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An Improvement on Likert Scale via Fuzzy Relation

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Abstract

Relation is a mathematical concept that is often used in modeling relationships in physical and social sciences and the likes. The advent of fuzzy sets which model imprecision and uncertainties that occur in social animals necessitates the introduction of fuzzy relations. Besides, events and situations possess properties which are often contradictory such as good and bad, positive and negative, profit and loss and so on. We dug several literatures, particularly the analysis and hypothesis testing of several results from questionnaires, compared one of them with our newly proposed Likert scale which has fuzzy influence. The results of the two cases was analysed and presented with a bar chart, and it was clear that unlike the usual Likert scale that could be restricted from outliers, the one with fuzzy influence took care of uncertainties.

1 Introduction

Set theory has found applications in a wide range of areas, including computer science, logic, and physics. It continues to be an active area of research today.

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Its historical background is marked by intense debates and controversies, which ultimately led to the development of a rigorous and formalized theory that remains central to modern mathematics.

Definition 1.1 [Zadeh (1965)]**.** Fuzziness occur when the boundary of an information is not clear or ascertained. The words like good, tall, young, high are fuzzy as there is no single quantitative value that defines each of the mentioned words.

Definition 1.2 [Zadeh (1965)]**.** Let X be a nonempty set. A fuzzy set A is given by $A = \{x, f_A(x) : x \in X\}$, where $f_A(x) : X \longrightarrow [0, 1]$ is the membership function of the fuzzy set.

Remark 1.1. It is also regarded as the collection of objects having degree of membership. Thus the nearer the value of $f_A(x)$ to the unity, the higher the degree of membership of x in A. If $f_A(x) = 0$, it means $x \notin A$ and $f_A(x) = 1$, it implies $x \in A$. The fuzzy empty set is defined by $f_\emptyset(x) = 0$ for every $x \in X$.

Example 1.1. We can consider $X = \{f, g, h, i, j\}$ as the set of students in a secondary school and let A be the fuzzy set of *brilliant* students.

$$A = \{(f, 0.3), (g, 0.1), (h, 1), (i, 0.8), (j, 0.8)\}.$$

Here, A indicates the brilliance of f is 0.3, g is 0.1 and so on.

Example 1.2. Consider a universal set $X = \{1, 6, 16, 26, 36, 46, 56, 66, 76, 86\}$ of different ages in years.

The fuzzy set of ages of young women can be
$\{(1, 0), (6, 0), (16, 0.4), (26, 1), (36, 0.8), (46, 0.5), (56, 0.1), (66, 0), (76, 0), (86, 0)\}$.

Definition 1.3 [Zadeh (1965)]**.** Let Z be a fuzzy set and f its membership function. The set $Z_\alpha = \{x \in Z : f(x) \geq \alpha \ \ for \ \ \alpha \in [0, 1]\}$ is called the $\alpha - cut$ of Z.

Example 1.3. Let $X = \{10, 12, 6, 7, 5, 3, 1, 2\}$ and A a fuzzy subset of X with

membership function f, where A is the set of all memberships of X not too far from 5. Then
$f_A = \{(10, 0.3), (12, 0.1)(6, 0.9), (7, 0.8), (5, 1), (3, 0.8), (1, 0.6), (2, 0.7)\}$
$A_{0.6} = \{6, 7, 5, 3, 1, 2\}$
$A_{0.7} = \{6, 7, 5, 3, 2\}$.

1.1 Operation on Fuzzy Sets

Definition 1.1.1 [Zadeh (1965)]**.** Two fuzzy sets A and B are said to be equal that is $A = B$ if $f_A = f_B$ or $f_A(x) = f_B(x)$ for all $x \in X$ the universe of discourse.

Example 1.1.1. Let $X = \{e, f, g\}$ and $A = \{(e, 0.3), (f, 0.8), (g, 0.5)\}$, $B = \{(e, 0.3), (f, 0.8), (g, 0.5)\}$ are fuzzy subsets of X. Then $f_A(x) = f_B(x)$ for all $x \in X$.

Definition 1.1.2 [Zadeh (1965)]**.** The complement of fuzzy set $f_A : X \longrightarrow [0.1]$ is given by $f_{A^{\complement}}(x) = 1 - f_A$.

Example 1.1.2. Let $X = \{a, b, c\}$ where A=$\{(x, 0.3), (y, 0.8), (z, 0.75)\}$, then $A^c = \{(x, 0.7), (y, 0.2), (z, 0.25)\}$.

Definition 1.1.3 [Zadeh (1965)]**.** Let U and V be any two fuzzy subsets of a set X. Then, U is contained in V or U is a subset of V, that is $U \subseteq V$ if $f_U \leq f_V$.

Example 1.1.3. Let $X = \{x, y, z\}$ where $U = \{(x, 0.3), (y, 0.8), (z, 0.5)\}$ and $V = \{(x, 0.8), (y, 1), (z, 0.5)\}$. Then, $f_U \leq f_V \quad \forall \ x \in X$.

Definition 1.1.4 [Zadeh (1965)]**.** Let A and B be any two fuzzy subsets of a set X and f_A and f_B are their respective membership functions then, the union of A and B with respect to their membership function is given by $f_{A\cup B}(x) = \bigvee \{f_A(x), F_B(x)\}$. Here $\bigvee$ is read "Maximum".

