Earthline Journal of Mathematical Sciences
http://www.earthlinepublishers.com
Volume 1, Number 2, 2019, Pages 143-155
https://doi.org/10.34198/ejms.1219.143155

Differential Subordination Results for Fractional Integral Associated with Generalized Mittag-Leffler Function

Abbas Kareem Wanas[1] and Alaa Hussein Khaleel[2]

Department of Mathematics, College of Science, University of Al-Qadisiyah, Iraq
e-mail: abbas.kareem.w@qu.edu.iq[1]; alaa.kahlil@qu.edu.iq[2]

Abstract

In this work, by making use of fractional integral, we define a certain class of holomorphic functions defined by generalized Mittag-Leffler function in the open unit disk U. Also, we establish some results for this class related to integral representation, inclusion relationship and argument estimate.

1. Introduction

Let $\mathcal{A}$ indicate the family of all functions f of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are holomorphic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$.

For functions f given by (1.1) and $g \in \mathcal{A}$ given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n,$$

Received: January 22, 2019; Accepted: February 11, 2019

2010 Mathematics Subject Classification: 30C45.

Keywords and phrases: holomorphic functions, generalized Mittag-Leffler function, fractional integral, subordination, integral representation.

the Hadamard product $f * g$ of f and g is defined by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_n b_n z^n = (g * f)(z).$$

Given two functions f and g which are analytic in U, we say that f is subordinate to g, written $f \prec g$ or $f(z) \prec g(z)$ $(z \in U)$, if there exists a Schwarz function w which is analytic in U with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$, $(z \in U)$. In particular, if the function g is univalent in U, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$.

The Mittag-Leffler function $E_\alpha(z)$, $(z \in \mathbb{C})$ (see [7, 8]) is defined by

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \quad (\alpha \in \mathbb{C}, \operatorname{Re}(\alpha) > 0).$$

Several researchers have investigated properties of Mittag-Leffler function and generalized Mittag-Leffler function, see for example [4, 5, 9, 10]. Moreover, Srivastava and Tomovski [13] introduced the function $E_{\alpha,\beta}^{\gamma,k}(z)$, $(z \in \mathbb{C})$ in the form:

$$E_{\alpha,\beta}^{\gamma,k}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{nk}\, z^n}{\Gamma(\alpha n + \beta)\, n!},$$

where $\alpha, \beta, \gamma \in \mathbb{C}$, $\operatorname{Re}(\alpha) > \max\{0, \operatorname{Re}(k) - 1\}$, $\operatorname{Re}(k) > 0$ and $(x)_n$ is the Pochhammer symbol defined by

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)} = \begin{cases} 1 & (n = 0), \\ x(x+1)\cdots(x+n-1) & (n \in \mathbb{N}). \end{cases}$$

Definition 1.1 [1]. For $f \in \mathcal{A}$ the operator $\mathcal{H}_{\alpha,\beta}^{\gamma,k} : \mathcal{A} \to \mathcal{A}$ is defined by

$$\mathcal{H}_{\alpha,\beta}^{\gamma,k} f(z) = Q_{\alpha,\beta}^{\gamma,k}(z) * f(z) \quad (z \in U),$$

where

$$Q_{\alpha,\beta}^{\gamma,k}(z) = \frac{\Gamma(\alpha+\beta)}{(\gamma)_k}\left(E_{\alpha,\beta}^{\gamma,k}(z) - \frac{1}{\Gamma(\beta)}\right),$$

$$\beta,\ \gamma \in \mathbb{C},\ \operatorname{Re}(\alpha) > \max\{0,\ \operatorname{Re}(k) - 1\},\ \operatorname{Re}(k) > 0.$$

By some easy calculations, we have

$$\mathcal{H}_{\alpha,\beta}^{\gamma,k} f(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(\gamma + nk)\Gamma(\alpha + \beta)}{\Gamma(\gamma + k)\Gamma(\beta + \alpha n)\, n!} a_n z^n.$$

Definition 1.2 [12]**.** The fractional integral of order λ, $(\lambda > 0)$ is defined for a function *f* by

$$D_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\lambda}}\, d\zeta,$$

where *f* is analytic function in a simply-connected region of the z-plane containing the origin and the multiplicity of $(z-\zeta)^{\lambda-1}$ is removed by requiring $\log(z-\zeta)$ to be real, when $(z-\zeta) > 0$.

