Earthline Journal of Mathematical Sciences (ISSN: 2581-8147)
http://www.earthlinepublishers.com
Volume 1, Number 2, 2019, Pages 157-169
https://doi.org/10.34198/ejms.1219.157169

The Convex (δ, L) Weak Contraction Mapping Theorem and its Non-Self Counterpart in Graphic Language

Clement Boateng Ampadu

31 Carrolton Road, Boston, MA 02132-6303, USA; e-mail: drampadu@hotmail.com

Abstract

Let (X, d) be a metric space. A map $T : X \mapsto X$ is said to be a (δ, L) weak contraction [1] if there exists $\delta \in (0, 1)$ and $L \geq 0$ such that the following inequality holds for all $x, y \in X$:

$$d(Tx, Ty) \leq \delta d(x, y) + Ld(y, Tx).$$

On the other hand, the idea of convex contractions appeared in [2] and [3]. In the first part of this paper, motivated by [1]-[3], we introduce a concept of convex (δ, L) weak contraction, and obtain a fixed point theorem associated with this mapping. In the second part of this paper, we consider the map is a non-self map, and obtain a best proximity point theorem. Finally, we leave the reader with some open problems.

1. Introduction and Preliminaries

The higher-order fixed point theory [4] is inspired by [5]. In particular, the idea of higher-order Banach mapping was defined as follows:

Definition 1.1. [5] Let (X, d) be a metric space. A map $T : X \mapsto X$ is called an *rth-order Banach mapping* if for all $x, y \in X$, $0 \leq c_q < 1$, $0 \leq q \leq r - 1$, and $r \in \mathbb{N}$, the following inequality holds

Received: February 5, 2019; Accepted: February 25, 2019; Published online: February 26, 2019

2010 Mathematics Subject Classification: 47H10.

Keywords and phrases: (δ, L) weak contraction, convex contraction, non-self map, graph, fixed point theorem, best proximity point theorem.

$$d(T^r x, T^r y) \leq \sum_{q=0}^{r-1} c_q d(T^q x, T^q y)$$

with $\sum_{q=0}^{r-1} c_q < 1$.

Remark 1.2. A map $T : X \mapsto X$ is called a *convex contraction* [2]-[3], if $r = 2$ in the definition immediately above.

By these observations we introduce the following

Definition 1.3. Let (X, d) be a metric space. A map $T : X \mapsto X$ is called an *rth-order* (δ, L) *weak contraction mapping* if for all $x, y \in X$, $0 < \delta_q < 1$, $L_q \geq 0$, $0 \leq q \leq r - 1$, and $r \in \mathbb{N}$, the following inequality holds

$$d(T^r x, T^r y) \leq \sum_{q=0}^{r-1} \{\delta_q d(T^q x, T^q y) + L_q d(T^q y, T^{q+1} x)\}$$

with $\sum_{q=0}^{r-1} \delta_q < 1$.

Remark 1.4. If $r = 2$ in the definition immediately above, then we say $T : X \mapsto X$ is a convex (δ, L) weak contraction mapping. Note that if $r = 1$ in the above definition, then $T : X \mapsto X$ is a (δ, L) weak contraction [1].

Also we recall the following results associated with the (δ, L) weak contraction

Theorem 1.5. [1] *Let* (X, d) *be a complete metric space and* $T : X \mapsto X$ *be an almost contraction, that is, a mapping for which there exist a constant* $\delta \in [0, 1)$ *and some* $L \geq 0$ *such that for all* $x, y \in X$

$$d(Tx, Ty) \leq \delta d(x, y) + Ld(y, Tx).$$

Then

(a) $Fix(T) = \{x \in X : Tx = x\} \neq \emptyset$.

(b) *For any* $x_0 \in X$, *the Picard iteration* $\{x_n\}_{n=0}^{\infty}$ *given by* $x_{n+1} = Tx_n$, $n = 1, 2, \ldots$ *converges to some* $x^* \in Fix(T)$.