Example 1.1.4. Let $X = \{x_1, x_2, x_3\}$ where $A = \{(x_1, 0.6), (x_2, 0.8), (x_3, 0.1)\}$ and $B = \{(x_1, 0.9), (x_2, 0.3), (X_3, 0.1)\}$. Then

$$A \cup B = \{(x_1, 0.9), (x_2, 0.8), (x_3, 0.1)\}.$$

Definition 1.1.5 [Zadeh (1965)]**.** Let A and B be any two fuzzy subsets of a set X and f_A and f_B be their respective membership functions. Then the intersection of A and B with respect to their membership function is given by $f_{A\cap B}(x) = \bigwedge \{f_A(x), f_B(x)\}$ for all $x \in X$.

Example 1.1.5. Let $X = \{x_1, x_2, x_3\}$ where $A = \{(x_1, 0.5), (x_2, 0.7), (x_3, 0)\}$ and $B = \{(x_1, 0.8), (x_2, 0.2), (X_3, 1)\}$. Then

$$A \cap B = \{(x_1, 0.5), (x_2, 0.2), (x_3, 0)\}.$$

Definition 1.1.6 [Zadeh (1965)]**.** If A and B are fuzzy sets. Their algebraic product denoted by AB is defined in terms of the membership functions of A and B by the relation $f_{AB} = f_A f_B$.

Definition 1.1.7 [Zadeh (1965)]**.** Let A and B be two fuzzy sets. The algebraic sum of A and B is denoted by $A+B$ and defined by $f_{A+B} = f_A + f_B$ provided that the sum $f_A + f_B$ is less than or equal to unity. The algebraic sum is meaningful only when the condition $f_A(x) + f_B(x) \leq 1$ is satisfied for all x.

Definition 1.1.8 [Zadeh (1965)]**.** The absolute difference of the fuzzy sets A and B is denoted by $|A - B|$ and defined by $f_{|A-B|} = |f_A - f_B|$.

1.2 Fuzzy Matrix

Definition 1.2.1 [Zadeh (1965)]**.** A fuzzy matrix is a matrix (a_{ij}) whose entries are fuzzy membership values.

Example 1.2.1.

1. Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, A is a 3×3 square fuzzy matrix.

2. Let $B = \begin{bmatrix} 1 & 0.3 \\ 0.4 & 0.02 \end{bmatrix}$, B is a 2×2 square fuzzy matrix.

3. Let $C = \begin{bmatrix} 0 & 0.3 & 1 & 0.2 \\ 0.2 & 1 & 0.1 & 0.8 \end{bmatrix}$, C is a 2×4 fuzzy matrix.

4. Let $T = \begin{bmatrix} 0.2 & 0 & 0 \\ 0 & 0.4 & 0 \\ 0 & 0 & 0.9 \end{bmatrix}$, T is a diagonal fuzzy matrix.

1.3 Operations on Fuzzy Matrix

1.3.1 Sum of Two Fuzzy Matrix

The sum of two fuzzy matrices A and B is given by Sum $A + B = \bigvee \{a_{ij}, b_{ij}\}$ for all $a_{ij} \in A$ and $b_{ij} \in B$ where $\bigvee$ is read "Maximum".

Example 1.3.1.1. Consider the fuzzy matrices A and B below

$A =$

	a	b	c
a	0.8	0.3	0.9
b	0.5	0.2	0.3
c	0.9	0.7	0.3

$B =$

	a	b	c
a	0.2	0.1	0.0
b	0.9	0.9	0.8
c	0.0	0.7	0.4

$A + B =$

	a	b	c
a	0.8	0.3	0.9
b	0.9	0.9	0.8
c	0.9	0.7	0.4

1.3.2 Product of two fuzzy matrix

Definition 1.3.2.1 [Zadeh (1965)]**.** The product of two fuzzy matrices A and B is defined as $AB = \bigvee \{\bigwedge \{a_{ij}, b_{i,j}\}\}$ which is called the Max-min product AB

where $\bigwedge$ is read "minimum".

Using the fuzzy matrices A and B given in Example 1.3.1.1, to find the product of AB, we have

a	0.8	0.3	0.9
b	0.2	0.9	0.0
min	0.2	0.3	0.0
max		0.3	

a	0.8	0.3	0.9
b	0.1	0.9	0.7
min	0.1	0.3	0.7
max			0.7

a	0.8	0.3	0.9
b	0.0	0.8	0.4
min	0.0	0.3	0.4
max			0.4

a	0.5	0.1	0.5
b	0.2	0.9	0.0
min	0.2	0.1	0.0
max	0.2		

a	0.5	0.1	0.5
b	0.1	0.9	0.7
min	0.1	0.1	0.5
max			0.5

a	0.5	0.1	0.5
b	0.0	0.8	0.4
min	0.0	0.1	0.4
max			0.4

a	0.9	0.7	0.3
b	0.2	0.9	0.0
min	0.2	0.7	0.0
max		0.7	

a	0.9	0.7	0.3
b	0.1	0.9	0.7
min	0.1	0.7	0.3
max		0.7	

a	0.9	0.7	0.3
b	0.0	0.8	0.4
min	0.0	0.7	0.3
max		0.7	

Therefore,

$AB =$

	a	b	c
a	0.3	0.7	0.4
b	0.2	0.5	0.4
c	0.7	0.7	0.7

2 Fuzzy Relation

A fuzzy relation is the cartesian product of mathematical fuzzy sets. Two fuzzy sets are taken as inputs, the fuzzy relation is then equal to the cross product of the sets which is created by vector multiplication. Usually, a rule base is stored in matrix notation which allows the fuzzy controller to update its internal values. From a historical perspective, the first fuzzy relation was mentioned in the year 1971 by Lofti A. Zadeh. A practical approach to describing a fuzzy relation is based on a $2D$ table. At first, a table is created which consists of fuzzy values from $0-1$. Next is to apply the "if then rules" to the values. The resulting numbers are stored in the table as an array. Fuzzy relations can be utilized in fuzzy databases. Fuzzy relation R is a mapping from the cartesian space AXB to interval [0,1] where the strength of mapping is expressed by the membership function of the relation : $R \subseteq AXB$ and $\mu_R = AXB \to [0,1]$.