We now, by making use of Definition 1.1 and Definition 1.2, we have

$$D_z^{-\lambda} \mathcal{H}_{\alpha,\beta}^{\gamma,k} f(z) = \frac{1}{\Gamma(2+\lambda)} z^{1+\lambda} + \sum_{n=2}^{\infty} \frac{\Gamma(\gamma + nk)\Gamma(\alpha + \beta)}{\Gamma(n+1+\lambda)\Gamma(\gamma + k)\Gamma(\beta + \alpha n)} a_n z^{n+\lambda}. \quad (1.2)$$

It is easily verified from (1.2) that

$$z\left(D_z^{-\lambda} \mathcal{H}_{\alpha,\beta}^{\gamma,k} f(z)\right)' = \left(\frac{\gamma + k}{k}\right) D_z^{-\lambda} \mathcal{H}_{\alpha,\beta}^{\gamma+1,k} f(z)$$

$$- \left(\frac{\gamma - \lambda k}{k}\right) D_z^{-\lambda} \mathcal{H}_{\alpha,\beta}^{\gamma,k} f(z), \quad \operatorname{Re}(\gamma - \lambda k) \neq 0. \quad (1.3)$$

Let *T* be the class of functions *h* of the form:

$$h(z) = 1 + \sum_{n=1}^{\infty} h_n z^n,$$

which are analytic and convex univalent in *U* and satisfy the condition:

$$\operatorname{Re}\{h(z)\} > 0, \quad (z \in U).$$

We will require the following lemmas in proving our main results.

Lemma 1.1 [3]. *Let* $u, v \in \mathbb{C}$ *and suppose that* ψ *is convex and univalent in U with* $\psi(0) = 1$ *and* $\operatorname{Re}\{u\psi(z) + v\} > 0, (z \in U)$. *If q is analytic in U with* $q(0) = 1,$ *then the subordination*

$$q(z) + \frac{zq'(z)}{uq(z) + v} \prec \psi(z)$$

implies that $q(z) \prec \psi(z)$.

Lemma 1.2 [6]. *Let h be convex univalent in U and* $\mathcal{T}$ *be analytic in U with* $\operatorname{Re}(\mathcal{T}(z)) \geq 0, (z \in U)$. *If q is analytic in U and* $q(0) = h(0),$ *then the subordination*

$$q(z) + \mathcal{T}(z) zq'(z) \prec h(z)$$

implies that $q(z) \prec h(z)$.

Lemma 1.3 [2]. *Let q be analytic in U with* $q(0) = 1$ *and* $q(z) \neq 0$ *for all* $z \in U$. *If there exists two points* $z_1, z_2 \in U$ *such that*

$$-\frac{\pi}{2} b_1 = \arg(q(z_1)) < \arg(q(z)) < \arg(q(z_2)) = \frac{\pi}{2} b_2,$$

for some b_1 *and* b_2 $(b_1 > 0, b_2 > 0)$ *and for all* z $(|z| < |z_1| = |z_2|)$, *then*

$$\frac{z_1 q'(z_1)}{q(z_1)} = -i\left(\frac{b_1 + b_2}{2}\right)m \quad \text{and} \quad \frac{z_2 q'(z_2)}{q(z_2)} = i\left(\frac{b_1 + b_2}{2}\right)m,$$

where

$$m \geq \frac{1 - |\varepsilon|}{1 + |\varepsilon|} \quad \text{and} \quad \varepsilon = i \tan \frac{\pi}{4}\left(\frac{b_2 - b_1}{b_1 + b_2}\right).$$