(c) *The following estimate holds*

$$d(x_{n+i-1}, x^*) \leq \frac{\delta^i}{1-\delta} d(x_{n-1}, x_n)$$

$n = 0, 1, 2, \ldots; i = 1, 2, \ldots.$

Theorem 1.6. [6] *Let* (X, d) *be a complete metric space and* $T : X \mapsto X$ *be a weak contraction for which there exist a constant* $\theta \in (0, 1)$ *and some* $L_1 \geq 0$ *such that for all* $x, y \in X$

$$d(Tx, Ty) \leq \theta d(x, y) + L_1 d(x, Tx).$$

Then

(a) *T has a unique fixed point, that is,* $F(T) = \{x^*\}$.

(b) *For any* $x_0 \in X$, *the Picard iteration* $\{x_n\}_{n=0}^{\infty}$ *given by* $x_{n+1} = Tx_n$, $n = 1, 2, \ldots$ *converges to* x^*.

(c) *The a priori and a posteriori error estimates holds*

$$d(x_n, x^*) \leq \frac{\delta^n}{1-\delta} d(x_0, x_1)$$

for $n = 0, 1, 2, \ldots;$

$$d(x_n, x^*) \leq \frac{\delta}{1-\delta} d(x_{n-1}, x_n)$$

for $n = 1, 2, \ldots.$

(d) *The rate of convergence of the Picard iteration is given by*

$$d(x_n, x^*) \leq \theta d(x_{n-1}, x^*)$$

for $n = 1, 2, \ldots.$

Now let W and V be two nonempty subsets of a metric space (X, d) and let $S : W \mapsto V$ be a non-self map. If $W \cap V$ is nonempty, then the equation $Sx = x$ may not have a solution. Naturally the following arises

Question 1.7. How far is the distance between *x* and *Sx*?

The problem of global optimization for determining the minimum value of the distance $d(x, Sx) = \min\{d(x, y) : x \in W \text{ and } y \in V\}$ is the study of best proximity point theory. Since the early paper of [7], many best proximity point theorems have been obtained, and for example see references [9-23] contained in [8].

Notation 1.8. Throughout this paper

(a) W and V will denote nonempty subsets of a metric space (X, d).

(b) $d(W, V) := \inf\{d(x, y) : x \in W \text{ and } y \in V\}$.

(c) $W_0 = \{x \in W : d(x, y) = d(W, V) \text{ for some } y \in V\}$.

(d) $V_0 = \{y \in V : d(x, y) = d(W, V) \text{ for some } x \in W\}$.

The notion of proximal contraction appeared in [9], now we introduce the following

Definition 1.9. Let $S : W \mapsto V$ be a non-self mapping. We say S is a *proximal convex* (δ, L) *weak contraction* if there exists $\delta_0, \delta_1 \in (0, 1)$, $L_0, L_1 \geq 0$, and $u_1, u_2, x, y \in W$ such that $d(u_1, Sx) = d(W, V)$ and $d(u_2, Sy) = d(W, V)$ implies

$$d(Su_1, Su_2) \leq \delta_0 d(x, y) + L_0 d(y, u_1) + \delta_1 d(Sx, Sy) + L_1 d(Sy, Su_1).$$

The notion of G-proximal Kannan mapping appeared in [8], now we introduce the following

Definition 1.10. Let (X, d) be a metric space, and $G = (V(G), E(G))$ be a directed graph such that $V(G) = X$. A non-self mapping $S : W \mapsto V$ is called a *G-proximal convex* (δ, L) *weak contraction*, if there exists $\delta_0, \delta_1 \in (0, 1)$ and $L_0, L_1 \geq 0$, such that $(x, y) \in E(G)$, $d(u, Sx) = d(W, V)$ and $d(v, Sy) = d(W, V)$ implies

$$d(Su, Sv) \leq \delta_0 d(x, y) + L_0 d(y, u) + \delta_1 d(Sx, Sy) + L_1 d(Sy, Su),$$

where $x, y, u, v \in W$.

Definition 1.11. [8] Let (X, d) be a metric space and $G = (V(G), E(G))$ be a directed graph such that $V(G) = X$. A non-self mapping $S : W \mapsto V$ is called *proximally G-edge-preserving*, if for each $x, y, u, v \in W$, $(x, y) \in E(G)$, $d(u, Sx) = d(W, V)$ and $d(v, Sy) = d(W, V)$ implies $(u, v) \in E(G)$.

The rest of this paper is organized as follows. In the next section we obtain a fixed point theorem associated with the convex (δ, L) weak contraction, and a best proximity point theorem for its non-self version endowed with a graph. We close this paper with some open problems suggested in Section 3.

