathcal{C}_h$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\langle \rho T(\frac{u_{n+1} + u_n}{2}, \frac{v u_{n+1}}{v - u_{n+1}} \rangle + \langle u_{n+1} - u_n, v - u_{n+1} \rangle$$
$$\geq -\langle \rho A(u_{n+1}), \frac{v u_{n+1}}{v - u_{n+1}} \rangle - \rho\phi(v) - \rho\phi(u_{n+1}), \quad \forall v \in H.$$

which is called the mid-point proximal method for solving the problem (12.13).

If $A(.;.) = 0$, then Algorithm 12.1 reduces to:

Algorithm 12.5. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\langle \rho T(u_{n+1} + \eta(u_n - u_{n+1})), \frac{v u_{n+1}}{v - u_{n+1}} \rangle + \langle u_{n+1} - u_n, v - u_{n+1} \rangle$$
$$\geq -\rho\phi(v) - \rho\phi(u_{n+1}, \quad \forall v \in H.$$

for solving mixed harmonic variational inequality.

Definition 12.6. *$\forall u, v, z \in H$, an operator $T : H \rightarrow H$ is said to be:*
(i). harmonic monotone, if and only if,

$$\langle Tu - Tv, \frac{uv}{u-v} \rangle \geq 0.$$

(ii) harmonic pseudomonotone if and only if,

$$\langle Tu, \frac{uv}{u-v} \rangle \geq 0 \quad \Longrightarrow \quad \langle Tv, \frac{uv}{u-v} \rangle \geq 0.$$

(iii). partially relaxed strongly harmonic monotone, if there exists a constant $\alpha > 0$ such that

$$\langle Tu - Tv, \frac{zv}{z-v} \rangle \geq -\alpha \|z - u\|^2.$$

Note that for $z = u$, partially relaxed strongly harmonic monotonicity reduces to monotonicity. It is known that partially relaxed strongly harmonic monotonicity, but the converse is not true. It is known that harmonic monotonicity implies harmonic pseudomonotonicity; but the converse is not true. Consequently, the class of harmonic pseudomonotone operators is bigger than the one of harmonic monotone operators.

Lemma 12.2. $\forall u, v \in \mathcal{H}$,

$$2\langle u, v \rangle = \|u + v\|^2 - \|u\|^2 - \|v\|^2. \tag{12.24}$$

We now consider the convergence criteria of Algorithm 12.2. The analysis is in the spirit of Noor [49] . We include the proof for the sake of completeness and to convey an idea of the technique involved.

Theorem 12.5. *Let $u \in H$ be a solution of (12.13) and let u_{n+1} be the approximate solution obtained from Algorithm 12.2. If the operators T and A are monotone harmonic, then*

$$\|u_{n+1} - u\|^2 \leq \|u_n - u\|^2 - \|u_{n+1} - u_n\|^2. \tag{12.25}$$

Proof. Let $u \in H$ be a solution of (12.13). Then

$$\langle Tv, \frac{uv}{v-u}\rangle + \langle A(v), \frac{uv}{v-u}\rangle + \phi(v) - \phi(u) \geq 0, \quad \forall v \in \mathcal{C}_h \tag{12.26}$$

since T and A are harmonic monotone operators.

Now taking $v = u_{n+1}$ in (13.5), we have

$$\langle Tu, \frac{uu_{n+1}}{u_{n+1}-u}\rangle + \langle A(u_{n+1}), \frac{uu_{n+1}}{u_{n+1}-u} + \phi(u_{n+1}) - \phi(u)\rangle \geq 0. \tag{12.27}$$

Taking $v = u$ in (12.23), we get

$$\begin{aligned}&\langle \rho T(u_{n+1}, \frac{u_n u_{n+1}}{u_n - u_{n+1}}\rangle + \langle u_{n+1} - u_n, u - u_{n+1}\rangle \\ &+\langle A(u_{n+1}), \frac{u_n u_{n+1}}{u_n - u_{n+1}}\rangle + \phi(u) - \phi(u_{n+1}) \geq 0,\end{aligned}$$

which can be written as

$$\begin{aligned}&\langle u_{n+1} - u_n, u - u_{n+1}\rangle \geq \langle \rho T u_{n+1}, \frac{uu_{n+1}}{u_{n+1}-u}\rangle \\ &+\langle \rho A(u_{n+1}), \frac{uu_{n+1}}{u - u_{n+1}}\rangle + \phi(u) - \phi(u_{n+1}) \geq 0,\end{aligned} \tag{12.28}$$

where we have used (13.6).

Setting $u = u - u_{n+1}$ and $v = u_{n+1} - u_n$ in (12.24), we obtain

$$2\langle u_{n+1} - u_n, u - u_{n+1}\rangle = \|u - u_n\|^2 - \|u - u_{n+1}\|^2 - \|u_{n+1} - u_n\|^2. \tag{12.29}$$

Combining (13.7) and (13.8), we have

$$\|u_{n+1} - u\|^2 \leq \|u_n - u\|^2 - \|u_{n+1} - u_n\|^2,$$

the required result (12.25). □

Theorem 12.6. *Let H be a finite dimensional space and all the assumptions of Theorem 12.5 hold. Then the sequence $\{u_n\}_1^\infty$ given by Algorithm 12.2 converges to a solution u of (12.13).*

Proof. Let $u \in K$ be a solution of (12.13). From (12.25), it follows that the sequence $\{||u - u_n||\}$ is nonincreasing and consequently $\{u_n\}$ is bounded. Furthermore, we have

$$\sum_{n=0}^{\infty} \|u_{n+1} - u_n\|^2 \leq \|u_0 - u\|^2,$$

which implies that

$$\lim_{n\to\infty} \|u_{n+1} - u_n\| = 0. \tag{12.30}$$

Let $\hat{u}$ be the limit point of $\{u_n\}_0^\infty$; a subsequence $\{u_{n_j}\}_1^\infty$ of $\{u_n\}_0^\infty$ converges to $\hat{u} \in H$. Replacing w_n by u_{n_j} in (13.2), taking the limit $n_j \longrightarrow \infty$ and using (13.9), we have

$$\langle T\hat{u}, \frac{\hat{u}v}{v - \hat{u}}\rangle + \langle A(\hat{u}), \frac{\hat{u}v}{v - \hat{u}}\rangle + \phi(v) - \phi(\hat{u}) \geq 0, \quad \forall v \in H,$$

which implies that $\hat{u}$ solves the harmonic hemivariational inequality (2.1) and

$$\|u_{n+1} - u\|^2 \leq \|u_n - u\|^2.$$

Thus, it follows from the above inequality that $\{u_n\}_1^\infty$ has exactly one limit point $\hat{u}$ and

$$\lim_{n\to\infty} (u_n) = \hat{u}.$$

the required result. □

We again consider the auxiliary principle technique to suggest some hybrid inertial proximal point methods for solving the problem (12.13).

For a given $u \in H$ satisfying (2.1), consider the problem of finding $w \in H$ such that

$$\langle \rho T(w + \eta(u - w)), \frac{uw}{u - w}\rangle + \langle w - u + \alpha(u - u), v - w\rangle$$
$$+\langle A((w + \xi(w - u))), \frac{uw}{u - w}\rangle + \phi(v) - \phi(w) \geq 0, \quad \forall v \in H, \tag{12.31}$$

where $\rho > 0, \alpha, \xi, \eta, \in [0,1]$ are constants.

Clearly, for $w = u, \quad w$ is a solution of (12.13). This fact motivated us to to suggest the following inertial iterative method for solving (12.13).

Algorithm 12.6. *For given $u_0, u_1 \in H$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\langle \rho T(u_{n+1} + \eta(u_n - u_{n+1})), \frac{u_n u_{n+1}}{u_n - u_{n+1}}\rangle + \langle u_{n+1} - u_n + \alpha(u_n - u_{n-1}, v - u_{n+1}\rangle$$
$$\geq -\langle \rho A((u_{n+1+\xi(u_n - u_{n+1})}), \frac{u_n u_{n+1}}{u_n - u_{n+1}}\rangle + \phi(v) - \phi(u_{n+1}, \quad \forall v \in H$$

which is known as the hybrid inertial iterative method.

Note that for $\alpha = 0, \quad \xi = 0$, Algorithm 12.6 is exactly the Algorithm 12.1. Using essentially the technique of Theorem 12.5 and Noor [28], one can study the convergence analysis of Algorithm 12.6.

For different and appropriate values of the parameters, ξ, η, ζ, α, the operators T, A, function $\phi(.)$ and spaces, one can obtain a wide class of inertial type iterative methods for solving the harmonic variational inequalities and related optimization problems.

Conclusion:

Some new classes of mixed harmonic variational inequalities are introduced in this paper. It is shown that several important problems such as harmonic complementarity problems, system of harmonic absolute value problems and related problems can be obtained as special cases. The auxiliary principle technique is applied to suggest several inertial type methods for solving mixed harmonic variational inequalities with suitable modifications. We note that this technique is independent of the projection and the resolvent of the operator. Moreover, we have studied the convergence analysis of these new methods under weaker conditions. We have only considered the theoretical aspects of the hybrid inertial iterative methods. It is an interesting problem to implement these methods numerically and compare with other iterative schemes.

13 Mixed Biharmonic Variational Inequalities

It is well known that the biconvex functions and the harmonic convex functions are two different generalization of the convex functions. It is natural to consider the new concepts which includes these concepts as special cases. These facts motivated us to consider some new classes of convex sets and convex functions with respect to an arbitrary bifunction.