2. Main Results

We begin this section by defining the function class $\mathcal{M}(\lambda, \gamma, k, \mu, \alpha, \beta, \delta; h)$ as follows:

Definition 2.1. A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{M}(\lambda, \gamma, k, \mu, \alpha, \beta, \delta; h)$ if it satisfies the following differential subordination condition:

$$\frac{1}{1-\delta}\left(\frac{z\left(D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\gamma,k}f(z)\right)'}{D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\gamma,k}f(z)}-\delta\right) \prec h(z), \tag{2.1}$$

where $\alpha, \beta, \gamma \in \mathbb{C}$, $\operatorname{Re}(\alpha) > \max\{0, \operatorname{Re}(k)-1\}$, $\operatorname{Re}(k) > 0$, $\lambda > 0$, $0 \le \delta < p$ and $h \in T$.

In the first theorem, we find integral representation of the class $\mathcal{M}(\lambda, \gamma, k, \mu, \alpha, \beta, \delta; h)$.

Theorem 2.1. *Let* $f \in \mathcal{M}(\lambda, \gamma, k, \mu, \alpha, \beta, \delta; h)$. *Then*

$$D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\gamma,k}f(z) = z \cdot \exp\left[(1-\delta)\int_0^z \frac{h(w(s))-1}{s}\,ds\right],$$

where w is analytic in U with $w(0) = 0$ *and* $|w(z)| < 1, (z \in U)$.

Proof. Assume that $f \in \mathcal{M}(\lambda, \gamma, k, \mu, \alpha, \beta, \delta; h)$. It is easy to see that subordination condition (2.1) can be written as follows

$$\frac{z\left(D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\gamma,k}f(z)\right)'}{D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\gamma,k}f(z)} = (1-\delta)\,h(w(z))+\delta, \tag{2.2}$$

where w is analytic in U with $w(0) = 0$ and $|w(z)| < 1, (z \in U)$.

From (2.2), we find that

$$\frac{\left(D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\gamma,k}f(z)\right)'}{D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\gamma,k}f(z)}-\frac{1}{z} = (1-\delta)\frac{h(w(z))-1}{z}. \tag{2.3}$$

After integrating both sides of (2.3), we have

$$\log\left(\frac{D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\gamma,k}f(z)}{z}\right) = (1-\delta)\int_0^z \frac{h(w(s))-1}{s}\,ds. \tag{2.4}$$

Therefore, from (2.4), we obtain the required result.

Next, we establish the following inclusion relationship for the class $\mathcal{M}(\lambda, \gamma, k, \mu, \alpha, \beta, \delta; h)$.

Theorem 2.2. *Let* $\operatorname{Re}\left\{(1-\delta)h(z)+\delta+\frac{\gamma-\lambda k}{k}\right\}>0.$ *Then*

$$\mathcal{M}(\lambda, \gamma+1, k, \mu, \alpha, \beta, \delta; h) \subset \mathcal{M}(\lambda, \gamma, k, \mu, \alpha, \beta, \delta; h).$$

Proof. Let $f \in \mathcal{M}(\lambda, \gamma+1, k, \mu, \alpha, \beta, \delta; h)$ and put

$$q(z)=\frac{1}{1-\delta}\left(\frac{z\left(D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\gamma,k}f(z)\right)'}{D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\gamma,k}f(z)}-\delta\right). \tag{2.5}$$

Then q is analytic in U with $q(0)=1.$ According to (2.5) and using the relation (1.3), we obtain

$$\frac{\gamma+k}{k}\frac{D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\gamma+1,k}f(z)}{D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\gamma,k}f(z)}=(1-\delta)q(z)+\delta+\frac{\gamma-\lambda k}{k}. \tag{2.6}$$

