Earthline Journal of Mathematical Sciences
http://www.earthlinepublishers.com
Volume 1, Number 1, 2019, Pages 105-118
https://doi.org/10.34198/ejms.1119.105118

A New Symmetric p-stable Obrechkoff Method with Optimal Phase-lag for Oscillatory Problems

I. C. Felix[1,*], **O. O. Famoofo**[2] **and S. M. Akintewe**[3]

[1,2] School of Engineering and Applied Sciences, Lagos City Polytechnic, Ikeja, Lagos State, Nigeria; e-mail: isaac.c.felix@aims-senegal.org[1]; emiaserere@gmail.com[2]

[3] Department of Mathematical Science, Adekunle Ajasin University, Akungba-Akoko, Ondo State, Nigeria; e-mail: mercy.s.akintewe@gmail.com[3]

[*] Corresponding author

Abstract

In this paper, we derive a class of symmetric p-stable Obrechkoff methods via Padé approximation approach (PAA) for the numerical solution of special second order initial value problems (IVPs) in ordinary differential equations (ODEs). We investigate periodicity analysis on the proposed scheme to verify p-stability property. The new algorithms possess minimum phase-lag error which shows that they can accurately solve oscillatory problems. Reports on several numerical experiments are provided to illustrate the accuracy of the method.

1. Introduction

Our task in this paper is to approximate the solution of the special second order IVPs of the form

$$y'' = f(x, y), \quad y(x_0) = y_0, \quad y'(x_0) = y'_0, \tag{1}$$

Received: January 4, 2019; Accepted: January 18, 2019

2010 Mathematics Subject Classification: 65L05, 65L06.

Keywords and phrases: p-stability, Obrechkoff method, Padé approximation, symmetric, oscillatory, phase-lag error.

where $y(x) \in \mathbb{R}$, $f : \mathbb{R} \times \mathbb{R}^g \to \mathbb{R}^g$, and the first derivative does not appear explicitly. Such problems are often encountered in applied science and engineering. Popular examples include: mechanical systems without dissipation, satellite tracking, celestial mechanics, etc. The solution of the type (1) which is considered in this paper is a priori known to be periodic, and when integrated numerically, the desire is that the numerical solution also preserves the analogical periodicity of the analytic solution [10-23]. Furthermore, equation (1) is known to have inherent "periodic stiffness" [12] which makes it difficult to solve analytically. Numerical methods must be employed to obtain its approximate solution. The well-known Stomer-Cowell method with step number greater than two exhibits orbital instability making it unsuitable to numerically solve (1). Unlike other stability requirements, 2-step p-stable methods remain desirable for the solution of (1). There is vast literature on approximate solution of (1), see [7], [8], [9], [10], [11], [31], [32], [3-5], [2], [24-27], [6], [11], [32], [28-30], [10], [35-37], [20], [33]. The significance of this present work is five-fold; (i) to illustrate the strength of PAA in the development of numerical methods capable of handling IVPs that are periodic in nature, (ii) derive new p-stable methods and investigate their phase-lag properties, (iii) provide useful insight on p-stable Obrechkoff methods, (iv) remark that "*direct application of PAA framework*" on the development of p-stable numerical methods will often be limited to schemes of order $p \leq 4$, (v) wide application of the derived methods. In what follows, we have demonstrated the accuracy of our methods using stiff, linear and non-linear IVPs in ODEs.

Consider the Obrechkoff method of the form

$$\sum_{i=0}^{k} \alpha_i y_{n-j+1} = \sum_{i=1}^{l} h^{2i} \sum_{j=0}^{k} \beta_{ij} y_{n-j+1}^{(2i)}, \tag{2}$$

for the numerical solution of the problem (1). When the method (2) is applied on test equation (3)

$$y'' + \lambda^2 y = 0, \quad \lambda,\ y \in \mathbb{R} \tag{3}$$

we get the characteristics equation as

$$\rho(\xi) = \sum_{i=1}^{l} (-1)^i v^{2i} \sigma_i(\xi), \tag{4}$$

where $v - \lambda h = 0$ and

$$\rho(\xi) = \sum_{j=0}^{k} \alpha_j \xi^{k-j}, \quad \sigma_i(\xi) = \sum_{j=0}^{k} \beta_{ij} \xi^{k-j}, \quad i = 1, 2, ..., l. \tag{5}$$

Definition 1.1. The method in (2) is said to be *symmetric* if $\alpha_j = \alpha_{k-j}$, $\beta_j = \beta_{k-j}$, $j = 0, 1, 2, ..., k$.