Definition 13.1. *A set $K_{h\beta} \subseteq \mathbb{H}$ is said to be a harmonic biconvex set with respect to the bifunction $\beta(\cdot - \cdot) : H \times H \to H$, if*

$$\frac{u(u+\beta(v-u))}{v+\lambda\beta(u-v)} \in K_{h\beta}, \quad \forall u, v \in K_{h\beta}, \lambda \in [0,1].$$

If $\beta(u-v) = u-v$, then harmonic biconvex set reduces to harmonic convex set. Clearly, every harmonic set is a harmonic biconvex set but the converse is not true.

Definition 13.2. *A function f is a harmonic biconvex function with respect to an arbitrary bifunction $\beta(\cdot - \cdot)$, if*

$$f\left(\frac{u(u+\beta(v-u))}{u+\lambda\beta(u-v)}\right) \leq (1-\lambda)f(u) + \lambda(v), \qquad \forall u, v \in K_{h\beta}, \lambda \in [0,1].$$

Note that for $t = \frac{1}{2}$, we have Jensen type harmonic biconvex function.

$$f\left(\frac{2u(u+\beta(v-u))}{2u+\beta(u-v)}\right) \leq \left(\frac{1}{2}\right)[f(u)+f(v)], \qquad \forall u, v \in K_{h\beta}.$$

Using the technique developed in the previous sections, we can prove the the minimum $u \in K_{h\beta}$ of the differentiable harmonic biconvex function f can be characterized by the inequality.

Theorem 13.1. *Let f be a differentiable harmonic biconvex function on the biconvex set $K_{h\beta}$. Then $u \in K_{h\beta}$ is the minimum of f, if and only if, $u \in K_{h\beta}$ satisfies the inequality*

$$\langle f'(u), \frac{uv}{\beta(u-v)} \rangle \geq 0, \quad \forall v \in K_{h\beta}. \tag{13.1}$$

The inequality of the type (13.1) is called the harmonic bivariational inequality.

We now consider the functional $J[v]$ defined as

$$J[v] = F(v) + \phi(v), \quad \forall v \in H, \tag{13.2}$$

where F and ϕ are two functions.

Using the above above technique, one can easily prove the following results.

Theorem 13.2. *If the function F is a differentiable harmonic biconvex function and ϕ) is non differentiable harmonic biconvex function, then $u \in H$ is the minimum of the functional $J[v]$ defined by (13.2), if and only if, $u \in H$ satisfies the inequality*

$$\langle F'(u), \frac{uv}{\beta u - v} \rangle + \phi(v) - \phi(u) \geq 0, \quad \forall v \in H, \tag{13.3}$$

which is called the mixed harmonic bivariational inequality.

Inequalities of the types (13.1) and (13.3) may not arise as optimality conditions of the differentiability of the harmonic biconvex functions in many important and complicated problems. In such type of problems, one usually considers the more general inequalities, which contains the inequalities (13.1), (13.3) and their variant forms as special cases. Motivated by these facts, we now consider more general problem.

For given nonlinear operator T and bifunction $\beta(. - .) : H \times H \Longrightarrow H$, we consider the problem of fining $u \in H$, such that

$$\langle Tu, \frac{uv}{\beta(u-v)} \rangle + \phi(v) - \phi(u) \geq 0, \quad \forall v \in H, \tag{13.4}$$

which is called the mixed harmonic bivariational inequality.

Special Cases

We now discuss some special cases of the problem (13.4):

(I). If $\beta(u-v) = u-v, \quad \forall v \in H$, then (13.4) reduces to finding $u \in H$, such that

$$\langle Tu, \frac{uv}{u-v}\rangle + \phi(v) - \phi(u) \geq 0, \quad \forall v \in H.$$

is called the mixed harmonic variational inequality.

(II). For $Tu = f'(u)$, the problem (13.4) reduces to finding $u \in H$, such that

$$\langle f'(u), \frac{uv}{\beta(u-v)})\rangle + \phi(v) - \phi(u) \geq 0, \quad \forall v \in H,$$

which is the problem (12.24).

(III). For $Tu = T|u|$, the problem (13.4) reduces to finding $u \in H$, such that

$$\langle T|u|, \frac{uv}{\beta(u-v)}\rangle + \phi(v) - \phi(u) \geq 0, \quad \forall v \in H,,$$

is called the absolute value mixed harmonic bivariational inequality.

In brief, for suitable and appropriate choice of the bifunction, one can obtain several classes of mixed harmonic variational inequalities, complementarity problems, absolute value harmonic inequalities and harmonic optimization problems. This clear shows that the problem (13.4) is more general and flexible and includes the previous ones as special cases.

Approximate schemes and convergence

In this section, we suggest and analyze some iterative methods for solving mixed harmonic bivariational inequality (13.4) using the auxiliary principle technique.

We again use the auxiliary principle technique involving an arbitrary operator as introduced by Noor [50]. As pointed out in [92]. the operator need not be the differential of a convex function. We point out that, if the operator is the derivative of the strongly convex function, then this technique reduces to auxiliary technique involving Bregman distance function [100].

For a given $u \in H$ satisfying (13.4), consider the auxiliary problem of finding $w \in H$ such that

$$\begin{aligned}&\rho\langle T(w+\eta(u-w))), \frac{vw}{\beta(v-w)}\rangle \\ &+\langle M(w)-M(v), v-w\rangle+\rho(\phi(v)-\phi(w)) \geq 0, \quad \forall v \in H, \end{aligned} \tag{13.5}$$

where $\rho > 0$ is a constant and M is an arbitrary operator. Clearly, if $w = u$, then clearly w is solution of the problem (13.4). This observation enables us to suggest and analyze the following iterative method for the problem (13.4).

Algorithm 13.1. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\begin{aligned}&\rho\langle T(u_{n+1}+\eta(u_n-u_{n+1})), \frac{vu_{n+1}}{\beta(v-u_{n+1})}\rangle \\ &+\langle M(u_{n+1})-M(u_n), v-u_{n+1}\rangle+\rho(\phi(v)-\phi(u_{n+1})) \geq 0, \quad \forall v \in H, \end{aligned} \tag{13.6}$$

where $\rho \geq 0$, $\eta \in [0,1]$ are constants. Algorithm 13.1 is called the hybrid proximal point method.

For $\eta = 0$, Algorithm 13.1 reduces to the following method for solving the problem (13.4).

Algorithm 13.2. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\begin{aligned}&\rho\langle Tu_{n+1}, \frac{vu_{n+1}}{\beta(v-u_{n+1})}\rangle+\langle M(u_{n+1})-M(u_n), v-u_{n+1}\rangle \\ &+\rho(\phi(v)-\phi(u_{n+1})) \geq 0, \quad \forall v \in H, \end{aligned} \tag{13.7}$$

is the implicit proximal method.

For $\eta = 1$, Algorithm 13.1 reduces to the explicit method for solving the problem (13.4).

Algorithm 13.3. *For a given $u_0 \in K_{h\beta}$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\rho\langle Tu_n, \frac{vu_{n+1}}{\beta(v-u_{n+1})}\rangle + \langle M(u_{n+1}) - M(u_n), v - u_{n+1}\rangle \\ +\rho(\phi(v) - \phi(u_{n+1})) \geq 0, \quad \forall v \in H.$$

For $\eta = \frac{1}{2}$, Algorithm 13.1 reduces to the hybrid proximal point method for solving the problem (13.4).

Algorithm 13.4. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\rho\langle T\big(\frac{u_{n+1}+u_n}{2}\big), \frac{vu_{n+1}}{\beta(v-u_{n+1})}\rangle + \langle M(u_{n+1}) - M(u_n), v - u_{n+1}\rangle \\ +\rho(\phi(v) - \phi(u_{n+1})) \geq 0, \quad \forall v \in H.$$

In brief, for suitable and appropriate choice of the operators and the spaces, one can obtain a number of known and new algorithms for solving harmonic variational inequalities and related problems.

We now consider the convergence criteria of Algorithm 13.2.

Theorem 13.3. *Let $u \in H$ be a solution of (13.4) and let u_{n+1} be the approximate solution obtained from Algorithm 13.2. Let the operator T be harmonic monotone. If the operator M is strongly monotone with constant $\xi \geq 0$ and Lipschitz continuous with constant $\zeta \geq 0$, then*

$$\xi\|u_n - u_{n+1}\| \leq \zeta\|u - u_n\|. \tag{13.8}$$

Proof. Let $u \in H$ be a solution of (13.4). Then

$$-\langle Tv, \frac{uv}{\beta(u-v)}\rangle + \phi(v) - \phi(u) \geq 0, \quad \forall v \in H. \tag{13.9}$$

since the operator T is a monotone operator.

Now taking $v = u_{n+1}$ in (13.9), we have

$$-\langle T(u_{n+1}), \frac{uu_{n+1}}{\beta(u - u_{n+1})}\rangle + \phi(u_{n+1} - \phi(u) \geq 0. \tag{13.10}$$

Taking $v = u$ in (13.7), we get

$$\begin{aligned}&\rho\langle T(u_{n+1}), \frac{uu_{n+1}}{\beta(u - u_{n+1})}\rangle \\ &\qquad +\langle M(u_{n+1}) - M(u_n), u - u_{n+1}\rangle + \rho\phi(u) - \rho\phi(u_{n+1}) \geq 0,\end{aligned}$$

which can be written as

$$\langle M(u_{n+1}) - M(u_n), u - u_{n+1}\rangle \geq -\rho\langle T(u_{n+1}, \frac{uu_{n+1}}{\beta(u - u_{n+1})})\rangle \geq 0, \tag{13.11}$$

where we have used (13.10).