By logarithmically differentiating both sides of (2.6) with respect to z and multiplying by z, we get

$$q(z)+\frac{zq'(z)}{(1-\delta)q(z)+\delta+\frac{\gamma-\lambda k}{k}}=\frac{1}{1-\delta}\left(\frac{z\left(D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\gamma+1,k}f(z)\right)'}{D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\gamma+1,k}f(z)}-\delta\right)\prec h(z). \tag{2.7}$$

Since $\operatorname{Re}\left\{(1-\delta)h(z)+\delta+\frac{\gamma-\lambda k}{k}\right\}>0,$ then applying Lemma 1.1 to the subordination (2.7), yields $q(z)\prec h(z),$ which implies $f \in \mathcal{M}(\lambda, \gamma, k, \mu, \alpha, \beta, \delta; h).$

Theorem 2.3. *Let* $f \in \mathcal{A},\ 0 < a_1,\ a_2 \le 1$ *and* $0 \le \delta < 1.$ *If*

$$-\frac{\pi}{2}a_1 < \arg\left(\frac{z\left(D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\gamma+1,k}f(z)\right)'}{D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\gamma+1,k}g(z)}-\delta\right)<\frac{\pi}{2}a_2,$$

for some $g \in \mathcal{M}\left(\lambda, \gamma+1, k, \mu, \alpha, \beta, \delta; \frac{1+AZ}{1+Bz}\right), (-1 \le B < A \le 1),$ *then*

$$-\frac{\pi}{2} b_1 < \arg\left(\frac{z\left(D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\gamma,k} f(z)\right)'}{D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\gamma,k} g(z)} - \delta \right) < \frac{\pi}{2} b_2,$$

where b_1 *and* b_2 $(0 < b_1, b_2 \le 1)$ *are the solutions of the equations*:

$$a_1 = \begin{cases} b_1 + \frac{2}{\pi}\tan^{-1}\left(\frac{(1-|\varepsilon|)(b_1+b_2)\cos\frac{\pi}{2}t}{2(1+|\varepsilon|)\left(\frac{(1+A)(1-\delta)}{1+B} + \delta + \frac{\gamma-\lambda k}{k}\right) + (1-|\varepsilon|)(b_1+b_2)\sin\frac{\pi}{2}t} \right), & B \ne -1 \\ b_1 & B = -1 \end{cases} \tag{2.8}$$

and

$$a_2 = \begin{cases} b_2 + \frac{2}{\pi}\tan^{-1}\left(\frac{(1-|\varepsilon|)(b_1+b_2)\cos\frac{\pi}{2}t}{2(1+|\varepsilon|)\left(\frac{(1+A)(1-\delta)}{1+B} + \delta + \frac{\gamma-\lambda k}{k}\right) + (1-|\varepsilon|)(b_1+b_2)\sin\frac{\pi}{2}t} \right), & B \ne -1 \\ b_2 & B = -1 \end{cases} \tag{2.9}$$

with

$$\varepsilon = i\tan\frac{\pi}{2}\left(\frac{b_2-b_1}{b_1+b_2}\right) \text{ and } t = \frac{2}{\pi}\sin^{-1}\left(\frac{(A-B)(1-\delta)}{\left(\delta + \frac{\gamma-\lambda k}{k}\right)(1-B^2) + (1-\delta)(1-AB)} \right). \tag{2.10}$$

Proof. Define the function G by

$$G(z) = \frac{1}{1-\tau}\left(\frac{z\left(D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\gamma,k} f(z)\right)'}{D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\gamma,k} g(z)} - \tau\right), \tag{2.11}$$

where $g \in \mathcal{M}\left(\lambda, \gamma+1, k, \mu, \alpha, \beta, \delta; \frac{1+AZ}{1+Bz}\right), (-1 \le B < A \le 1),$ and $0 \le \tau < 1.$