Definition 1.2. The method in (2) is said to have order p if the truncation error associated with the linear difference operator is given as

$$TE = C_{p+2} h^{p+2} y(p+2), \quad x_{n-k+1} < \eta < x_{n+1}, \tag{6}$$

where C_{p+2} is the error constant dependent on h.

Definition 1.3. The method in (2) is said to have interval of periodicity $(0, v_0^2)$ if for all $v_0^2 \in (0, v_0^2)$ the roots of (4) are complex and at least two of them lie on the unit circle and others lie inside the unit circle.

Definition 1.4. The method in (2) is said to be *p-stable* if its interval of periodicity is $(0, \infty)$.

2. Derivation of the Method

In the spirit of [10], we consider the following algebraic expressions

$$\Pi(\xi, z) = (P_j(z) P_j(-z))\xi^2 - (P_j^2(z) + P_j^2(-z))\xi + (P_j(z) P_j(-z)) \tag{7}$$

with $z \in \mathbb{C}$ and P_j described by the expression

$$P_j(z) = 1 + \frac{j}{2j} z + \frac{j(j-1)}{(2j)(2j-1)} \frac{z^2}{2!} + \cdots + \frac{j(j-1)\cdots 2 \cdot 1}{(2j)(2j-1)\cdots(j+1)} \frac{z^j}{j!} \tag{8}$$

whose roots is given by

$$\xi_1 = (\xi_2)^{-1} = \frac{P_j(z)}{P_j(-z)}, \tag{9}$$

such that

$$\| R_{j,j}(z) - e^z \| \le O(\| z \|^{2j+1}), \quad z \to 0 \tag{10}$$

is known as the (j, j)-diagonal Padé approximation to e^z. Observe that (7) is of the form (4). In what follows, the basic idea is to obtain the stability function of an integration scheme and compare it with (7) to determine the corresponding coefficients of the numerical method.

Consider the symmetric Obrechkoff methods (2), when applied on (3) yield the following stability function

$$\Pi(\xi, z) = (\alpha_2 - z^2\beta_{10} - z^4\beta_{20})\xi^2 + (\alpha_1 - z^2\beta_{11} - z^4\beta_{21})\xi + (\alpha_2 - z^2\beta_{10} - z^4\beta_{20}). \quad (11)$$

Without loss of generality, we compare (11) with results from which $j = 2$ in (7) and obtain the following symmetric p-stable Obrechkoff method

$$y_{n+1} - 2y_n + y_{n-1} = \frac{h^2}{12\gamma}[10y_n'' + (y_{n+1}'' + y_{n-1}'')] + \frac{h^4}{72\gamma}\left[y_n'' - \frac{1}{2}(y_{n+1}'' + y_{n-1}'')\right], \quad (12)$$

where $\gamma \geq 10^3$. Figure 1 shows an indefinite progression of the new method along the positive real axis satisfying Definition 1.4,

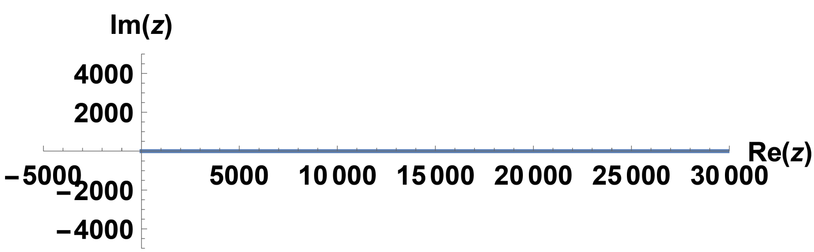


Figure 1. The stability plot of scheme (12).

Remarkably, the stability region of the famed most accurate p-stable formulas satisfying Definition 1.4 is given in Figure 2

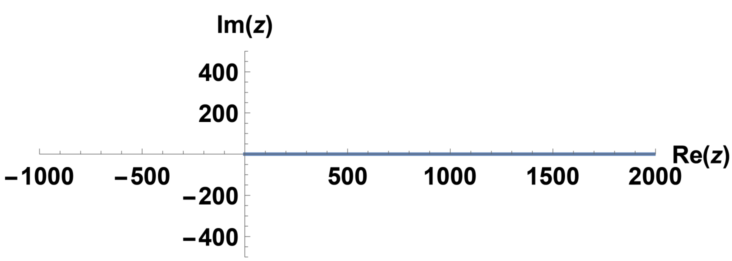


Figure 2. The stability plot of the famed most accurate p-stable method.