From the equation (13.13), we have

$$\begin{aligned} 0 &\leq \langle M(u_{n+1}) - M(u_n), u - u_{n+1}\rangle \\ &= \langle M(u_{n+1}) - M(u_n), u - u_n\rangle + \langle M(u_{n+1} - M(u_n), u_n - u_{n+1}\rangle, \end{aligned}$$

from which, it follows that

$$\langle M(u_{n+1} - M(u_n), u_{n+1} - u_n\rangle \leq \langle M(u_{n+1}) - M(u_n), u - u_n\rangle.$$

Using the strongly monotonicity and Lipschitz continuity of the operator M, we obtain

$$\xi\|u_n - u_{n+1}\|^2 \leq \zeta\|u_n - u_{n+1}\|\|u - u_n\|.$$

This implies that

$$\xi\|u_n - u_{n+1}\| \leq \zeta\|u - u_n\|,$$

which is the required result (13.8). □

Theorem 13.4. *Let H be a finite dimensional space and all the assumptions of Theorem 14.1 hold. Then the sequence $\{u_n\}_0^\infty$ given by Algorithm 13.2 converges to a solution u of the problem (13.4).*

Proof. Let $u \in H$ be a solution of the problem (13.4). From (13.8), it follows that the sequence $\{||u - u_n||\}$ is nonincreasing and consequently $\{u_n\}$ is bounded. Also, we have

$$\xi \sum_{n=0}^{\infty} \|u_{n+1} - u_n\| \leq \zeta \|u_0 - u\|,$$

which implies that

$$\lim_{n\to\infty} \|u_{n+1} - u_n\| = 0. \tag{13.12}$$

Let $\hat{u}$ be the limit point of $\{u_n\}_0^\infty$; a subsequence $\{u_{n_j}\}_1^\infty$ of $\{u_n\}_0^\infty$ converges to $\hat{u} \in H$. Replacing w_n by u_{n_j} in (13.5), taking the limit $n_j \longrightarrow \infty$ and using (13.12), we have

$$\langle T(\hat{u}), \frac{\hat{u}v}{\beta(\hat{u} - v)}\rangle + \phi(v) - \phi(\hat{u}) \geq 0, \quad \forall v \in H,$$

which implies that $\hat{u}$ solves the problem (13.4) and

$$\|u_{n+1} - u\|^2 \leq \|u_n - u\|^2.$$

Thus, it follows from the above inequality that $\{u_n\}_1^\infty$ has exactly one limit point $\hat{u}$ and

$$\lim_{n\to\infty} (u_n) = \hat{u}.$$

the required result. □

In recent years, much attention have been given to consider the inertial type algorithms for solving the variational inequalities. Such type of iterative methods were considered by Polyak [96] for speeding the convergence criteria of the iterative methods. For the applications of the inertial type methods in variational inequalities, see [2, 42, 83, 84, 88]. We gain apply the auxiliary principle technique to propose some inertial type approximate schemes for solving the problem (13.4).

For a given $u \in H$ satisfying (13.4), consider $w \in H$ such that

$$\rho\langle T(w+\eta(u-w))), \frac{vw}{\beta(v-w)}\rangle + \langle M(w) - M(v) + \alpha(u-u), v-w\rangle$$
$$+\rho(\phi(v)-\phi(w)) \geq 0, \quad \forall v \in H, \tag{13.13}$$

where $\rho > 0$ and $\alpha \geq 0$ are constants and M is an arbitrary operator. Clearly, if $w = u$, then clearly w is solution of the problem (13.4). This observation enables us to suggest and analyze the following iterative method for solving (13.4).

Algorithm 13.5. *For a given $u_0, u_1 \in H$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\rho\langle T(u_{n+1}+\eta(u_n-u_{n+1})), \frac{vu_{n+1}}{\beta(v-u_{n+1})}\rangle + \rho(\phi(v)-\phi(u_{n+1}))$$
$$+\langle M(u_{n+1}) - M(u_n) + \alpha(u_n-u_{n-1}), v-u_{n+1}\rangle \geq 0, \quad \forall v \in H, \tag{13.14}$$

where $\rho \geq 0$, $\eta \in [0,1]$ are constants and M is any arbitrary operator. Algorithm 13.5 is called the hybrid inertial proximal point method for solving the mixed harmonic bivariational inequality (13.4).

If $M = 1, \alpha = 0$, then Algorithm 13.5 becomes:

Algorithm 13.6. *For a given $u_0, u_1 \in H$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\rho\langle T(u_{n+1}+\eta(u_n-u_{n+1})), \frac{vu_{n+1}}{\beta(v-u_{n+1})}\rangle + \rho(\phi(v)-\phi(u_{n+1}))$$
$$+\langle u_n-u_{n-1}, v-u_{n+1}\rangle \geq 0, \quad \forall v \in H, \tag{13.15}$$

where $\rho \geq 0$, $\eta \in [0,1]$ are constants.

Remark 13.1. *For $\alpha = 0$, Algorithm 13.5 reduces to Algorithm 13.1. For suitable and appropriate choices of the parameters α, η, operator T, bifunction $\beta(.-.)$ and spaces, one can obtain several known and new inertial iterative schemes for solving harmonic bivariational inequalities and related optimization problems. Using the technique of Theorem 12.5, one can consider the convergence analysis of these Algorithms.*

14 Exponentially Mixed Variational Inequalities

In this section, we introduce and consider the exponentially mixed variational inequalities. First f all, we recall the concepts of the exponentially convex functions, which are mainly due to Noor and Noor [86–88].

Definition 14.1. *[7, 86–88] A function F is said to be exponentially convex function, if*

$$e^{F((1-t)u+tv)} \leq (1-t)e^{F(u)} + te^{F(v)}, \quad \forall u, v \in K, \quad t \in [0,1].$$

We remark that Definition 14.1 can be rewritten in the following equivalent way, which is due to Antczak [7].

Definition 14.2. *A function F is said to be exponentially convex function, if*

$$F((1-t)u+tv)) \leq \log[(1-t)e^{F(u)} + te^{F(v)}], \quad \forall u, v \in K, \quad t \in [0,1].$$

A function is called the exponentially concave function f, if $-f$ is exponentially convex function.

It is obvious that two concepts are equivalent. This equivalent have been used to discuss various aspects of the exponentially convex functions. It is worth mentioning that one can also deduce the concept of exponentially convex functions from r-convex functions, which were considered by Avriel [9] and Bernstein [15].

For the applications of the exponentially convex functions in the mathematical programming and information theory, see Antczak [7], Alirezaei and Mathar [1] and Pal et al. [98]. For the applications of the exponentially concave function in the communication and information theory, we have the following example.

Example 14.1. *[1] The error function*

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt,$$

becomes an exponentially concave function in the form $erf(\sqrt{x}), \quad x \geq 0$, *which describes the bit/symbol error probability of communication systems depending on the square root of the underlying signal-to-noise ratio. This shows that the exponentially concave functions can play important part in communication theory and information theory.*

Definition 14.3. *A function F is said to be exponentially affine convex function, if*

$$e^{F((1-t)u+tv))} = (1-t)e^{F(u)} + te^{F(v)}, \quad \forall u, v \in K, \quad t \in [0,1].$$

Definition 14.4. *The function F on the convex set K is said to be exponentially quasi convex, if*

$$e^{F(u+t(v-u))} \leq \max\{e^{F(u)}, e^{F(v)}\}, \quad \forall u, v \in K, t \in [0,1].$$

From the above definitions, we have

$$\begin{aligned} e^{F(u+t(v-u))} &\leq (1-t)e^{F(u)} + te^{F(v)}) \\ &\leq \max\{e^{F(u)}, e^{F(v)}\}. \end{aligned}$$

This shows that every exponentially convex function is a exponentially quasi-convex function. However, the converse is not true.

Let $I = [a, b]$ be the interval. We now define the exponentially convex functions on I.

Definition 14.5. *A F is exponentially convex function on the interval $I = [a, b]$, if and only if,*

$$\begin{vmatrix} 1 & 1 & 1 \\ a & x & b \\ e^{F(a)} & e^{F(x)} & e^{F(b)} \end{vmatrix} \geq 0; \quad a \leq x \leq b.$$

One can easily show that the following are equivalent:

1. F is exponentially convex function.
2. $e^{F(x)} \leq e^{F(a)} + \frac{e^{F(b)}-e^{F(a)}}{b-a}(x-a)$.
3. $\frac{e^{F(x)}-e^{F(a)}}{x-a} \leq \frac{e^{F(b)}-e^{F(a)}}{b-a}$.
4. $(b-x)e^{F(a)} + (a-b)e^{F(x)} + (x-a)e^{F(b)}) \geq 0$.
5. $\frac{F(a)}{(b-a)(a-x)} + \frac{e^{F(x)}}{(x-b)(a-x)} + \frac{e^{F(b)}}{(b-a)(x-b)} \leq 0$,

where $x = (1-t)a + tb \in [0,1]$.

We discuss the optimality condition for the differentiable exponentially convex functions using the technique of Noor and Noor [86, 87], which is the main motivation of our next result.

Theorem 14.1. *Let F be a differentiable exponentially convex function. Then $u \in K$ is the minimum of the function F, if and only if, $u \in K$ satisfies the inequality*

$$\langle e^{F(u)}F'(u), v-u\rangle \geq 0, \quad \forall u, v \in K. \tag{14.1}$$

Proof. Let $u \in K$ be a minimum of the function F. Then

$$F(u) \leq F(v), \forall v \in K.$$

from which, we have

$$e^{F(u)} \leq e^{F(v)}, \forall v \in K. \tag{14.2}$$

Since K is a convex set, so, $\forall u, v \in K, \quad t \in [0,1]$,

$$v_t = (1-t)u + tv \in K.$$

Taking $v = v_t$ in (14.2), we have

$$0 \leq \lim_{t\to 0}\{\frac{e^{F(u+t(v-u))} - e^{F(u)}}{t}\} = \langle e^{F(u)}F'(u), v-u\rangle, \tag{14.3}$$

the required (14.1).