Then G is analytic in U with $G(0) = 1.$ Therefore by making use of (1.3) and (2.11), we obtain

$$((1-\tau)G(z)+\tau)\, D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\gamma,k} g(z) = \frac{\gamma+k}{k} D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\gamma+1,k} f(z) - \frac{\gamma-\lambda k}{k} D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\gamma,k} f(z).$$

Differentiating above relation with respect to z and multiplying by z, we get

$$((1-\tau)G(z)+\tau)\, z\left(D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\gamma,k} g(z)\right)' + (1-\tau)\, zG'(z)\, D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\gamma,k} g(z)$$

$$= \frac{\gamma+k}{k} z\left(D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\gamma+1,k} f(z)\right)' - \frac{\gamma-\lambda k}{k} z\left(D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\gamma,k} f(z)\right)'. \tag{2.12}$$

Suppose that

$$H(z) = \frac{1}{1-\delta}\left(\frac{z\left(D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\gamma,k} g(z)\right)'}{D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\gamma,k} g(z)} - \delta\right).$$

Using (1.3) again, we have

$$\frac{\gamma+k}{k}\, \frac{D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\gamma+1,k} g(z)}{D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\gamma,k} g(z)} = (1-\delta)H(z) + \delta + \frac{\gamma-\lambda k}{k}. \tag{2.13}$$

From (2.12) and (2.13), we easily get

$$G(z) + \frac{zG'(z)}{(1-\delta)H(z) + \delta + \frac{\gamma-\lambda k}{k}} = \frac{1}{1-\tau}\left(\frac{z\left(D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\gamma+1,k} f(z)\right)'}{D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\gamma+1,k} g(z)} - \tau\right). \tag{2.14}$$

Notice that from Theorem 2.2, $g \in \mathcal{M}\left(\lambda, \gamma+1, k, \mu, \alpha, \beta, \delta; \frac{1+AZ}{1+Bz}\right)$ implies $g \in \mathcal{M}\left(\lambda, \gamma, k, \mu, \alpha, \beta, \delta; \frac{1+AZ}{1+Bz}\right)$. Thus,

$$H(z) \prec \frac{1+AZ}{1+Bz} \quad (-1 \le B < A \le 1).$$

By using the result of Silverman and Silvia [11], we have

$$\left| H(z) - \frac{1-AB}{1-B^2} \right| < \frac{A-B}{1-B^2} \quad (B \neq -1, \; z \in U) \tag{2.15}$$

and

$$\operatorname{Re}\{H(z)\} > \frac{1-A}{2} \quad (B = -1, \; z \in U). \tag{2.16}$$

It follows from (2.15) and (2.16) that

$$\left| (1-\delta)H(z) + \delta + \frac{\gamma-\lambda k}{k} - \frac{\left(\delta + \frac{\gamma-\lambda k}{k}\right)(1-B^2) + (1-\delta)(1-AB)}{1-B^2} \right|$$

$$< \frac{(A-B)(1-\delta)}{1-B^2}, \quad (B \neq -1, \; z \in U)$$

and

$$\operatorname{Re}\left\{(1-\delta)H(z) + \delta + \frac{\gamma-\lambda k}{k}\right\} > \frac{(1-A)(1-\delta)}{2} + \delta + \frac{\gamma-\lambda k}{k}, \quad (B = -1, \; z \in U).$$

Putting

$$(1-\delta)H(z) + \delta + \frac{\gamma-\lambda k}{k} = \rho e^{i\frac{\pi}{2}\phi},$$

where

$$-\frac{(A-B)(1-\delta)}{\left(\delta + \frac{\gamma-\lambda k}{k}\right)(1-B^2) + (1-\delta)(1-AB)}$$

$$< \phi < \frac{(A-B)(1-\delta)}{\left(\delta + \frac{\gamma - \lambda k}{k}\right)(1-B^2) + (1-\delta)(1-AB)}, \ (B \neq -1)$$

and $-1 < \phi < 1, \ (B = -1),$ then

$$\frac{(1-A)(1-\delta)}{1-B} + \delta + \frac{\gamma - \lambda k}{k} < \rho < \frac{(1+A)(1-\delta)}{1+B} + \delta + \frac{\gamma - \lambda k}{k}, \ (B \neq -1)$$

and

$$\frac{(1-A)(1-\delta)}{1-B} + \delta + \frac{\gamma - \lambda k}{k} < \rho < \infty, \ (B = -1).$$