Definition 2.1 (Fuzzy Relation)**.** Let $X, Y \subseteq R$ be nonempty sets. Then R is called a fuzzy relation in $X \times Y \subseteq R$ if $R = \{(a,b), R(a,b)|(a,b) \in X \times Y\}$. Fuzzy relation can be represented in matrix form, it can be transformed into fuzzy graph and vice-versa.

Example 2.1. Let $X = \{a,b,c\}$, $Y = \{e,f,g\}$, $Z = \{i,j,k\}$ be sets. The fuzzy relation R_0 and R_1 between X and Y are given below.

$R_0 =$

	e	f	g
a	0.8	0.4	1.0
b	0.5	0.1	0.4
c	1.0	0.6	0.2

$R_1 =$

	i	j	k
a	0.2	0.1	0.0
b	1.0	0.0	0.9
c	0.0	0.7	0.4

2.1 Operations on fuzzy relations

Consider the following two fuzzy relation R_1 and R_2 defined on a Cartesian spaces $X \times Y$ and $Y \times Z$ respectively.

2.1.1 Max-Min Composition

The max-min composition of R_1 and R_2 is a fuzzy set defined on Cartesian spaces $X \times Z$ as $R_1 \circ R_2 = [(x,z), \bigvee \{\bigwedge(R_1(x,y), R_2(y,z))\} \, |x \in X, y \in Y, z \in Z]$.

Example 2.1.1.1. Let $R_1(x,y)$ and $R_2(y,z)$ be defined as

$$R_1 = \begin{bmatrix} 0.5 & 0.4 \\ 1 & 0.2 \\ 0 & 0.6 \end{bmatrix} \qquad R_2 = \begin{bmatrix} 0.6 & 0.2 & 0.4 \\ 0.8 & 0.1 & 0.5 \end{bmatrix}$$

The Max-min composition $R_1 \circ R_2$ is determined as follows:

$$R_1 \circ R_2 = \begin{bmatrix} 0.5 & 0.4 \\ 1 & 0.2 \\ 0 & 0.6 \end{bmatrix} \circ \begin{bmatrix} 0.6 & 0.2 & 0.4 \\ 0.8 & 0.1 & 0.5 \end{bmatrix}$$

a	0.5	0.4
b	0.6	0.8
min	0.5	0.4
max	0.5	

a	0.5	0.4
b	0.2	0.1
min	0.2	0.1
max	0.2	

a	0.5	0.4
b	0.4	0.5
min	0.4	0.4
max	0.4	

a	1.0	0.2
b	0.6	0.8
min	0.6	0.2
max	0.6	

a	1.0	0.2
b	0.2	0.1
min	0.2	0.1
max	0.2	

a	1.0	0.2
b	0.4	0.5
min	0.4	0.2
max	0.4	

a	0.0	0.6
b	0.6	0.8
min	0.0	0.6
max		0.6

a	0.0	0.6
b	0.2	0.1
min	0.0	0.1
max		0.1

a	0.0	0.6
b	0.4	0.5
min	0.0	0.5
max		0.5

Hence the relational matrix for Max-Min composition is fuzzy relation is given by

$$R_1 \circ R_2 = \begin{bmatrix} 0.5 & 0.2 & 0.4 \\ 0.6 & 0.2 & 0.4 \\ 0.6 & 0.1 & 0.5 \end{bmatrix}.$$

2.1.2 Max-product composition

The max-product composition of R_1 and R_2 is a fuzzy set defined on Cartesian spaces $X \times Z$ as $R_1 \cdot R_2 = [(x,z), \bigvee \{(R_1(x,y) \cdot R_2(y,z))\} \,| x \in X, y \in Y, z \in Z]$.

Example 2.1.2.1. Using the same $R_1(x,y)$ and $R_2(y,z)$ as defined in Example 2.1.1.1 to determine the max-product composition $R_1 \cdot R_2$

a	0.5	0.4
b	0.6	0.8
product	0.3	0.32
max		0.32

a	0.5	0.4
b	0.2	0.1
product	0.1	0.04
max	0.1	

a	0.5	0.4
b	0.4	0.5
product	0.2	0.2
max	0.2	

a	1.0	0.2
b	0.6	0.8
product	0.6	0.16
max	0.6	

a	1.0	0.2
b	0.2	0.1
product	0.2	0.02
max	0.2	

a	1.0	0.2
b	0.4	0.5
min	0.4	0.1
max	0.4	

a	0.0	0.6
b	0.6	0.8
product	0.0	0.48
max		0.48

a	0.0	0.6
b	0.2	0.1
product	0.0	0.06
max		0.06

a	0.0	0.6
b	0.4	0.5
product	0.0	0.3
max		0.3

Therefore,

$$R_1 \cdot R_2(x,z) = \begin{bmatrix} 0.32 & 0.1 & 0.2 \\ 0.6 & 0.2 & 0.4 \\ 0.48 & 0.06 & 0.3 \end{bmatrix}.$$

2.1.3 Max average composition

The max-average composition of R_1 and R_2 is a fuzzy set defined on Cartesian spaces $X \times Z$ as $R_1 \circ R_2 = R(x,z) = \bigvee \left\{\frac{1}{2}(R_1(x,y) + R_2(y,z))\right\}$.