2. Main Result

2.1. *A fixed point theorem*

Theorem 2.1. *Let* (X, d) *be a metric space, and* $T : X \mapsto X$ *be a convex* (δ, L) *weak contraction mapping, that is, T satisfies*

$$d(T^2x, T^2y) \leq \delta_0 d(x, y) + L_0 d(y, Tx) + \delta_1 d(Tx, Ty) + L_1 d(Ty, T^2x)$$

for all $x, y \in X$ *with* $0 < \delta_0, \delta_1 < 1,$ $L_0, L_1 \geq 0,$ *and* $\delta_0 + \delta_1 < 1.$ *If* (X, d) *is complete, then the fixed point of T exists. If, in addition, T is a convex* (δ, L) *weak contraction such that there exists* $0 < \delta_0, \delta_1 < 1,$ $L_0^*, L_1^* \geq 0,$ *with* $\delta_0 + \delta_1 < 1$ *satisfying*

$$d(T^2x, T^2y) \leq \delta_0 d(x, y) + L_0^* d(x, Tx) + \delta_1 d(Tx, Ty) + L_1^* d(Tx, T^2x),$$

then the fixed point is unique.

Proof. Define $x_{n+1} = Tx_n = T^2x_{n-1}$ for all $n \in \mathbb{N},$ and observe we have the following

$$\begin{aligned} d(x_{n+1}, x_{n+2}) &= d(T^2x_{n-1}, T^2x_n) \\ &\leq \delta_0 d(x_{n-1}, x_n) + L_0 d(x_n, Tx_{n-1}) + \delta_1 d(Tx_{n-1}, Tx_n) \\ &\quad + L_1 d(Tx_n, T^2x_{n-1}) \\ &= \delta_0 d(x_{n-1}, x_n) + L_0 d(x_n, x_n) + \delta_1 d(x_n, x_{n+1}) + L_1 d(x_{n+1}, x_{n+1}) \\ &= \delta_0 d(x_{n-1}, x_n) + \delta_1 d(x_n, x_{n+1}) \\ &\leq (\delta_0 + \delta_1) \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} \\ &= (\delta_0 + \delta_1) d(x_n, x_{n+1}). \end{aligned}$$

Set $h := (\delta_0 + \delta_1)$, and observe by induction we have $d(x_n, x_{n+1}) \leq h^n d(x_0, x_1)$ for all $n \in \mathbb{N}$. For $n, m \in \mathbb{N}$ with $n < m$ we deduce the following

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + \cdots + d(x_{m-1}, x_m) \\ &\leq h^n d(x_0, x_1) + \cdots + h^{m-1} d(x_0, x_1) \\ &\leq (h^n + h^{n+1} + \cdots) d(x_0, x_1) \\ &= \frac{h^n}{1-h} d(x_0, x_1). \end{aligned}$$

Since $h < 1$, if we take limits in the above inequality as $n, m \to \infty$ we deduce that $\{x_n\}$ is Cauchy, and since X is complete, there is $v \in X$ such that $\lim_{n\to\infty} x_n = v$. Now we show the fixed point exist. Suppose v is a fixed point of T but not of T^2, then we know $d(v, Tv) = 0$, but $d(v, T^2 v) > 0$. Now observe we have the following

$$\begin{aligned} 0 &< d(v, T^2 v) \\ &\leq d(v, x_{n+1}) + d(x_{n+1}, T^2 v) \\ &= d(v, x_{n+1}) + d(T^2 x_{n-1}, T^2 v) \\ &\leq d(v, x_{n+1}) + \delta_0 d(x_{n-1}, v) + L_0 d(v, Tx_{n-1}) + \delta_1 d(Tx_{n-1}, Tv) + L_1 d(Tv, T^2 x_{n-1}) \\ &= d(v, x_{n+1}) + \delta_0 d(x_{n-1}, v) + L_0 d(v, x_n) + \delta_1 d(x_n, Tv) + L_1 d(Tv, x_{n+1}). \end{aligned}$$