An application of Lemma 1.2 with $\mathcal{T}(z) = \frac{1}{(1-\delta)H(z) + \delta + \frac{\gamma - \lambda k}{k}},$ yields $G(z) \prec h(z).$

If there exist two points $z_1, z_2 \in U$ such that

$$-\frac{\pi}{2} b_1 = \arg(G(z_1)) < \arg(G(z)) < \arg(G(z_2)) = \frac{\pi}{2} b_2,$$

then by Lemma 1.3, we get

$$\frac{z_1 G'(z_1)}{G(z_1)} = -\frac{mi}{2}(b_1 + b_2) \quad \text{and} \quad \frac{z_2 G'(z_2)}{G(z_2)} = \frac{mi}{2}(b_1 + b_2),$$

where

$$m \geq \frac{1 - |\varepsilon|}{1 + |\varepsilon|} \quad \text{and} \quad \varepsilon = i \tan \frac{\pi}{4} \left(\frac{b_2 - b_1}{b_1 + b_2} \right).$$

Now, for the case $B \neq -1,$ we obtain

$$\arg\left(\frac{1}{1-\tau} \left(\frac{z_1 \left(D_z^{-\lambda} \mathcal{H}_{\alpha,\beta}^{\gamma+1,k} f(z_1) \right)'}{D_z^{-\lambda} \mathcal{H}_{\alpha,\beta}^{\gamma+1,k} g(z_1)} - \tau \right) \right)$$

$$= \arg\left(G(z_1) + \frac{z_1 G'(z_1)}{(1-\delta)H(z_1) + \delta + \frac{\gamma - \lambda k}{k}} \right)$$

$$= \arg(G(z_1)) + \arg\left(1 + \frac{z_1 G'(z_1)}{\left[(1-\delta)H(z_1) + \delta + \frac{\gamma - \lambda k}{k}\right] G(z_1)} \right)$$

$$= -\frac{\pi}{2} b_1 + \arg\left(1 - \frac{mi}{2\rho}(b_1 + b_2) e^{-i\frac{\pi}{2}\phi} \right)$$

$$= -\frac{\pi}{2} b_1 + \arg\left(1 - \frac{m}{2\rho}(b_1 + b_2)\cos\frac{\pi}{2}(1-\phi) + \frac{mi}{2\rho}(b_1 + b_2)\sin\frac{\pi}{2}(1-\phi) \right)$$

$$\leq -\frac{\pi}{2} b_1 - \tan^{-1}\left(\frac{m(b_1 + b_2)\sin\frac{\pi}{2}(1-\phi)}{2\rho + m(b_1 + b_2)\cos\frac{\pi}{2}(1-\phi)} \right)$$

$$\leq -\frac{\pi}{2} b_1 - \tan^{-1}\left(\frac{(1-|\varepsilon|)(b_1 + b_2)\cos\frac{\pi}{2}t}{2(1+|\varepsilon|)\left(\frac{(1+A)(1-\delta)}{1+B} + \delta + \frac{\gamma - \lambda k}{k}\right) + (1-|\varepsilon|)(b_1 + b_2)\sin\frac{\pi}{2}t} \right)$$

$$= -\frac{\pi}{2} a_1,$$

where a_1 and t are given by (2.8) and (2.10), respectively.