Example 2.1.3.1. Using the same $R_1(x,y)$ and $R_2(y,z)$ as defined in example 2.1.1.1 to determine the max-average composition $R_1 \circ R_2$

a	0.5	0.4
b	0.6	0.8
Average	0.55	0.6
max		0.6

a	0.5	0.4
b	0.2	0.1
average	0.35	0.25
max	0.35	

a	0.5	0.4
b	0.4	0.5
average	0.45	0.45
max	0.45	

a	1.0	0.2
b	0.6	0.8
average	0.8	0.5
max	0.8	

a	1.0	0.2
b	0.2	0.1
average	0.6	0.15
max	0.6	

a	1.0	0.2
b	0.4	0.5
average	0.7	0.35
max	0.7	

a	0.0	0.6
b	0.6	0.8
average	0.3	0.7
max		0.7

a	0.0	0.6
b	0.2	0.1
average	0.1	0.35
max		0.35

a	0.0	0.6
b	0.4	0.5
product	0.2	0.55
max		0.55

Therefore,

$$R_1 \cdot R_2(x,z) = \begin{bmatrix} 0.55 & 0.35 & 0.45 \\ 0.8 & 0.6 & 0.7 \\ 0.7 & 0.35 & 0.55 \end{bmatrix}.$$

3 Likert Scale Using the Normal Statistical Approach and Fuzzy Approach

A Likert or summative scale is a common approach in survey research, invented by American Social Scientist Rensis Likert. It uses a point answer range to gauge respondents opinion or feelings about a particular context. In a plot to necessitate the importance of fuzziness on responders feelings, we come up with ten questions pointing at the cause of mass failure of students in mathematics in a named secondary school.

3.1 Likert scale and statistical approach on a collected data

The usual statistical survey about a particular context is to issue questionnaires and analyze people's response using any of the statistical method or hypothesis testing. The results obtained can thus be represented using any data presentation tool (Bar Chart, Histogram, Pie chart). For the purpose of the research, we intuitively picked a problem based on the cause of mass failure of students in mathematics.

The table below shows the questions embedded in the questionnaire given.

QUESTIONAIRES TO EVALUATE THE CAUSE OF THE MASS FAILURE OF STUDENTS IN MATHEMATICS

QUESTIONS	STRONGLY AGREE (SA)	AGREE (A)	DISAGREE (D)	STRONGLY DISAGREE(SD)
Q1. MATHEMATICS IS FUN				
Q2. MATHEMATICS TEACHERS ARE DIFFICULT TO RELATE WITH				
Q3. I ACTIVELY PARTICIPATE IN MATH CLASS				
Q4. I HAVE A GOOD RELATIONSHIP WITH MY MATHEMATICS TEACHER				
Q5. I HAVE GOOD TEXTBOOKS THAT MAKE ME EXPLORE MATHEMATICS QUESTIONS				
Q6. I SOLVE MATHEMATICS QUESTIONS ONLY FEW DAYS BEFORE EXAM				
Q7. I STUDY HARDER TO IMPROVE MY PERFORMANCE WHEN I GET LOW GRADES				
Q8. I ACTIVELY DO EVERY MATHEMATICS ASSIGNMENT GIVEN IN CLASS				
Q9. EXTRACURRICULAR ACTIVITIES DO NOT HAMPER ME FROM SOLVING MATHEMATICS QUESTIONS.				
Q10. I STRIVE TO UNDERSTAND EVERYTHING TAUGHT IN CLASS.				

(Fig 1: Questionnaire)

From the questionnaire above, as distributed to 10 persons, we have the table below following from their responses.

Questions	SA	A	D	SD
1	2	4	3	1
2	6	2	1	1
3	3	6	1	0
4	5	1	2	2
5	2	2	4	2
6	1	3	5	1
7	3	4	0	3
8	0	5	4	1
9	1	7	2	0
10	9	0	0	1

(Fig 2)

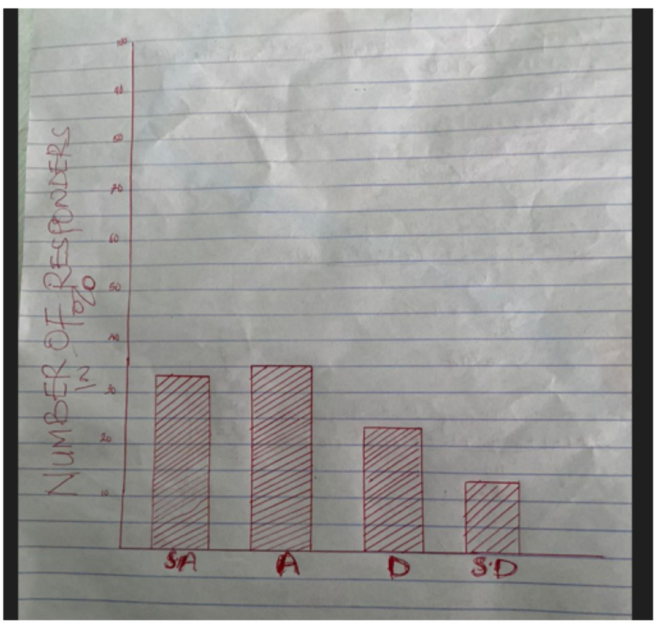
Analytically, the percentage of responders who strongly agree to the questions above is $\frac{2+6+3+5+2+1+3+0+1+9}{100} \times 100 = 32\%$.