Also, in [10] the following methods are proposed

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h^2 \sum_{j=0}^{k} \beta_j f(t_{n+j}, y_{n+j}) + h^2 \beta f(\tilde{t}_n, \tilde{y}_n) \tag{13}$$

$$\tilde{y}_n = \sum_{j=0}^{k} \gamma_j y_{n+j} + h^2 \sum_{j=0}^{k} \delta_j f(t_{n+j}, y_{n+j}), \tag{14}$$

where h is the step size and

$$t_j = t_0 + jh, \quad \tilde{t}_n = t_n + \left(\sum_{j=0}^{k} \gamma_j j \right) h. \tag{15}$$

When applied to the scalar test (3) yields a difference equation with the following characteristics polynomial

$$\sum_{j=0}^{k} [\alpha_j - z^2(\beta_j + \beta\gamma_j) - z^4 \beta\delta_j] R^j = 0. \tag{16}$$

The interval of periodicity of this scheme is found to be indeed p-stable. Figure 3 shows the stability region of the integration scheme. Clearly, this grows indefinitely along the positive real axis with however some insignificant jumps at the origin which could be as a result of parameter choice.

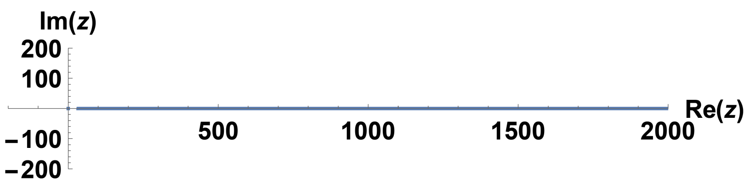


Figure 3. Stability plot showing the interval of periodicity of scheme (13).

Next, we carry out phase-lag analysis on the method (12) following [20-22], see also [27-29] and [31-33]. The phase-lag analysis of numerical method is described by

$$PL(H) = \frac{2\sum_{j=1}^{\frac{k'}{2}} A_j(H^2)\cos(jH) + A_0(H^2)}{2\sum_{j=1}^{\frac{k'}{2}} j^2 A_j(H^2)} \tag{17}$$

and when expanded has a non-vanishing phase-lag order d and phase-lag constant c_{d+2}.

$$PL(H) = \frac{2A_{\frac{k'}{2}}(H^2)\cos\left(\frac{k'}{2}H\right) + \cdots + 2A_j(H^2)\cos(jH) + A_0(H^2)}{2\left(\frac{k'}{2}\right)^2 + \cdots + 2j^2A_j(H^2) + \cdots + 2A_1(H^2)}$$

$$= -c_{d+2}H^{d+2} + O(H^{d+4}). \tag{18}$$

In what follows, the application of (18) on the derived method (12) yields the following phase-lag

$$-\frac{1}{120}z^6 + 0(z^8) = -c_{d+2}z^{d+2} + 0(z^{d+4}), \tag{19}$$

where $-c_{d+2}$ is the phase-lag error (PLE) constant, and d, the phase-lag order.

Remark. For all values of j in (7), p-stable numerical methods obtained by directly applying PAA will be limited to order of accuracy $p \leq 4$.

3. Numerical Experiment and Results

In this section, we carry out numerical experiments in order to show the accuracy of the new algorithm. Consider the following second order IVPs.

Example 1. (An orbital problem): Source [12]

$$y'' = -y + 0.001e^{ix}$$

$$y(0) = 1, \quad y'(0) = 0.9995i, \quad i^2 = -1 \tag{20}$$

and has a theoretical solution

$$y(x) = u(x) + v(x)$$

$$u(x) = \cos x + 0.0005x\sin x$$

$$v(x) = \sin x - 0.0005x\cos x. \tag{21}$$

The initial value problem (20) represents a motion on a perturbed circular orbits in the complex plane in which the point $y(x)$ slowly spirals outward such that its distance from the origin at a any given time x is described by

$$\Psi(x) = \sqrt{u^2(x) + v^2(x)}. \tag{22}$$

Using the predictor

$$y_{n+2} - 2y_{n+1} + y_n = h^2 f_{n+1}, \tag{23}$$

the system of equation in (20) can be approximated in the interval [0, 40] which corresponds to 20 orbits of the points $y(x)$. The integration is then carried out with uniform meshsizes

$$h = \frac{\pi}{2^q}, \quad q = 3(1)10. \tag{24}$$

The numerical results from the new p-stable Obrechkoff methods comparing with existing p-stable numerical methods from the literature are presented in Table 1 and Table 2 respectively.