Conversely, assume that (14.1) holds. We have to show that $u \in K$ is the minimum of the exponentially convex function F. Since F is differentiable exponentially convex function, so

$$e^{F(u+t(v-u))} \leq e^{F(u)} + t(e^{F(v)} - e^{F(u)}, \quad u, v \in K, t \in [0,1],$$

from which, using (14.3), we have

$$e^{F(v)} - e^{F(u)} \geq \lim_{t\to 0}\{\frac{e^{F(u+t(v-u))} - e^{F(u)}}{t}\} = \langle e^{F(u)}F'(u), v-u\rangle \geq 0,$$

from which, we have

$$F(u) \leq F(v), \quad \forall v \in K.$$

This shows that $u \in K$ is the minimum of the differentiable exponentially convex function, the required result. □

The inequality of the type (14.1) is called the exponentially variational inequality. In many applications, the inequality of the type (14.1) may not arise as the optimality condition of the differentiable exponentially convex functions. This fact motivated us to introduce a more general variational inequality of which the inequality (14.1) is a special case.

For a given nonlinear operator $T : H \longrightarrow H$ and the function ϕ, consider the problem of finding $u \in h$, such that

$$\langle e^{Tu}, v-u\rangle + \phi(v) - \phi(u) \geq 0, \quad \forall v \in H, \tag{14.4}$$

which is called the exponentially mixed variational inequality.

We now consider some important special cases of the problem (14.4)'

1. Let the function ϕ be the indicator function of the closed convex set in H. Then problem(14.4) reduces to finding $u \in K$ such that

$$\langle e^{Tu}, v-u\rangle \geq 0, \quad \forall v \in K, \tag{14.5}$$

is called the exponentially variational inequality.

2. Clearly $e^{Tu} = e^{F(u)}F'(u) = \left(e^{F(u)}\right)'$, we obtain the inequality (14.5).

3. If K^* is the dual cone of the convex cone, then problem (14.5) is equivalent to finding $u \in K$ such that

$$e^{Tu} \in K^* \quad \text{and} \quad \langle e^{Tu}, u\rangle = 0, \tag{14.6}$$

which is called the exponentially complementarity problem and appears to be a new one.

4. If $e^{Tu} = \Phi(u)$, then the problem (14.4) is equivalent to fining $u \in H$ such that

$$\langle \Phi(u), v-u\rangle + \phi(v) - \phi(u) \geq 0, \quad \forall v \in H. \tag{14.7}$$

which is called the classical mixed variational inequality.

We now define some new concepts

Definition 14.6. *An exponentially operator T is said to:*

(i) exponentially monotone, if

$$\langle e^{Tu} - e^{Tv}, u-v\rangle \geq 0, \quad \forall u, v \in K.$$

(ii). exponentially strongly monotone, if there exists a constant $\eta > 0$ such that

$$\langle e^{Tu} - e^{Tv}, u-v\rangle \geq \eta\|u-v\|^2, \quad \forall u, v \in K.$$

(iii). Lipschitz continuous, if there exists a constant $\beta > 0$ *such that*

$$\|e^{Tu} - e^{Tv}\| \leq \beta\|u - v\|, \quad \forall u, v \in K.$$

(iv). exponentially pseudo monotone, if

$$\langle e^{Tu}, v - u\rangle \geq 0 \quad \Longrightarrow \quad \langle e^{Tv}, v - u\rangle \geq 0, \quad \forall u, v \in K.$$

(v). An operator $T : K \to H$ *is called exponentially hemicontinuous, if,* $\forall u, v \in K$*, the mapping* $t \in [0, 1]$ *implies that* $\langle e^{T(u+t(v-u))}, v-u\rangle$ *is continuous.*

14.1 Iterative methods and convergence

If the function $\phi(.)$ in the exponentially mixed variational inequality (14.4) is not lower-semicontinuous, then one cannot show that the mixed variational inequality (14.4) is not equivalent the fixed point problem. To overcome this drawback, we suggest and analyze some iterative methods for solving the exponentially mixed variational inequality (14.4) using the auxiliary principle technique.

For a given $u \in H$ satisfying the exponentially mixed variational inequality (14.4)), consider the auxiliary problem of finding $w \in H$ such that

$$\langle \rho e^{T(\zeta w+(1-\zeta)u)}, v - w\rangle + \langle w - u + +\alpha(u - u)), v - w\rangle$$
$$+\rho\phi(v) - \rho\phi(w) \geq 0, \quad \forall v \in H, \tag{14.8}$$

where $\rho > 0$, α and $\zeta \in [0, 1]$ are constants. Problem (14.8) is known as the auxiliary exponentially mixed variational inequality. We note that, if $w = u$, then clearly w is a solution of the problem (14.4). This observation enables us to suggest and analyze the following iterative method for solving the problem(14.4).

Algorithm 14.1. *For a given* $u_0, u_1 \in H$*, compute the approximate solution* u_{n+1} *by the iterative scheme*

$$\langle \rho e^{T(\zeta u_{n+1}+(1-\zeta)u_n)}, v - u_{n+1}\rangle$$
$$+\langle u_{n+1} - u_n + \alpha(u_n - u_{n-1}), v - u_{n+1}\rangle + \rho(\phi(v) - \phi(u_{n+1})) \geq 0, \quad \forall v \in H,$$

where $\rho > 0$ *is a constants.*

Algorithm 14.14 is an hybrid implicit inertial method for solving the problem (14.4).

If $\zeta = \frac{1}{2}$ then Algorithm 14.14 reduces to

Algorithm 14.2. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\langle \rho e^{T(\frac{u_{n+1}+u_n}{2})}, v - u_{n+1}\rangle$$
$$+\langle u_{n+1} - u_n + \alpha(u_n - u_{n-1}), v - u_{n+1}\rangle + \phi(v) - \phi(u_{n+1}) \geq 0, \quad \forall v \in H$$

which is known as the mid point proximal inertial method for solving the exponentially mixed variational inequalities.

If $\zeta = 0$, then Algorithm 14.14 collapses to:

Algorithm 14.3. *For a given $u_0, u_1 \in H$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\langle \rho e^{Tu_n}, v - u_{n+1}\rangle + \langle u_{n+1} - u_n + \alpha(u_n - u_{n-1}), v - u_{n+1}\rangle$$
$$+\rho\phi(v) - \rho\phi(u_{n+1}) \geq 0, \quad \forall v \in H,$$

which is known as implicit method.

For $\zeta = 1$, Algorithm 14.14 becomes

Algorithm 14.4. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative schemes*

$$\rho\langle e^{Tu_{n+1}}, v - u_{n+1}\rangle + \langle u_{n+1} - u_n) + \alpha(u_n - u_{n-1}), v - u_{n+1}\rangle$$
$$+\rho\phi(v) - \rho\phi(u_{n+1}) \geq 0, \quad \forall v \in H. \tag{14.9}$$

Algorithm 14.4 is called the inertial proximal method for solving the problem (14.4). For $\alpha = 0$, Algorithm 14.4 becomes:

Algorithm 14.5. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative schemes*

$$\rho\langle e^{Tu_{n+1}}, v - u_{n+1}\rangle + \langle u_{n+1} - u_n), v - u_{n+1}\rangle + \rho\phi(v) - \rho\phi(u_{n+1}) \geq 0, \quad \forall v \in H.$$

We now study the convergence criteria of Algorithm 14.4 under suitable conditions.

Theorem 14.2. *Let $u \in H$ be a solution of (14.4) and let u_{n+1} be the approximate solution obtained from Algorithm 14.4. If the $T : H \longrightarrow R$ is pseudo exponentially monotone with respect to the function ϕ, then*

$$\begin{aligned} \|u_{n+1} - u\|^2 \leq \; & \|u_n - u\|^2 - \|u_{n+1} - u_n - \alpha_n(u_n - u_{n-1})\|^2 \\ & +\alpha_n\{\|u_n - u\|^2 - \|u_{n-1} - u\|^2 + 2\|u_n - u_{n-1}\|^2\}. \end{aligned} \quad (14.10)$$

Proof. Let $u \in H$ be a solution of (14.4). Then

$$\langle e^{Tu}, v - u\rangle + \phi(v) - \phi(u) \geq 0, \quad \forall v \in H,$$

implies that

$$-\langle e^{Tv}, u - v\rangle + \phi(v) - \phi(u) \geq 0 \quad \forall v \in H, \quad (14.11)$$

since T is pseudo exponentially monotone.

Taking $v = u_{n+1}$ in (14.11), we have

$$\langle e^{Tu_{n+1}}, u - u_{n+1}\rangle + \phi(u_{n+1}) - \phi(u) \geq 0. \quad (14.12)$$

Now taking $v = u$ in (14.9), we obtain

$$\begin{aligned} & \langle \rho e^{Tu_{n+1}}, u - u_{n+1}\rangle + \langle u_{n+1} - u_n - \alpha_n(u_n - u_{n-1}), u - u_{n+1}\rangle \\ & +\rho\phi(u) - \rho\phi(u_{n+1}) \geq 0. \end{aligned} \quad (14.13)$$

From (14.12) and (14.13), we have

$$\langle u_{n+1} - u_n - \alpha_n(u_n - u_{n-1}), \bar{u} - u_{n+1}\rangle \geq -\langle \rho e^{Tu_{n+1}}, \beta(u - u_{n+1})\rangle \geq 0. (14.14)$$

One can write (14.13) in the form

$$\langle u_{n+1} - u_n, \beta(-u_{n+1})\rangle \geq \alpha_n \langle u_n - u_{n-1}, u - u_n + u_n - u_{n+1}\rangle. \qquad (14.15)$$

Using the inequality $2\langle u, v\rangle = \|u+v\|^2 - \|u\|^2 - \|v\|^2, \forall u, v \in H$ and rearranging the terms in (14.15), one can easily obtain the required result 14.10. □

Theorem 14.3. *Let H be a finite dimensional space. Let u_{n+1} be the approximate solution obtained from Algorithm 14.4 and $\bar{u} \in H$ be a solution of (14.4). If there exists $\alpha \in (0,1)$ such that $0 \leq \alpha_n \leq \alpha, \quad \forall n \in N$ and*

$$\sum_{n=1}^{\infty} \alpha_n \|u_n - u_{n-1}\|^2 \leq \infty,$$

then $\lim_{n \longrightarrow \infty} u_n = u$.