Similarly,the percentage of responders who agree to the questions above is $\frac{4+2+6+1+2+3+4+5+7+0}{100} \times 100 = 34\%$.

Also, the percentage of responders who disagree to the questions above is $\frac{3+1+1+2+4+5+0+4+2+0}{100} \times 100 = 22\%$.

And the percentage of responders who strongly disagree to the questions above is $\frac{1+1+0+2+2+1+3+1+0+1+}{100} \times 100 = 12\%$.

The graph below shows the percentage of responders as they strongly agree, agree, disagree and strongly disagree to the questions.



(Fig 3)

3.2 Likert scale and fuzzy approach on a collected data

The same questionnaire issued above, with a touch of fuzziness is analysed. The template of the questionnaire was given below

QUESTIONAIRES TO EVALUATE THE CAUSE OF THE MASS FAILURE OF STUDENTS IN MATHEMATICS

WITH DEGREE OF ACCEPTANCE BETWEEN 0 AND 1

QUESTIONS	STRONGLY AGREE (SA (0-1)	AGREE (A) (0-1)	DISAGREE (D) (0-1)	STRONGLY DISAGREE(SD) (0-1)
Q1. MATHEMATICS IS FUN				
Q2. MATHEMATICS TEACHERS ARE DIFFICULT TO RELATE WITH				
Q3. I ACTIVELY PARTICIPATE IN MATH CLASS				
Q4. I HAVE A GOOD RELATIONSHIP WITH MY MATHEMATICS TEACHER				
Q5. I HAVE GOOD TEXTBOOKS THAT MAKE ME EXPLORE MATHEMATICS QUESTIONS				
Q6. I SOLVE MATHEMATICS QUESTIONS ONLY FEW DAYS BEFORE EXAM				
Q7. I STUDY HARDER TO IMPROVE MY PERFORMANCE WHEN I GET LOW GRADES				
Q8. I ACTIVELY DO EVERY MATHEMATICS ASSIGNMENT GIVEN IN CLASS				
Q9. EXTRACURRICULAR ACTIVITIES DO NOT HAMPER ME FROM SOLVING MATHEMATICS QUESTIONS.				
Q10. I STRIVE TO UNDERSTAND EVERYTHING TAUGHT IN CLASS.				

(Fig 4)

Following the response to the questionnaire in the fig 1 above, the table below gives the fuzzy response of the responsders.

R_1	Q_1	Q_2	Q_3	Q_4	Q_5	Q_6	Q_7	Q_8	Q_9	Q_{10}	R_2	SA	A	D	SD
1	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	Q_1	0.4	0.2	0.1	0.6
2	1.0	0.7	0.8	0.4	0.6	0.9	0.8	0.1	0.2	0.1	Q_2	0.3	0.4	0.4	0.5
3	0.0	1.0	0.8	0.4	0.3	0.2	0.1	0.3	0.4	0.2	Q_3	0.6	0.6	0.8	0.4
4	0.2	0.2	0.3	0.6	0.1	0.6	0.2	0.6	0.3	0.3	Q_4	0.5	0.8	0.6	0.3
5	0.4	1.0	0.8	0.9	0.6	0.5	0.6	0.4	0.7	0.4	Q_5	1.0	0.1	0.3	0.2
6	0.3	0.2	0.3	0.6	0.7	0.5	0.2	0.3	0.4	1.0	Q_6	0.2	0.3	0.2	0.1
7	0.4	0.1	0.3	0.2	0.4	0.5	0.6	0.7	0.8	0.8	Q_7	0.0	0.5	0.9	1.0
8	0.1	0.3	0.2	0.5	0.4	0.2	0.3	0.1	0.6	0.6	Q_8	0.6	0.7	1.0	0.9
9	0.6	0.1	0.3	0.4	0.5	0.8	0.9	1.0	0.3	0.4	Q_9	0.8	0.0	0.8	0.8
10	0.0	0.2	0.4	0.6	0.7	0.8	0.3	0.2	0.5	0.2	Q_{10}	1.0	0.4	0.7	0.5

(Fig 5)

From the table above, we have the following theorem of fuzzy relation.

Theorem 1. *Let* $X = [x_1, x_2, x_3, \cdots, x_{10}]$ *be the set of responders to the given questionnaires. Let* $Y = [q_1, q_2, q_3, \cdots, q_{10}]$ *be the given questions, and* $Z = [p_1, p_2, p_3, \cdots, p_{10}]$ *be the scale (0-1) with which the responders intuitively strongly agree(SA), agree(A), disagree(D) or strongly disagree(SD) to the questions which are represented by fuzzy membership values. Let* $R_1 \subseteq X \times Y$ *be the degree or rate of optimism of the responders as to answering the questions, and* $R_2 \subseteq Y \times Z$ *as generated from the questionnaire, define the fuzzy relation.*

Analyzing the fuzzy relation, we have the following analysis by max-min. It is noteworthy to say that the researcher could make the choice of using max product or max average for the analysis.