Taking limits in the above inequality and using the fact that $d(v, Tv) = 0$, we deduce that $d(v, T^2 v)$ is bounded above and below by zero, hence the assumption that $d(v, T^2 v) > 0$ cannot be true, it must be the case that $d(v, T^2 v) = 0$, that is, $T^2 v = v$. It now follows that v is also a fixed point of T^2. Now we show the fixed point is unique. Suppose $a = Ta = T^2 a$ and $b = Tb = T^2 b$, but $a \neq b$. From the second part of the theorem we deduce the following

$$\begin{aligned} d(a, b) &= d(T^2 a, T^2 b) \\ &\leq \delta_0 d(a, b) + L_0^* d(a, Ta) + \delta_1 d(Ta, Tb) + L_1^* d(Ta, T^2 a) \end{aligned}$$

$$= \delta_0 d(a, b) + \delta_1 d(a, b)$$

$$\leq (\delta_0 + \delta_1) d(a, b).$$

Since $1 - (\delta_0 + \delta_1) \neq 0$ and $d > 0$, from the above inequality we must have $d(a, b) = 0$ and hence $a = b$, which contradicts the assumption that $a \neq b$. Thus, the fixed point is unique.

2.2. ***A best proximity point theorem***

Theorem 2.2. *Let (X, d) be a complete metric space, $G = (V(G), E(G))$ be a directed graph such that $V(G) = X$. Let W and V be nonempty closed subsets of X with W_0 nonempty. Let $S : W \mapsto V$ be a non-self mapping satisfying the following properties*:

(a) *S is proximally G-edge-preserving*, *continuous and G-proximal convex* (δ, L) *weak contraction such that $S(W_0) \subset V_0$.*

(b) *there exist $x_0, x_1 \in W_0$ such that*

$$d(x_1, Sx_0) = d(W, V),\ d(x_2, Sx_1) = d(x_2, S^2x_0) = d(W, V) \text{ and } (x_0, x_1) \in E(G).$$

Then S has a best proximity point in W, *that is*, *there exists an element $w \in W$ such that $d(w, Sw) = d(W, V)$ and $d(w, S^2w) = d(W, V)$. Further the sequence $\{x_n\}$ defined by*

$$d(x_n, Sx_{n-1}) = d(W, V) \text{ and } d(x_{n+1}, Sx_n) = d(x_{n+1}, S^2x_{n-1}) = d(W, V)$$

for all $n \in \mathbb{N}$ converges to the element w.

Proof. From condition (b), there exist $x_0, x_1 \in W_0$ such that

$$d(x_1, Sx_0) = d(W, V),\ d(x_2, Sx_1) = d(x_2, S^2x_0) = d(W, V) \text{ and } (x_0, x_1) \in E(G). \quad (1)$$

Since $S(W_0) \subseteq V_0$, we have $Sx_2 \in V_0$ and hence there exists $x_3 \in W_0$ such that

$$d(x_3, Sx_2) = d(W, V) \text{ and } d(x_4, Sx_3) = d(x_4, S^2x_2) = d(W, V). \quad (2)$$

By the proximally *G*-edge preserving of *S* and using both (1) and (2), we get

$$(x_3, x_4), (x_2, x_3) \in E(G).$$

By continuing this process, we can form the sequence $\{x_n\}$ in W_0 such that

$$d(x_n, Sx_{n-1}) = d(W, V) \text{ and } d(x_{n+1}, Sx_n) = d(x_{n+1}, S^2 x_{n-1}) = d(W, V)$$

$$\text{with } (x_{n-1}, x_n) \in E(G), \text{ for all } n \in \mathbb{N}. \quad (3)$$

Next we show that *S* has a best proximity point in *W*. Suppose there exists $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1}$. By using (3), we obtain that

$$d(x_{n_0}, Sx_{n_0}) = d(x_{n_0+1}, Sx_{n_0}) = d(W, V)$$

and

$$d(x_{n_0}, S^2 x_{n_0}) = d(x_{n_0+1}, S^2 x_{n_0}) = d(x_{n_0+2}, S^2 x_{n_0}) = d(W, V)$$