Proof. Let $u \in H$ be a solution of (14.4). First we consider the case $\alpha_n = 0$. In this case, we see from 14.10 that the sequence $\{\|u - u_n\|\}$ is nonincreasing and consequently $\{u_n\}$ is bounded. Also from (14.10), we have

$$\sum_{n=0}^{\infty} \|u_{n+1} - u_n\|^2 \leq \|u_0 - u\|^2,$$

which implies that

$$\lim_{n \longrightarrow \infty} \|u_{n+1} - u_n\| = 0. \qquad (14.16)$$

Let $\hat{u}$ be the cluster point of $\{u_n\}$ and the subsequence $\{u_{n_j}\}$ of the sequence $\{u_n\}$ converge to $\hat{u} \in H$. Replacing u_n by u_{n_j} in (14.9) and taking the limit $n_j \longrightarrow \infty$ and using (14.16), we have

$$\langle e^{T\hat{u}}, v - \hat{u}\rangle + \phi(v) - \phi(\hat{u}) \geq 0, \quad \forall v \in H,$$

which implies that $\hat{u}$ solves the exponentially variational inequality (14.4) and

$$\|u_{n+1} - u_n\|^2 \leq \|u_n - \bar{u}\|^2.$$

Thus it follows from the above inequality that the sequence $\{u_n\}$ has exactly one cluster point $\hat{u}$ and $\lim_{n\longrightarrow\infty} u_n = \hat{u}$.

Now we consider the case $\alpha_n > 0$. From (14.10), we have

$$\sum_{n+1}^{\infty} \|u_{n+1} - u_n - \alpha_n(u_n - u_{n-1})\|^2 \leq \|u_0 - u\|^2 + \sum_{n=1}^{\infty}\{\alpha\|u_n - u\|^2 + 2\|u_n - u_{n-1}\|^2\} \leq \infty,$$

which implies that

$$\lim_{n\longrightarrow\infty} \|u_{n+1} - u_n - \alpha_n(u_n - u_{n-1})\|^2 = 0.$$

Repeating the above arguments as in the case $\alpha_n = 0$, one can easily show that

$$\lim_{n\rightarrow\infty} u_n = \hat{u},$$

the required result. □

For the readers convenience and for the sake of completeness, we recall some basic properties of the Bregman convex functions. For strongly convex function F, we define the Bregman distance function as

$$B(v, u) = F(v) - F(u) - \langle F'(u), v - u\rangle \geq \alpha\|v - u\|^2, \forall u, v \in K. \tag{14.17}$$

It is important to emphasize that various types of function F gives different Bregman distance. We give the following important examples of some practical important types of function F and their corresponding Bregman distance.

Examples

1. If $f(v) = \|v\|^2$, then $B(v, u) = \|v - u\|^2$, which is the squared Euclidean distance.

2. If $f(v) = \sum_{i=1}^{n} a_i \log(v_i)$, which is known as Shannon entropy, then its corresponding Bregman distance is given as
$$B(v,u) = \sum_{i=1}^{n} \Big(v_i \log(\frac{v_i}{u_i}) + u_i - v_i \Big).$$
This distance is called Kullback Leibler distance (KL) and as become a very important tool in several areas of applied mathematics such as machine learning.

3. If $f(v) = -\sum_{i=1}^{n} \log(v_i)$, which is called Burg entropy, then its corresponding Bregman distance is given as
$$B(v,u) = \sum_{i=1}^{n} \Big(\log(\frac{v_i}{u_i}) + \frac{v_i}{u_i} - 1 \Big).$$
This is called Itakura Saito distance (IS),which is very important in the information theory, data analysis and machine learning.

For a given $u \in H$ satisfying (14.4), we consider the auxiliary exponentially variational inequality problem of finding $w \in H$, such that

$$\langle \rho e^{T(w)}, v - w\rangle + \langle E'(w) - E'(u), v - w\rangle + \rho\phi(v) - \rho\phi(w) \geq 0, \quad \forall v \in H, \tag{14.18}$$

where $\rho > 0$ is a constant and $E'(u)$ is the differential of a strongly convex function E at $u \in H$. From the strongly convexity of the differentiable function $E(u)$, it follows that problem (14.18) has a unique solution.

It is clear that if $w = u$, then w is a solution of problem (14.4). This observation enables to suggest and analyze the following iterative method for solving (**??**).

Algorithm 14.6. *For a given $u_0 \in H$, calculate the approximate solution u_{n+1} by the iterative scheme*

$$\begin{aligned} &\langle \rho e^{Tu_{n+1}}, v - u_{n+1}\rangle + \langle E'(u_{n+1}) - E'(u_n), v - u_{n+1}\rangle \\ &\quad + \rho\phi(v) - \rho\phi(u_{n+1}) \geq 0, \quad \forall v \in H, \end{aligned} \tag{14.19}$$

where $\rho > 0$ is a constant.

We now study the convergence criteria of Algorithm 14.6 and this is the main motivation of next result.

Theorem 14.4. *Let the operator T be general monotone and let $E(u)$ be strongly convex function with modulus $\beta > 0$. Then the approximate solution u_{n+1} obtained from Algorithm 14.6 converges to a solution $u \in H$ of the problem (14.4).*

Proof. Let $u \in K$ be a solution of (14.4). Then, using the exponentially monotonicity of T, we have

$$-\langle e^{Tv}, u - v\rangle + \phi(v) - \phi(u) \geq 0, \quad \forall v \in H. \tag{14.20}$$

Taking $v = u_{n+1}$ in (14.20) and $v = u$ in (14.19), we have

$$-\langle e^{Tu_{n+1}}, u - u_{n+1}\rangle + \phi(u_{n+1}) - \phi(u) \geq 0 \tag{14.21}$$

$$\langle \rho e^{Tu_{n+1}}, u - u_{n+1}\rangle + \langle E'(u_{n+1}) - E'(u_n), u - u_{n+1}\rangle$$
$$+\rho(\phi(u) - \phi(u_{n+1})) \geq 0. \tag{14.22}$$

Now we consider the generalized Bergman function as

$$B(u, z) = E(u) - E(z) - \langle E'(z), u - z\rangle \geq \beta\|u - z\|^2, \tag{14.23}$$

where we have used the fact that the function $E(u)$ is strongly convex.

Combining (14.21), (14.22) and (14.23), we have

$$\begin{aligned} B(u, u_n) - B(u, u_{n+1}) &= E(u_{n+1}) - E(u_n) - \langle E'(u_n), u - u_n\rangle \\ &\quad +\langle E'(u_{n+1}), u - u_{n+1}\rangle \\ &= E(u_{n+1}) - E(u_n) - \langle E'(u_n) - E'(u_{n+1}), u - u_{n+1}\rangle \\ &\quad -\langle E'(u_n), u_{n+1} - u_n\rangle \\ &\geq \beta\|u_{n+1} - u_n\|^2 + \langle E'(u_{n+1}) - E'(u_n), u - u_{n+1}\rangle \\ &\geq \beta\|u_{n+1} - u_n\|^2 - \langle \rho e^{Tu_{n+1}}, u - u_{n+1}\rangle \\ &\geq \beta\|u_{n+1} - u_n\|^2. \end{aligned}$$

If $u_{n+1} = u_n$, then clearly u_n is a solution of (14.4). Otherwise, for $\beta > 0$, the sequences $B(u, u_n) - B(u, u_{n+1})$ is nonnegative and we must have

$$\lim_{n\to\infty} (\|u_{n+1} - u_n\|) = 0.$$

It follows that the the sequence $\{u_n\}$ is bounded. Let $\bar{u}$ be a cluster point of the subsequence $\{u_{n_i}\}$, and let $\{u_{n_i}\}$ be a subsequence converging toward $\bar{u}$. Now using the technique of Zhu and Marcotte [115], it can be shown that the entire sequence $\{u_n\}$ converges to the cluster point $\bar{u}$ satisfying (14.4). □

We now consider another iterative method for solving the exponentially variational inequality (14.4).

For a given $u \in H$ satisfying (14.4), we consider the auxiliary exponentially variational inequality problem of finding $w \in H$, such that

$$\langle \rho e^{Tu}, v - w\rangle + \langle E'(w) - E'(u), v - w\rangle + \rho(\phi(v) - \phi(w)) \geq 0, \quad \forall v \in H, \tag{14.24}$$

where $\rho > 0$ is a constant and $E'(u)$ is the differential of a strongly convex function E at $u \in H$. From the strongly convexity of the differentiable function $E(u)$, it follows that problem (14.24) has a unique solution. It is clear that if $w = u$, then w is a solution of problem (14.4). This observation enables to suggest and analyze the following iterative method for solving (14.4).