The degree to which responders strongly agree to the questions by max-min is

Q1	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
S.A	0.4	0.3	0.6	0.5	1.0	0.2	0.0	0.6	0.8	1.0
min	0.1	0.2	0.3	0.4	0.5	0.2	0.0	0.6	0.8	1.0
max	1.0									

Q2	1.0	0.7	0.8	0.4	0.6	0.9	0.8	0.1	0.2	0.1
S.A	0.4	0.3	0.6	0.5	1.0	0.2	0.0	0.6	0.8	1.0
min	0.4	0.3	0.6	0.4	0.6	0.2	0.0	0.1	0.2	0.1
max	0.6									

Q3	0.0	1.0	0.8	0.4	0.3	0.2	0.1	0.3	0.4	0.2
S.A	0.4	0.3	0.6	0.5	1.0	0.2	0.0	0.6	0.8	1.0
min	0.0	0.3	0.6	0.4	0.3	0.2	0.0	0.3	0.4	0.2
max	0.6									

Q4	0.2	0.2	0.3	0.6	0.1	0.6	0.2	0.6	0.3	0.3
S.A	0.4	0.3	0.6	0.5	1.0	0.2	0.0	0.6	0.8	1.0
min	0.2	0.2	0.3	0.5	0.1	0.2	0.0	0.6	0.3	0.3
max	0.6									

Q5	0.4	1.0	0.8	0.9	0.6	0.5	0.6	0.4	0.7	0.4
S.A	0.4	0.3	0.6	0.5	1.0	0.2	0.0	0.6	0.8	1.0
min	0.4	0.3	0.6	0.5	0.6	0.2	0.0	0.4	0.7	0.4
max	0.7									

Q6	0.3	0.2	0.3	0.6	0.7	0.5	0.2	0.3	0.4	1.0
S.A	0.4	0.3	0.6	0.5	1.0	0.2	0.0	0.6	0.8	1.0
min	0.3	0.2	0.3	0.5	0.7	0.2	0.0	0.3	0.4	1.0
max	1.0									

Q7	0.4	0.1	0.3	0.2	0.4	0.5	0.6	0.7	0.8	0.8
S.A	0.4	0.3	0.6	0.5	1.0	0.2	0.0	0.6	0.8	1.0
min	0.4	0.1	0.3	0.2	0.4	0.2	0.0	0.6	0.8	0.8
max	0.8									

Q8	0.1	0.3	0.2	0.5	0.4	0.2	0.3	0.1	0.6	0.6
S.A	0.4	0.3	0.6	0.5	1.0	0.2	0.0	0.6	0.8	1.0
min	0.1	0.3	0.2	0.5	0.4	0.2	0.0	0.1	0.6	0.6
max	0.6									

Q9	0.6	0.1	0.3	0.4	0.5	0.8	0.9	1.0	0.3	0.4
S.A	0.4	0.3	0.6	0.5	1.0	0.2	0.0	0.6	0.8	1.0
min	0.4	0.1	0.3	0.4	0.5	0.2	0.0	0.6	0.3	0.4
max	0.6									

Q10	0.0	0.2	0.4	0.6	0.7	0.8	0.3	0.2	0.5	0.2
S.A	0.4	0.3	0.6	0.5	1.0	0.2	0.0	0.6	0.8	1.0
min	0.0	0.2	0.4	0.5	0.7	0.2	0.0	0.2	0.5	0.2
max	0.7									

The percentage at which the responders strongly agree to the questions becomes

$$\frac{Addition\ of\ all\ max\ values}{10} \times 100$$

So we have

$$\frac{1.0 + 0.6 + 0.6 + 0.6 + 0.7 + 1.0 + 0.8 + 0.6 + 0.6 + 0.7}{10} \times 100 = 82\%.$$

The degree to which responders agree to the questions by max - min is :

Q1	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
A	0.2	0.4	0.6	0.8	0.1	0.3	0.5	0.7	0.0	0.4
min	0.1	0.2	0.3	0.4	0.1	0.3	0.5	0.7	0.0	0.4
max	0.7									

Q2	1.0	0.7	0.8	0.4	0.6	0.9	0.8	0.1	0.2	0.1
A	0.2	0.4	0.6	0.8	0.1	0.3	0.5	0.7	0.0	0.4
min	0.2	0.4	0.6	0.4	0.1	0.3	0.5	0.1	0.0	0.1
max	0.6									

Q3	0.0	1.0	0.8	0.4	0.3	0.2	0.1	0.3	0.4	0.2
A	0.2	0.4	0.6	0.8	0.1	0.3	0.5	0.7	0.0	0.4
min	0.0	0.4	0.6	0.4	0.1	0.2	0.1	0.3	0.0	0.2
max	0.6									

Q4	0.2	0.2	0.3	0.6	0.1	0.6	0.2	0.6	0.3	0.3
A	0.2	0.4	0.6	0.8	0.1	0.3	0.5	0.7	0.0	0.4
min	0.2	0.2	0.3	0.6	0.1	0.3	0.2	0.6	0.0	0.3
max	0.6									

Q5	0.4	1.0	0.8	0.9	0.6	0.5	0.6	0.4	0.7	0.4
A	0.2	0.4	0.6	0.8	0.1	0.3	0.5	0.7	0.0	0.4
min	0.2	0.4	0.6	0.8	0.1	0.3	0.5	0.4	0.0	0.4
max	0.8									