and so x_{n_0} is a best proximity point of *S* and of S^2. Now we suppose that $x_{n-1} \neq x_n$ for all $n \in \mathbb{N}$. We show that $\{x_n\}$ is a Cauchy sequence in *W*. As *S* is *G*-proximal convex (δ, L) weak contraction, and for each $n \in \mathbb{N}$, $(x_{n-1}, x_n) \in E(G)$, $d(x_n, Sx_{n-1}) = d(W, V)$ and $d(x_{n+1}, Sx_n) = d(x_{n+1}, S^2 x_{n-1}) = d(W, V)$, then we have

$$\begin{aligned} d(x_{n+1}, x_{n+2}) &\leq \delta_0 d(x_{n-1}, x_n) + L_0 d(x_n, x_n) + \delta_1 d(x_n, x_{n+1}) + L_1 d(x_{n+1}, x_{n+1}) \\ &= \delta_0 d(x_{n-1}, x_n) + \delta_1 d(x_n, x_{n+1}) \\ &\leq (\delta_0 + \delta_1) \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} \\ &= (\delta_0 + \delta_1) d(x_n, x_{n+1}). \end{aligned}$$

Now set $h := \delta_0 + \delta_1$. By the above inequality we have

$$d(x_1, x_2) \leq h d(x_0, x_1)$$

and hence

$$d(x_2, x_3) \leq h^2 d(x_0, x_1).$$

By induction, we deduce the following

$$d(x_n, x_{n+1}) \leq h^n d(x_0, x_1) \quad (4)$$

for all $n \in \mathbb{N} \cup \{0\}$. From (4), for each $m, n \in \mathbb{N}$ with $m > n$, we deduce the following

$$\begin{aligned} d(x_n, x_m) &\le d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ &\le h^n d(x_0, x_1) + h^{n+1} d(x_0, x_1) + \cdots + h^{m-1} d(x_0, x_1) \\ &= d(x_0, x_1) \sum_{i=n}^{m-1} h^i \\ &= \frac{h^n}{1-h} d(x_0, x_1). \end{aligned}$$

Since $h \in (0, 1)$, it follows that $\{x_n\}$ is a Cauchy sequence in W. Since W is closed, there exists $w \in W$ such that $x_n \to w$. By continuity of S and of S^2, we have $Sx_n \to Sw$ and $S^2 x_n \to S^2 w$ as $n \to \infty$. As the metric function is continuous, we obtain

$$d(x_{n+1}, Sx_n) \to d(w, Sw) \text{ as } n \to \infty$$

and

$$d(x_{n+2}, Sx_{n+1}) = d(x_{n+2}, S^2 x_n) \to d(w, S^2 w) \text{ as } n \to \infty.$$

Similarly, by (3), we have

$$d(w, Sw) = d(W, V) \text{ and } d(w, S^2 w) = d(W, V).$$

It follows that $w \in W$ is a best proximity point of S and of S^2. Moreover, the sequence $\{x_n\}$ defined by

$$d(x_{n+1}, Sx_n) = d(W, V) \text{ and } d(x_{n+2}, Sx_{n+1}) = d(x_{n+2}, S^2 x_n) = d(W, V),$$
$$n \in \mathbb{N} \cup \{0\}$$

converges to an element w, and the proof is completed. □

3. Open Problem

Definition 3.1. [10] Let (X, d) be a metric space. A map $T : X \mapsto X$ is called a *Reich contraction* if there exist nonnegative constants a, b, c with $a + b + c < 1$ such that the following holds for all $x, y \in X$

$$d(Tx, Ty) \leq ad(x, y) + bd(x, Tx) + cd(y, Ty).$$

Note if T is a Reich contraction, then it also satisfies the following

$$d(Tx, Ty) \leq (a + b + c)\max\{d(x, y), d(x, Tx), d(y, Ty)\}.$$

Therefore we have the following

Definition 3.2. [11] Let (X, d) be a metric space. A map $T : X \mapsto X$ is also a *Reich contraction* if there exists $k \in \left[0, \frac{1}{3}\right)$ such that the following holds for all $x, y \in X$

$$d(Tx, Ty) \leq k[d(x, y) + d(x, Tx) + d(y, Ty)].$$

Related to the Reich contraction, the following was obtained

Theorem 3.3. [10] *Let (X, d) be a metric space, and $T : X \mapsto X$ be a Reich contraction. Then T has a unique fixed point, provided (X, d) is complete.*

Now we introduced the following, as the Berinde characterization of the Reich contraction