Algorithm 14.7. *For a given $u_0 \in H$, calculate the approximate solution u_{n+1} by the iterative scheme*

$$\langle \rho e^{Tu_n}, v - u_{n+1}\rangle + \langle E'(u_{n+1}) - E'(u_n), v - u_{n+1}\rangle + \rho(\phi(v) - \phi(u_{n+1})) \geq 0, \quad \forall v \in H,$$

which is called the explicit method for solving the exponentially variational inequality (14.4).

For a given $u \in H$ satisfying (14.4), we consider the auxiliary exponentially variational inequality problem of finding $w \in H$, such that

$$\langle \rho e^{T(\zeta w + (1-\zeta)u)}, v - w\rangle + \langle E'(w) - E'(u), v - w\rangle$$
$$+\rho(\phi(v) - \phi(w)) \geq 0, \quad \forall v \in H, \tag{14.25}$$

where $\rho > 0$ is a constant and $E'(u)$ is the differential of a strongly convex function. It is clear that if $w = u$, then w is a solution of problem (14.4). This observation enables to suggest and analyze the following iterative method for solving (14.4).

Algorithm 14.8. *For a given $u_0 \in H$, calculate the approximate solution u_{n+1} by the iterative scheme*

$$\langle \rho e^{T(\zeta u_{n+1}+(1-\zeta)u_n}, v - u_{n+1}\rangle + \langle E'(u_{n+1}) - E'(u_n), v - u_{n+1}\rangle \\ +\rho(\phi(v) - \phi(u_{n+1})) \geq 0, \quad \forall v \in H, \tag{14.26}$$

which is called the hybrid implicit iterative method for solving the exponentially variational inequality (14.4).

For different and suitable choice of the parameter ζ, one can some new iterative methods for approximating the solution of the exponentially variational inequalities. All these method do not do not require the evaluation of the projection and its variant form. Using the technique of this paper, one can study the convergence of other iterative methods.

We now consider the case, if the function ϕ in the mixed variational inequality (14.4) is lower-semicontinuous. Applying Lemma 2.1, we can prove that the mixed variational inequality (14.4) is equivalent to the fixed point problem.

Theorem 14.5. *The element $u \in H$ be the solution of the problem (14.4), if and only if, $u \in H$ satisfies*

$$u = J_\phi[u - \rho e^{Tu}]. \tag{14.27}$$

Theorem 14.5 implies that the exponentially mixed variational inequality (14.4) is equivalent to the fixed point problem (14.27). This equivalent fixed point formulation (14.27) will play an important role in deriving the main results.

We define the function F associated with (14.27) as

$$F(u) = J_\phi[u - \rho e^{Tu}]. \tag{14.28}$$

To prove the unique existence of the solution of the problem (14.4), it is enough to show that the map F defined by (14.28) has a fixed point.

Theorem 14.6. *Let the operator $\mathcal{T}$ be strongly monotone with constants $\alpha > 0$ and Lipschitz continuous with constants $\beta > 0$, respectively. If there exists a parameter $\rho > 0$, such that*

$$\rho < \frac{2\alpha}{\beta^2}, \tag{14.29}$$

then there exists a unique solution of the problem (14.4).

Proof. From Theorem 14.5, it follows that problems (14.4) and (14.27) are equivalent. Thus it is enough to show that the map $F(u)$, defined by (14.28) has a fixed point.

For all $u \neq u \in \Omega(\mu)$, we have

$$\begin{aligned} \|F(u) - F(v))\| &= \|J_\phi[u - \rho e^{Tu}] - J_\phi[v - \rho e^{Tv}]\| \\ &\leq \|u - v - \rho(e^{Tu} - e^{Tv})\|. \end{aligned} \tag{14.30}$$

Since the operator $\mathcal{T}$ is strongly monotone with constant $\alpha > 0$ and Lipschitz continuous with constant $\beta > 0$, respectively, it follows that

$$\begin{aligned} \|u - v - \rho(e^{Tu} - e^{Tv})\|^2 &\leq \|u - v|^2 - 2\langle u - v\rho(e^{Tu} - e^{Tv})\rangle + \rho^2\|e^{Tu} - e^{Tv}\|^2 \\ &\leq (1 - 2\alpha\rho + \rho^2\beta^2)\|u - v\|^2. \end{aligned} \tag{14.31}$$

From (14.30) and (14.31), we have

$$\begin{aligned} \|F(u) - F(v)\| &\leq \sqrt{(1 - 2\alpha\rho + \rho^2\beta^2)}\|u - v\| \\ &= \theta\|u - v\|, \end{aligned}$$

where

$$\theta = \sqrt{(1 - 2\alpha\rho + \rho^2\beta^2)}$$

From (14.29), it follows that $\theta < 1$, which implies that the map $F(u)$ defined by (14.27) has a fixed point, which is the unique solution of (14.4). □

The fixed point formulation (14.27) is applied to propose and suggest the iterative methods for solving the problem (14.4).

This alternative equivalent formulation (14.27) is used to suggest the following iterative methods for solving the problem (14.4) using the updating technique of the solution.

Algorithm 14.9. *For a given $\mu_0 \in \mathcal{H}$, compute the approximate solution $\{\mu_{n+1}\}$ by the iterative schemes*

$$y_n = J_\phi[u_n - \rho e^{Tu_n}] \tag{14.32}$$

$$w_n = J_\phi[y_n - \rho e^{Ty_n}] \tag{14.33}$$

$$\mu_{n+1} = J_\phi[w_n - \rho e^{Tw_n}]. \tag{14.34}$$

Algorithm 14.9 is a three step forward-backward splitting algorithm for solving exponentially mixed variational inequality (14.4). This method is very much similar to that of Glowinski et al. [28] for variational inequalities, which they suggested by using the Lagrangian technique.

We now study the convergence analysis of Algorithm 14.9, which is the main motivation of our next result.

Theorem 14.7. *Let the operator $\mathcal{T}$ satisfy all the assumptions of Theorem 14.6. Then the approximate solution $\{u_n\}$ obtained from Algorithm 14.9 converges to the exact solution $u \in H$) of the exponentially mixed variational inequality (14.4) strongly in H.*

Proof. From Theorem 14.5, we see that there exists a unique solution $u \in H$) of the exponentially mixed variational inequalities (14.4). Let $u \in H$ be the unique solution of (14.4). Then, using Lemma 3.1, we have

$$u = J_\phi[u_n - \rho e^{Tu_n}] \tag{14.35}$$

$$= J_\phi[u_n - \rho e^{Tu_n}] \tag{14.36}$$

$$= J_\phi[u_n - \rho e^{Tu_n}]. \tag{14.37}$$

From (14.34) and (14.35), we have

$$\begin{aligned} \|u_{n+1} - u\| &= J_\phi[w_n - \rho e^{Tw_n}] - J_\phi[u_n - \rho e^{Tu_n}]\| \\ &\leq \|w_n - u - \rho(e^{w_n} - e^{Tu})\| \\ &\leq \theta\|w_n - u\|, \end{aligned} \tag{14.38}$$

where $\theta = \sqrt{1 - 2\alpha\rho + \rho^2\beta^2}$.

In a similar way, from (14.32) and (14.36), we have

$$\|w_n - u\| \leq \theta\|y_n - u - (e^{Ty_n} - e^{Tu})). \tag{14.39}$$

From (14.32) and (14.37), we obtain

$$\|y_n - \mu\| \leq \theta\|u_n - u\|. \tag{14.40}$$

From (14.39) and (14.40), we obtain

$$\|w_n - u\| \leq \theta\|u_n - u\|. \tag{14.41}$$

Form the above we equations, have

$$\|u_{n+1} - u\| \leq \theta\|u_n - u\|.$$

From (14.29), it follows that $\theta < 1$, Consequently the sequence $\{u_n\}$ converges strongly to μ. From (14.40), and (14.41), it follows that the sequences $\{y_n\}$ and $\{w_n\}$ also converge to μ strongly in H. This completes the proof. □

We now suggested and analyzed the three step scheme for solving the exponentially mixed variational inequality (14.4). These three step schemes also are called the novel Noor iterations. For the applications of novel Noor iterations in signal recovery, polynomiography, fixed point theory, compress programming, nonlinear equations, compressive sensing and image in painting, see [12–14, 22–24, 40, 44, 95, 101, 103] and the references therein.

Algorithm 14.10. *For a given $\mu_0 \in \mathcal{H}$, compute the approximate solution $\{\mu_{n+1}\}$ by the iterative schemes*

$$\begin{aligned} y_n &= (1-\gamma_n)u_n + \gamma_n J_\phi[u_n - \rho e^{Tu_n}] \\ w_n &= (1-\beta_n)y_n + \beta_n J_\phi[y_n - \rho(e^{Ty_n} + e^{Tu_n})] \\ \mu_{n+1} &= (1-\alpha_n)w_n + \alpha_n J_\phi[w_n - \rho(e^{Tw_n} + e^{Ty_n})]. \end{aligned}$$

Convergence analysis of Algorithm 14.10 can be studied using the techniques as developed in [25, 53, 110, 114]. For $\gamma_n = 0$, Algorithm 14.10 reduces to:

Algorithm 14.11. *For a given $\mu_0 \in \Omega(\mu)$, compute $\{\mu_{n+1}\}$ by the iterative schemes*

$$\begin{aligned} w_n &= (1-\beta_n)u_n + \beta_n J_\phi[u_n - \rho e^{Tu_n}] \\ u_{n+1} &= (1-\alpha_n)w_n + \alpha_n J_\phi[w_n - \rho(e^{Tw_n} + e^{Tu_n})], \end{aligned}$$

which is known as the Ishikawa iterative scheme for the problem (14.4).

Note that for $\gamma_n = 0$ *and* $\beta_n = 0$, Algorithm 14.10 is called the Mann iterative method, that is.