Q6	0.3	0.2	0.3	0.6	0.7	0.5	0.2	0.3	0.4	1.0
A	0.2	0.4	0.6	0.8	0.1	0.3	0.5	0.7	0.0	0.4
min	0.2	0.2	0.3	0.6	0.1	0.3	0.2	0.3	0.0	0.4
max	0.6									

Q7	0.4	0.1	0.3	0.2	0.4	0.5	0.6	0.7	0.8	0.8
A	0.2	0.4	0.6	0.8	0.1	0.3	0.5	0.7	0.0	0.4
min	0.2	0.1	0.3	0.2	0.1	0.3	0.5	0.7	0.0	0.4
max	0.7									

Q8	0.1	0.3	0.2	0.5	0.4	0.2	0.3	0.1	0.6	0.6
A	0.2	0.4	0.6	0.8	0.1	0.3	0.5	0.7	0.0	0.4
min	0.1	0.3	0.2	0.5	0.1	0.2	0.3	0.1	0.0	0.4
max	0.5									

Q9	0.6	0.1	0.3	0.4	0.5	0.8	0.9	1.0	0.3	0.4
A	0.2	0.4	0.6	0.8	0.1	0.3	0.5	0.7	0.0	0.4
min	0.2	0.1	0.3	0.4	0.1	0.3	0.5	0.7	0.0	0.4
max	0.7									

Q10	0.0	0.2	0.4	0.6	0.7	0.8	0.3	0.2	0.5	0.2
A	0.2	0.4	0.6	0.8	0.1	0.3	0.5	0.7	0.0	0.4
min	0.0	0.2	0.4	0.6	0.1	0.3	0.3	0.2	0.0	0.2
max	0.6									

The percentage at which the responders agree to the questions becomes

$$\frac{Addition\ of\ all\ max\ values}{10} \times 100$$

So we have

$$\frac{0.7+0.6+0.6+0.6+0.8+0.6+0.7+0.5+0.7+0.6}{10} = 64\%.$$

The degree to which responders disagree to the questions by max - min is

Q1	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
D	0.1	0.4	0.8	0.6	0.3	0.2	0.9	1.0	0.8	0.7
min	0.1	0.2	0.3	0.4	0.3	0.2	0.7	0.8	0.8	0.7
max	0.8									

Q2	1.0	0.7	0.8	0.4	0.6	0.9	0.8	0.1	0.2	0.1
D	0.1	0.4	0.8	0.6	0.3	0.2	0.9	1.0	0.8	0.7
min	0.1	0.4	0.8	0.4	0.3	0.2	0.8	0.1	0.2	0.1
max	0.8									

Q3	0.0	1.0	0.8	0.4	0.3	0.2	0.1	0.3	0.4	0.2
D	0.1	0.4	0.8	0.6	0.3	0.2	0.9	1.0	0.8	0.7
min	0.0	0.4	0.8	0.4	0.3	0.2	0.1	0.3	0.4	0.2
max	0.8									

Q4	0.2	0.2	0.3	0.6	0.1	0.6	0.2	0.6	0.3	0.3
D	0.1	0.4	0.8	0.6	0.3	0.2	0.9	1.0	0.8	0.7
min	0.1	0.2	0.3	0.6	0.1	0.2	0.2	0.6	0.3	0.3
max	0.6									

Q5	0.4	1.0	0.8	0.9	0.6	0.5	0.6	0.4	0.7	0.4
D	0.1	0.4	0.8	0.6	0.3	0.2	0.9	1.0	0.8	0.7
min	0.1	0.4	0.8	0.6	0.3	0.2	0.6	0.4	0.7	0.4
max	0.8									

Q6	0.3	0.2	0.3	0.6	0.7	0.5	0.2	0.3	0.4	1.0
D	0.1	0.4	0.8	0.6	0.3	0.2	0.9	1.0	0.8	0.7
min	0.1	0.2	0.3	0.6	0.3	0.2	0.2	0.3	0.4	0.7
max	0.7									

Q7	0.4	0.1	0.3	0.2	0.4	0.5	0.6	0.7	0.8	0.8
D	0.1	0.4	0.8	0.6	0.3	0.2	0.9	1.0	0.8	0.7
min	0.1	0.1	0.3	0.2	0.3	0.2	0.6	0.7	0.8	0.7
max	0.8									

Q8	0.1	0.3	0.2	0.5	0.4	0.2	0.3	0.1	0.6	0.6
D	0.1	0.4	0.8	0.6	0.3	0.2	0.9	1.0	0.8	0.7
min	0.1	0.3	0.2	0.5	0.3	0.2	0.3	0.1	0.6	0.6
max	0.6									

Q9	0.6	0.1	0.3	0.4	0.5	0.8	0.9	1.0	0.3	0.4
D	0.1	0.4	0.8	0.6	0.3	0.2	0.9	1.0	0.8	0.7
min	0.1	0.1	0.3	0.4	0.3	0.2	0.9	1.0	0.3	0.4
max	1.0									

Q10	0.0	0.2	0.4	0.6	0.7	0.8	0.3	0.2	0.5	0.2
D	0.1	0.4	0.8	0.6	0.3	0.2	0.9	1.0	0.8	0.7
min	0.0	0.2	0.4	0.6	0.3	0.2	0.3	0.2	0.5	0.2
max	0.6									