Also,

$$\arg\left(\frac{1}{1-\tau}\left(\frac{z_2\left(D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\gamma+1,k} f(z_2)\right)'}{D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\gamma+1,k} g(z_2)} - \tau \right)\right)$$

$$\geq \frac{\pi}{2} b_2 + \tan^{-1}\left(\frac{(1-|\varepsilon|)(b_1 + b_2)\cos\frac{\pi}{2}t}{2(1+|\varepsilon|)\left(\frac{(1+A)(1-\delta)}{1+B} + \delta + \frac{\gamma - \lambda k}{k}\right) + (1-|\varepsilon|)(b_1 + b_2)\sin\frac{\pi}{2}t} \right)$$

$$= \frac{\pi}{2} a_2,$$

where a_2 and t are given by (2.9) and (2.10), respectively.

Similarly, for the case $B = -1$, we have

$$\arg\left(\frac{1}{1-\tau}\left(\frac{z_1\left(D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\gamma+1,k} f(z_1)\right)'}{D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\gamma+1,k} g(z_1)} - \tau\right)\right) \leq -\frac{\pi}{2} b_1$$

and

$$\arg\left(\frac{1}{1-\tau}\left(\frac{z_2\left(D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\gamma+1,k} f(z_2)\right)'}{D_z^{-\lambda}\mathcal{H}_{\alpha,\beta}^{\gamma+1,k} g(z_2)} - \tau\right)\right) \geq \frac{\pi}{2} b_2.$$

The above two cases contradict the assumptions. Consequently, the proof of the theorem is complete.

References

[1] A. A. Attiya, Some applications of Mittag-Leffler function in the unit disk, *Filomat* 30(7) (2016), 2075-2081.

[2] A. Ebadian, S. Shams, Z. G. Wang and Y. Sun, A class of multivalent analytic functions involving the generalized Jung-Kim-Srivastava operator, *Acta Univ. Apulensis* 18 (2009), 265-277.

[3] P. J. Eenigenburg, S. S. Miller, P. T. Mocanu and M. O. Reade, On a Briot-Bouquet differential subordination, in: *General Inequalities* 3, vol. 64 of International Series of Numerical Mathematics, pp. 339-348, Birkhäuser, Basel, 1983.

[4] M. Garg, P. Manoha and S. L. Kalla, A Mittag-Leffler-type function of two variables, *Integral Transforms Spec. Funct.* 24(11) (2013), 934-944.

[5] V. Kiryakova, The multi-index Mittag-Leffler functions as an important class of special functions of fractional calculus, *Comput. Math. Appl.* 59(5) (2010), 1885-1895.

[6] S. S. Miller and P. T. Mocanu, Differential subordinations and univalent functions, *Michigan Math. J.* 28 (1981), 157-171.

[7] G. M. Mittag-Leffler, Sur la nouvelle fonction $E_\alpha(x)$, *C. R. Acad. Sci. Paris* 137 (1903), 554-558.

[8] G. M. Mittag-Leffler, Sur la représentation analytique d'une fonction monogène (cinquieme note), *Acta Math.* 29 (1905), 101-181.

[9] J. C. Prajapati, R. K. Jana, R. K. Saxena and A. K. Shukla, Some results on the generalized Mittag-Leffler function operator, *J. Inequal. Appl.* 2013, 2013:33, 6 pp.

[10] A. K. Shukla and J. C. Prajapati, On a generalization of Mittag-Leffler function and its properties, *J. Math. Anal. Appl.* 336 (2007), 797-811.

[11] H. Silverman and E. M. Silvia, Subclasses of starlike functions subordinate to convex functions, *Canad. J. Math.* 37 (1985), 48-61.

[12] H. M. Srivastava and S. Owa (eds.), *Current Topics in Analytic Function Theory*, Singapore: World Scientific Publishing Company, 1992.

[13] H. M. Srivastava and Z. Tomovski, Fractional calculus with an integral operator containing a generalized Mittag-Leffler function in the kernal, *Appl. Math. Comp.* 211 (2009), 198-210.