Algorithm 14.12. *For a given $\mu_0 \in \Omega(\mu)$, compute $\{\mu_{n+1}\}$ by the iterative schemes*

$$u_{n+1} = (1-\beta_n)u_n + \beta J_\phi[u_n - \rho e^{Tu_n}].$$

We suggest another perturbed iterative scheme for solving the exponentially mixed variational inequality (14.4).

Algorithm 14.13. *For a given $\mu_o \in \mathcal{H}$, compute the approximate solution $\{\mu_n\}$ by the iterative schemes*

$$\begin{aligned} y_n &= (1-\gamma_n)u_n + \gamma_n J_\phi[u_n - \rho e^{Tu_n}] + \gamma_n h_n \\ w_n &= (1-\beta_n)y_n + \beta_n J_\phi[y_n - \rho e^{Ty_n}] + \beta_n f_n \\ \mu_{n+1} &= (1-\alpha_n)w_n + \alpha_n J_\phi[w_n - \rho e^{Tw_n}] + \alpha_n e_n, \end{aligned}$$

where $\{e_n\}$, $\{f_n\}$, *and* $\{h_n\}$ *are the sequences of the elements of* $\mathcal{H}$ *introduced to take into account possible inexact computations and* J_ϕ *is the corresponding perturbed resolvent operator and the sequences* $\{\alpha_n\}$, $\{\beta_n\}$ *and* $\{\gamma_n\}$ *satisfy*

$$0 \leq \alpha_n, \beta_n, \gamma_n \leq 1; \quad \forall n \geq 0, \quad \sum_{n=0}^{\infty} \alpha_n = \infty.$$

For $\gamma_n = 0$, we obtain the perturbed Ishikawa iterative method and for $\gamma_n = 0$ and $\beta_n = 0$, we obtain the perturbed Mann iterative schemes for solving exponentially mixed variational inequality (14.4).

Algorithm 14.14. *For a given* $u_0 \in H$, *compute* u_{n+1} *by the iterative scheme*

$$u_{n+1} = J_\phi[u_n - \rho e^{Tu_n}], \quad n = 0, 1, 2, ...$$

which is known as the projection method and has been studied extensively.

Algorithm 14.15. *For a given* $u_0 \in H$, *compute* u_{n+1} *by the iterative scheme*

$$u_{n+1} = J_\phi[u_n - \rho e^{Tu_{n+1}}], \quad n = 0, 1, 2, ...$$

which is known as the implicit projection method and is equivalent to the following two-step method.

Algorithm 14.16. *For a given* $u_0 \in H$, *compute* u_{n+1} *by the iterative scheme*

$$\begin{aligned} w_n &= J_\phi[u_n - \rho e^{Tu_n}] \\ \mu_{n+1} &= J_\phi[u_n - \rho e^{Tw_n}], \quad n = 0, 1, 2, ... \end{aligned}$$

We also propose the following iterative method.

Algorithm 14.17. *For a given* $\mu_0 \in \Omega(\mu)$, *compute* μ_{n+1} *by the iterative scheme*

$$u_{n+1} = J_\phi[u_{n+1} - \rho e^{Tu_{n+1}}], \quad n = 0, 1, 2, ...$$

which is known as the modified resolvent method and is equivalent to the iterative method.

Algorithm 14.18. *For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme*

$$\begin{aligned} w_n &= J_\phi[u_n - \rho e^{Tu_n}] \\ \mu_{n+1} &= J_\phi[w_n - \rho e^{Tw_n}], \quad n = 0, 1, 2, ... \end{aligned}$$

which is two-step predictor-corrector method for solving the problem (14.4).

We can rewrite the equation (14.27) as:

$$u = J_\phi[\frac{u+u}{2} - \rho e^{Tu}] \tag{14.42}$$

This fixed point formulation is used to suggest the following implicit method.

Algorithm 14.19. *[61]. For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme*

$$u_{n+1} = J_\phi[\frac{u_{n+1} + u_n}{2} - \rho e^{Tu_{n+1}}] \tag{14.43}$$

Applying the predictor-corrector technique, we suggest the following inertial iterative method for solving the problem (14.4).

Algorithm 14.20. *For a given $\mu_0 \in \Omega(\mu)$, compute μ_{n+1} by the iterative scheme*

$$\begin{aligned} w_n &= J_\phi[\frac{u+u}{2} - \rho e^{Tu}] \\ u_{n+1} &= J_\phi[\frac{w_n + u_n}{2} - \rho e^{Tw_n}]. \end{aligned}$$

From equation (14.27), we have

$$\mu = J_\phi[u - \rho e^{(\frac{u+u}{2})}]. \tag{14.44}$$

This fixed point formulation (14.27) is used to suggest the implicit method for solving the problem (14.4) as

Algorithm 14.21. *For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme*

$$u_{n+1} = J_\phi[u_n - \rho e^{T(\frac{u_n + u_{n+1}}{2})}]. \tag{14.45}$$

We can use the predictor-corrector technique to rewrite Algorithm 14.21 as:

Algorithm 14.22. *For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme*

$$\begin{aligned} w_n &= J_\phi u_n - \rho e^{Tu_n}], \\ u_{n+1} &= J_\phi[u_n - \rho e^{T(\frac{u_n+w_n}{2})}]. \end{aligned}$$

is known as the mid-point implicit method for solving the problem (14.4).

We again use the above fixed formulation to suggest the following implicit iterative method.

Algorithm 14.23. *For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme*

$$u_{n+1} = J_\phi[u_{n+1} - \rho e^{T(\frac{u_n+u_{n+1}}{2})}]. \tag{14.46}$$

Using the predictor-corrector technique, Algorithm 14.23 can be written as:

Algorithm 14.24. *For a given $\mu_0 \in H$, compute μ_{n+1} by the iterative scheme*

$$\begin{aligned} w_n &= J_\phi[u_n - \rho e^{Tu_n}], \\ u_{n+1} &= J_\phi[w_n - \rho e^{T(\frac{u_n+w_n}{2})}], \end{aligned}$$

which appears to be new one.

It is obvious that the above Algorithms have been suggested using different variant of the fixed point formulations (14.27). It is natural to combine these fixed point formulation to suggest a hybrid implicit method for solving the problem (14.4) and related optimization problems.

One can rewrite (14.27) as

$$u = J_\phi[\frac{u+u}{2} - \rho e^{T(\frac{u+u}{2})}]. \tag{14.47}$$

This equivalent fixed point formulation enables us to suggest the following implicit method for solving the problem (14.4).

Algorithm 14.25. *For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme*

$$u_{n+1} = J_\phi[\frac{u_n + u_{n+1}}{2} - \rho e^{T(\frac{u_n+u_{n+1}}{2})}]. \tag{14.48}$$

To implement the implicit method, one uses the predictor-corrector technique. We use Algorithm 14.17 as the predictor and Algorithm 14.25 as corrector. Thus, we obtain a new two-step method for solving the problem (14.4).

Algorithm 14.26. *For a given $\mu_0 \in H$, compute u_{n+1} by the iterative scheme*

$$\begin{aligned} w_n &= J_\phi[u_n - \rho e^{Tu_n}] \\ \mu_{n+1} &= J_\phi\Big[\frac{w_n + u_n}{2} - \rho e^{T(\frac{w_n+u_n}{2}}\Big], \end{aligned}$$

which is a new predictor-corrector two-step method.

For a parameter ξ, one can rewrite the (14.27) as

$$u = J_\phi[(1-\xi)u + \xi u) - \rho e^{Tu}].$$

This equivalent fixed point formulation enables to suggest the following inertial method for solving the problem (14.4).

Algorithm 14.27. *For a given $\mu_0, \mu_1 \in \Omega(\mu)$, compute μ_{n+1} by the iterative scheme*

$$\mu_{n+1} = J_\phi[(1-\xi)\mu_n + \xi\mu_{n-1} - \rho e^{Tu_n}], \quad n = 0, 1, 2, ...$$

It is noted that Algorithm 14.27 is equivalent to the following two-step method.

Algorithm 14.28. *For a given $u_0 \in H$, compute u_{n+1} by the inertial iterative scheme*

$$\begin{aligned} w_n &= (1-\xi)u_n + \xi u_{n-1} \\ \mu_{n+1} &= J_\phi[w_n - \rho e^{Tu_n}]. \end{aligned}$$

Using this idea, we can suggest the following iterative methods for solving exponentially mixed variational inequalities.

Algorithm 14.29. *For a given* $u_0, u_1 H$, *compute* u_{n+1} *by the iterative scheme*

$$u_{n+1} = (1-\alpha_n)u_n + \xi_n(u_n - u_{n-1}) + \alpha_n J_\phi[u_{n+1} - \rho e^{Tu_{n+1}}], \quad n = 0, 1, 2, ...$$

which is called the inertial proximal point method and appears to be new one.

Here $\alpha_n, \xi_n \geq 0$ are constants and term $\xi_n(u_n - u_{n-1})$ is called the inertial term.

Algorithm 14.30. *For a given* $u_0, \mu_1 \in \Omega(\mu)$, *compute* μ_{n+1} *by the iterative scheme*

$$\begin{aligned} y_n &= (1-\xi)u_n + \xi u_{n-1} \\ u_{n+1} &= u_n + y_n + J_\phi[y_n - \rho e^{Ty_n)}], \quad n = 0, 1, 2, ... \end{aligned}$$

We now suggest multi-step inertial methods for solving the exponentially mixed variational inequalities (14.4).