Definition 3.4. [11] Let (X, d) be a metric space. A map $T : X \mapsto X$ is called a *Berinde weak Reich contraction*, if there exists $\delta \in \left[0, \frac{1}{3}\right)$ and $L \geq 0$ such that the following holds for all $x, y \in X$

$$d(Tx, Ty) \leq \delta[d(x, y) + d(x, Tx) + d(y, Ty)] + Ld(y, Tx).$$

Note that if $a = Ta$ and $b = Tb$, but $a \neq b$, then we have the following inequality from Definition 3.4

$$\begin{aligned} d(a, b) &= d(Ta, Tb) \\ &\leq \delta[d(a, b) + d(a, Ta) + d(b, Tb)] + Ld(b, Ta) \\ &= \delta[d(a, b) + d(a, a) + d(b, b)] + Ld(b, a) \\ &= \delta d(a, b) + Ld(b, a) \\ &= (\delta + L)d(a, b). \end{aligned}$$

Unless $1-(\delta+L)>0$ we cannot conclude $a=b$. So we introduced the following as a way to "force" uniqueness of the fixed point.

Definition 3.5. [11] Let (X, d) be a metric space. A map $T: X \mapsto X$ is called a $(\delta, 1-3\delta)$ *weak Reich contraction* if the following holds for all $x, y \in X$ and $\delta \in \left(0, \frac{1}{3}\right)$

$$d(Tx, Ty) \leq \delta[d(x, y)+d(x, Tx)+d(y, Ty)]+(1-3\delta)\, d(y, Tx).$$

Note that if $L=0$ in Definition 3.4, then we recover Definition 3.2. Note that for any $\delta \in \left(0, \frac{1}{3}\right)$, Definition 3.5 does not reduce to Definition 3.2.

The following result is previously known

Theorem 3.6. [11] *Let* (X, d) *be a metric space, and* $T: X \mapsto X$ *be a* $(\delta, 1-3\delta)$ *weak Reich contraction. Then T has a unique fixed point provided X is complete.*

Our first open problem introduces a so-called convex (δ, L) weak Reich contraction mapping theorem

Conjecture 3.7. *Let* (X, d) *be a metric space, and* $T: X \mapsto X$ *be a convex* (δ, L) *weak Reich contraction mapping, that is, T satisfies*

$$\begin{aligned} d(T^2x, T^2y) \leq\ & \delta_0[d(x, y)+d(x, Tx)+d(y, Ty)]+L_0 d(y, Tx) \\ & +\delta_1[d(Tx, Ty)+d(Tx, T^2x)+d(Ty, T^2y)]+L_1 d(Ty, T^2x) \end{aligned}$$

for all $x, y \in X$ *with* $0<\delta_0, \delta_1<\frac{1}{3}, L_0, L_1 \geq 0,$ *and* $\delta_0+\delta_1<1.$ *If* (X, d) *is complete, then the fixed point of T exists. If in addition, T is a convex* (δ, L) *weak Reich contraction such there exists* $0<\delta_0, \delta_1<\frac{1}{3},\ L_0^*, L_1^* \geq 0,$ *with* $\delta_0+\delta_1<1$ *satisfying*

$$\begin{aligned} d(T^2x, T^2y) \leq\ & \delta_0[d(x, y)+d(x, Tx)+d(y, Ty)]+L_0^* d(x, Tx) \\ & +\delta_1[d(Tx, Ty)+d(Tx, T^2x)+d(Ty, T^2y)]+L_1^* d(Tx, T^2x), \end{aligned}$$

then the fixed point is unique.

Sequel to the second open problem, we will need the following

Definition 3.8. Let $S : W \mapsto V$ be a non-self mapping. We say S is a *proximal convex* (δ, L) *weak Reich contraction* if there exist $\delta_0, \delta_1 \in \left(0, \frac{1}{3}\right)$, $L_0, L_1 \geq 0$, and $u_1, u_2, x, y \in W$ such that $d(u_1, Sx) = d(W, V)$ and $d(u_2, Sy) = d(W, V)$ implies

$$d(Su_1, Su_2) \leq \delta_0[d(x, y) + d(x, u_1) + d(y, u_2)] + L_0 d(y, u_1)$$
$$+ \delta_1[d(Sx, Sy) + d(Sx, Su_1) + d(Sy, Su_2)] + L_1 d(Sy, Su_1).$$