The percentage at which the responders disagree to the questions becomes

$$\frac{Addition\ of\ all\ max\ values}{10} \times 100$$

So we have

$$\frac{0.8+0.8+0.8+0.6+0.8+0.7+0.8+0.6+1.0+0.6}{10} \times 100 = 76\%.$$

The degree to which responders strongly disagree to the questions by max - min is :

Q1	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
SD	0.6	0.5	0.4	0.3	0.2	0.1	1.0	0.9	0.8	0.5
min	0.1	0.2	0.3	0.3	0.2	0.1	0.7	0.8	0.8	0.5
max	0.8									

Q2	1.0	0.7	0.8	0.4	0.6	0.9	0.8	0.1	0.2	0.1
SD	0.6	0.5	0.4	0.3	0.2	0.1	1.0	0.9	0.8	0.5
min	0.6	0.5	0.4	0.3	0.2	0.1	0.8	0.1	0.2	0.1
max	0.8									

Q3	0.0	1.0	0.8	0.4	0.3	0.2	0.1	0.3	0.4	0.2
SD	0.6	0.5	0.4	0.3	0.2	0.1	1.0	0.9	0.8	0.5
min	0.0	0.5	0.4	0.3	0.2	0.1	0.1	0.3	0.4	0.2
max	0.5									

Q4	0.2	0.2	0.3	0.6	0.1	0.6	0.2	0.6	0.3	0.3
SD	0.6	0.5	0.4	0.3	0.2	0.1	1.0	0.9	0.8	0.5
min	0.2	0.2	0.3	0.3	0.1	0.1	0.2	0.6	0.3	0.3
max	0.6									

Q5	0.4	1.0	0.8	0.9	0.6	0.5	0.6	0.4	0.7	0.4
SD	0.6	0.5	0.4	0.3	0.2	0.1	1.0	0.9	0.8	0.5
min	0.4	0.5	0.4	0.3	0.2	0.1	0.6	0.4	0.7	0.4
max	0.7									

Q6	0.3	0.2	0.3	0.6	0.7	0.5	0.2	0.3	0.4	1.0
SD	0.6	0.5	0.4	0.3	0.2	0.1	1.0	0.9	0.8	0.5
min	0.3	0.2	0.3	0.3	0.2	0.1	0.2	0.3	0.4	0.5
max	0.5									

Q7	0.4	0.1	0.3	0.2	0.4	0.5	0.6	0.7	0.8	0.8
SD	0.6	0.5	0.4	0.3	0.2	0.1	1.0	0.9	0.8	0.5
min	0.4	0.1	0.3	0.2	0.2	0.1	0.6	0.7	0.8	0.5
max	0.8									

Q8	0.1	0.3	0.2	0.5	0.4	0.2	0.3	0.1	0.6	0.6
SD	0.6	0.5	0.4	0.3	0.2	0.1	1.0	0.9	0.8	0.5
min	0.1	0.3	0.2	0.3	0.2	0.1	0.3	0.1	0.6	0.5
max	0.6									

Q9	0.6	0.1	0.3	0.4	0.5	0.8	0.9	1.0	0.3	0.4
SD	0.6	0.5	0.4	0.3	0.2	0.1	1.0	0.9	0.8	0.5
min	0.6	0.1	0.3	0.3	0.2	0.1	0.9	0.9	0.3	0.4
max	0.9									

Q10	0.0	0.2	0.4	0.6	0.7	0.8	0.3	0.2	0.5	0.2
SD	0.6	0.5	0.4	0.3	0.2	0.1	1.0	0.9	0.8	0.5
min	0.0	0.2	0.4	0.3	0.2	0.1	0.3	0.2	0.5	0.2
max	0.5									

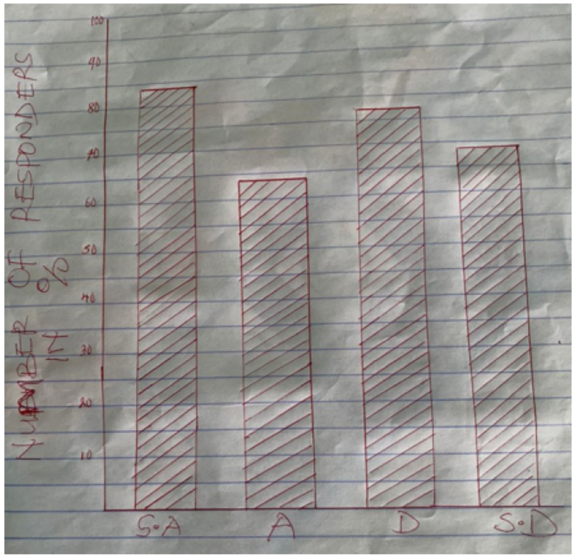
The percentage at which the responders strongly disagree to the questions becomes

$$\frac{Addition\ of\ all\ max\ values}{10} \times 100$$

So we have

$$\frac{0.8 + 0.8 + 0.5 + 0.6 + 0.7 + 0.5 + 0.8 + 0.6 + 0.9 + 0.5}{10} \times 100 = 67\%.$$

Graphically, we have :



(Fig 6)

4 Conclusion

We have been able to construct a phenomenal that goes beyond classical set (Fuzziness). To a widely used data collection tool (Likert Scale), we inject certain fuzzy parameter (Relation) which takes care of uncertainties, statistical analysis was carried out on the results generated from the two cases (Likert scale with(out) fuzziness) and it is evident that the one with fuzziness give a more effective and robust result.

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