Algorithm 14.31. *For given* $\mu_0, \mu_1 \in \Omega(\mu)$, *compute* μ_{n+1} *by the recurrence relation*

$$\begin{aligned} w_n &= u_n - \Theta_n (u_n - u_{n-1}) \\ y_n &= (1-\beta_n)w_n + \beta_n J_\phi\left[\frac{w_n + u_n}{2} - \rho e^{T(\frac{w_n+u_n}{2})}\right], \\ u_{n+1} &= (1-\alpha_n)y_n + \alpha_n J_\phi\left[\frac{\omega_n + y_n}{2} - \rho e^{T(\frac{y_n+\omega_n}{2})}\right], \end{aligned}$$

where $\beta_n, \alpha_n, \Theta_n \in [0,1], \forall n \geq 1.$

Algorithm 14.31 is a three-step modified inertial method for solving exponentially variational inclusion (14.4).

Similarly a four-step inertial method for solving the exponentially mixed variational inequalities (14.4) is suggested.

Algorithm 14.32. *For given $\mu_0, \mu_1 \in \Omega(\mu)$, compute μ_{n+1} by the recurrence relation*

$$\begin{aligned} w_n &= u_n - \Theta_n (u_n - u_{n-1}), \\ t_n &= (1-\gamma_n)\omega_n + \gamma_n J_\phi\left[\frac{w_n + u_n}{2} - \rho e^{T(\frac{w_n+u_n}{2})}\right], \\ y_n &= (1-\beta_n)t_n + \beta J_\phi\left[\frac{t_n + w_n}{2} - \rho e^{T(\frac{t_n+w_n}{2})}\right], \\ u_{n+1} &= (1-\alpha_n)y_n + \alpha_n J_\phi\left[\frac{y_n + t_n}{2} - \rho e^{T(\frac{y_n+t_n}{2})}\right], \end{aligned}$$

where $\alpha_n, \beta_n, \gamma_n, \Theta_n \in [0,1], \quad \forall n \geq 1$.

One can use the use the resolvent method, dynamical systems and merit function method to suggest several new methods for solving the exponentially variational inequalities. We have only convey the main idea of the exponentially mixed variational inequalities.

Remark 14.1. *In this section, we have shown that the minimum of the differentiable exponentially convex functions can be characterized by a new class of inequalities, which is called the exponential mixed variational inequality. These facts motivated us to consider a class of exponentially mixed variational inequalities. Some important special cases are discussed as applications of the exponentially variational inequalities. Several multistep iterative methods for solving the exponentially variational inequalities are proposed and analyzed using the fixed point methods and auxiliary principle techniques. Convergence criteria of the proposed iterative methods considered under some suitable weak conditions. It is worth mentioning that the three step iterative methods proposed and investigated by Noor [58, 63] are known as Noor iterations, which contain Mann and Ishikawa iterations as special cases. For the applications, modifications and generalizations of Noor iterations, see [12–14, 22–24, 44, 46, 80, 95, 99, 101, 103, 112] and the references therein. For the novel applications of the Noor iterations in green innovations utilising fractal and power for solar panel optimization, see Natarajaan et al. [46] and for logistic map in Noor orbit, see Chugh et al. [24].*

One can explore the applications of the exponentially mixed variational inequalities The ideas and techniques paper may be starting point of further research activities in these dynamical fields.

15 Change of variable Method for variational inequalities

In this section, we consider the change of variable method for solving variational inequalities (2.2). This technique is mainly due to Noor [51] and Noor et al. [70]. For the sake of completeness and to convey the main idea, we include some details. We note that the complementarity problem (2.4) can be rewritten in the following form:

$$w = u \in K, \quad v = Tu \in K^*, \quad \langle Tu, u\rangle = 0. \tag{15.1}$$

which is useful in developing a fixed point formulation.

It is well known that, for $z \in H$, we have

$$z = P_K z + P_{-K^*} z = P_K z + P_{K^*}(-z). \tag{15.2}$$

Following the idea of Noor and Al-Said [70], we consider the following change of variables

$$w = u = \frac{|z| + z}{2} = z^+ = P_K(z) \tag{15.3}$$

and

$$v = \frac{|z| - z}{2\rho} = \rho^{-1} z^-. \tag{15.4}$$

From (15.2), (15.3) and (15.4), we have

$$u = P_K z \tag{15.5}$$

$$z = z^+ - z^- = P_K z + P_{K^*}(-z) = u - \rho T P_K z. \tag{15.6}$$

Combining (15.5) and (15.6), we obtain

$$u = P_K[u - \rho Tu]. \tag{15.7}$$

Thus, we have shown that the complementarity problem (2.4) is equivalent to the fixed point problem (15.7). This implies that $u \in K$ is the solution of the variational inequality (2.2). That is, $u \in K$ satisfies the inequality

$$\langle Tu, v - u\rangle \geq 0, \quad \forall v \in K. \tag{15.8}$$

In recent years, this technique have been used to develop modulus based methods for solving the system of absolute value equations, which is another area in numerical analysis and optimization. This approach can be extended for solving the mixed variational inequalities, which needs further research efforts.

16 Generalizations and Applications

We would like to mention that some of the results obtained and presented in this paper can be extended for more multivalued mixed variational inequalities. To be more precise, let $C(H)$ be a family of nonempty compact subsets of H. Let $T, V : H \longrightarrow C(H)$ be the multivalued operators. For a given nonlinear bifunction $N(.,.) : H \times H \longrightarrow H$ and operators $g, h : H \longrightarrow H$,consider the problem of finding $u \in \Omega(u), w \in T(u), y \in V(u)$ such that

$$\langle N(w, y), h(v) - g(u)\rangle + \phi(h(v)) - \phi(g(u)) \geq 0, \quad \forall \quad v \in H, \tag{16.1}$$

which is called the multivalued mixed general variational inequality. We would like to mention that one can obtain various classes of mixed variational inequalities for appropriate and suitable choices of the bifunction $N(.,.)$, the operators g, h, and convex-valued set $\Omega(u)$.

Note that, if $N(w, y) = Tu, \quad h = I, g = I$, then the problem (16.1) is equivalent to find $u \in H$, such that

$$\langle Tu, v - u\rangle + \phi(v) - \phi(u) \geq 0 \quad \forall v \in H,$$

which is exactly the mixed variational inequality (2.1).

Using Lemma 3.1, one can prove that the problem (16.1) is equivalent to finding $u \in H$ such that

$$g(u) = J_{\phi}[h(u) - \rho N(w, y)] \tag{16.2}$$

which can be written as

$$u = u - g(u) + J_{\phi}[h(u) - \rho N(w, y)].$$

Thus one can consider the mapping F associated with the problem (16.1) as

$$F(u) = u - g(u) + J_{\phi}[h(u) - \rho N(w, y)],$$

which can be used to discuss the uniqueness of the solution of the problem (16.1). From (16.1) and (16.2, it follows that the multivalued mixed general variational inequalities are equivalent to the fixed problems. Consequently, all results obtained for the problem (2.1) continue to hold for the problem (16.1) with suitable modifications and adjustments. The development of efficient implementable numerical methods for solving the multivalued mixed general variational inequalities and non optimization problems requires further efforts.

Conclusion

In this paper, we have reviewed the state-of-the-art in the theory, computation and applications of the mixed variational inequalities. The study of this area is a fruitful and growing field of intellectual endeavour. While our main aim in this study has been to describe the basic ideas and techniques which have been used to develop the up-to-date theory of the mixed variational inequalities, the foundation we have laid is quite broad and general. The general theories and results surveyed in this paper can be used to formulate variational principles and computational methods for a wide range of moving, free and equilibrium problems arising in fluid flow through porous media, elasticity, transportation science and economics. It is true that each of these areas of applications require special consideration of peculiarities of the physical problem at hand and the inequalities that model it. However, many of the concepts and techniques we have discussed are fundamental to all of these applications. There are several topics that we have not dealt with in this paper that pertain to mixed variational inequalities. Optimal shape optimization is a branch of the calculus of variations and has many practical applications in industry. In spite of their importance, little research has been carried out in this direction. This subject is very recent and offers great opportunities for further research. It is worth mentioning that this field has been continuing and will continue to foster new, innovative and novel applications in various branches of pure and applied sciences.

In this paper, we have used the equivalence between the mixed variational inequalities and fixed point formulation to suggest some new iterative methods for solving the mixed variational inequalities. Convergence analysis of the proposed method is investigated under suitable conditions. These new implicit methods include extragradient method and modified double projection methods as special cases. Some examples are given to illustrate the efficiency which shows that the proposed methods are robust and perform better than the known methods. Comparison of the proposed methods with other methods need further efforts. Dynamical system technique is also used to suggest iterative methods along with convergence criteria. The alternative fixed point approach is used to consider the sensitivity analysis of the mixed variational inequalities. Biconvex functions and mixed bivariational inequalities are also studied. Several classes of the mixed variational classes associated with harmonic convex, biconvex harmonic, exponentially convex functions are introduced and investigated. Fixed point, resolvent method, dynamical system and auxiliary principle techniques are the main tool to suggest iterative methods for solving mixed variational inequalities. Using the ideas and techniques of this paper, one can suggest and investigate several new implicit methods for solving various classes of variational inequalities and related problems. This paper is continuation of our previous research in [5, 19, 56, 60, 64, 72, 77–80, 90, 91, 91, 93]. One can extend these results for mixed general variational inequalities [5, 19, 59, 62, 64, 67, 71, 72, 77, 90, 91, 93] and related optimization problems applying the ideas and techniques developed in this paper with suitable and appropriate modifications.

Contributions of the authors:

All the authors contributed equally in writing, editing, reviewing and agreed for the final version for publication.

Data availability:

Data sharing not applicable to this article as no data sets were generated or analyzed during the current study

Conflict of Interest:

All authors have no conflict of interest.

Acknowledgement

We wish to express our deepest gratitude to our respected visionary teachers, students colleagues, collaborators, referees, reviewers and friends, who have direct or indirect contributions in the process of this paper.

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