Definition 3.9. Let (X, d) be a metric space, and $G = (V(G), E(G))$ be a directed graph such that $V(G) = X$. A non-self mapping $S : W \mapsto V$ is called a *G-proximal convex* (δ, L) *weak Reich contraction*, if there exists $\delta_0, \delta_1 \in \left(0, \frac{1}{3}\right)$ and $L_0, L_1 \geq 0$ such that $(x, y) \in E(G)$, $d(u, Sx) = d(W, V)$, and $d(v, Sy) = d(W, V)$ implies

$$d(Su, Sv) \leq \delta_0[d(x, y) + d(x, u) + d(y, v)] + L_0 d(y, u)$$
$$+ \delta_1[d(Sx, Sy) + d(Sx, Su) + d(Sy, Sv)] + L_1 d(Sy, Su),$$

where $x, y, u, v \in W$.

Now we have the following which can be regarded as the non-self counterpart to Conjecture 3.7 in graphic language

Conjecture 3.10. *Let* (X, d) *be a complete metric space*, $G = (V(G), E(G))$ *be a directed graph such that* $V(G) = X$. *Let W and V be nonempty closed subsets of X with* W_0 *nonempty. Let* $S : W \mapsto V$ *be a non-self mapping satisfying the following properties*:

(a) *S is proximally G-edge-preserving*, *continuous and G-proximal convex* (δ, L) *weak Reich contraction such that* $S(W_0) \subset V_0$

(b) *there exist* $x_0, x_1 \in W_0$ *such that*

$$d(x_1, Sx_0) = d(W, V),\ d(x_2, Sx_1) = d(x_2, S^2 x_0) = d(W, V) \text{ and } (x_0, x_1) \in E(G).$$

Then S has a best proximity point in W, that is, there exists an element $w \in W$ *such that* $d(w, Sw) = d(W, V)$ *and* $d(w, S^2 w) = d(W, V)$. *Further the sequence* $\{x_n\}$ *defined* *by*

$$d(x_n, Sx_{n-1}) = d(W, V) \text{ and } d(x_{n+1}, Sx_n) = d(x_{n+1}, S^2x_{n-1}) = d(W, V)$$

for all $n \in \mathbb{N}$ *converges to the element w.*

References

[1] V. Berinde, Approximation fixed points of weak contractions using the Picard iteration, *Nonlinear Anal. Forum* 9(1) (2004), 43-53.

[2] Vasile I. Istratescu, Some fixed point theorems for convex contraction mappings and convex nonexpansive mappings (I), *Libertas Math.* 1 (1981), 151-163.

[3] Clement Boateng Ampadu, A new proof of the convex contraction mapping theorem in metric spaces, *Internat. J. Math. Arch.*, to appear. https://drive.google.com/file/d/0BwtkpMtWoUlEV0d4QUhnaVlqOHc/view

[4] Clement Ampadu, *Fixed Point Theory for Higher-Order Mappings*, lulu.com, 2016. ISBN: 5800118959925

[5] Jeffery Ezearn, Higher-order Lipschitz mappings, *Fixed Point Theory Appl.* 2015, 2015:88, 18 pp.

[6] Vasile Berinde, On the approximation of fixed points of weak contractive mappings, *Carpathian J. Math.* 19(1) (2003), 7-22.

[7] K. Fan, Extensions of two fixed point theorems of F.E. Browder, *Math. Z.* 112 (1969) 234-240.

[8] Chalongchai Klanarong and Suthep Suantai, Best proximity point theorems for *G*-proximal generalized contraction in complete metric spaces endowed with graphs, *Thai J. Math.* 15(1) (2017), 261-276.

[9] S. S. Basha, Best proximity points: optimal solutions, *J. Optim. Theory Appl.* 151(1) (2011), 210-216.

[10] S. Reich, Some remarks concerning contraction mappings, *Canad. Math. Bull.* 14 (1971), 121-124.

[11] Clement Boateng Ampadu, An almost Berinde Reich mapping theorem with unique fixed point, *Global J. Pure Appl. Math.*, to appear. https://drive.google.com/file/d/1TecJ2bVkpKRsCtCZZ8fhlZ8h04yptsFU/view