Table 1. $\Psi(x_f) = 1.00197197653449$.

q	h	New symmetric Obrechkoff method (12) (Ψ)	Error
3	$\frac{\pi}{2^3}$	1.001848523273178	1.24E-4
4	$\frac{\pi}{2^4}$	1.001801747000679	1.70E-4
5	$\frac{\pi}{2^5}$	1.001987397382566	1.54E-5
6	$\frac{\pi}{2^6}$	1.001974909077729	2.93E-6
7	$\frac{\pi}{2^7}$	1.001973299992589	1.32E-6
8	$\frac{\pi}{2^8}$	1.001972374396628	3.98E-7
9	$\frac{\pi}{2^9}$	1.001972355120986	3.79E-7
10	$\frac{\pi}{2^{10}}$	1.001972345485908	3.69E-7

Table 2. Continuation of Table 1.

q	h	Method in [12]	Error	Method in [10]	Error	Method in [2]	Error
3	$\frac{\pi}{2^3}$	0.965645	3.63E-2	0.994863	7.11E-3	NA	NA
4	$\frac{\pi}{2^4}$	0.993734	8.23E-3	0.997223	4.75E-3	1.004118	2.15E-3
5	$\frac{\pi}{2^5}$	0.999596	2.38E-3	0.997578	4.39E-3	1.002856	8.84E-4
6	$\frac{\pi}{2^6}$	NA	NA	0.997687	4.39E-3	1.002400	4.3E-4
7	$\frac{\pi}{2^7}$	NA	NA	0.997730	4.24E-3	NA	NA
8	$\frac{\pi}{2^8}$	NA	NA	0.997748	4.22E-3	NA	NA
9	$\frac{\pi}{2^9}$	1.001829	1.43E-4	0.997757	4.22E-3	1.002057	8.50E-5
10	$\frac{\pi}{2^{10}}$	NA	NA	0.997761	4.21E-3	NA	NA

In what follows, we apply the derived method (12) to investigate the orbital property of problem (20),

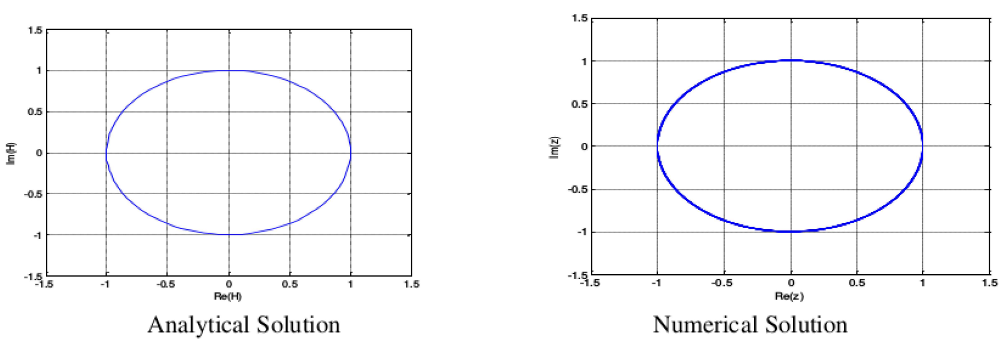


Figure 4. Numerical simulation of an orbital problem comparing with its analytical solution.

Example 2. (Stiff Oscillatory Problem: Source [14])

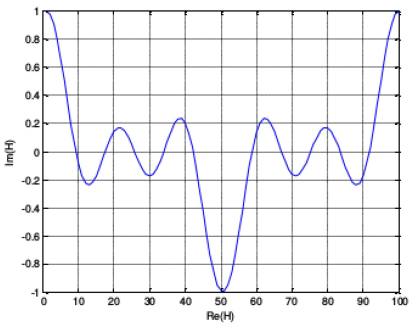
Consider the stiff oscillatory problem defined by

$$y''(t) + m^2 y(t) = 8\left(\cos(t) + \frac{2}{3}\cos(3t)\right), \quad y(0) = 1,\ y'(0) = 0 \quad \text{with} \quad m = 5 \tag{25}$$

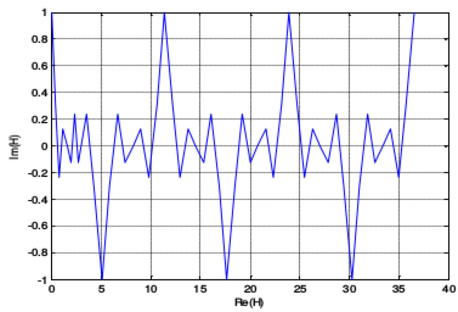
whose exact solution is

$$y(t) = \frac{1}{3}(\cos t + \cos 3t + \cos 5t). \tag{26}$$

In similar manner, we implement the system (25) using meshsize $h = \frac{\pi}{8}$ at $t = 10\pi$.



Analytical Solution



Numerical Solution

Figure 5. Numerical simulation of stiff oscillatory problem comparing with its analytical solution.

Example 3. (Undamped Duffing Problem: Source [3])

Consider again a non-linear duffing problem,

$$y'' + y + y^3 = \delta\cos(\mu t), \quad y(0) = A, \quad y'(0) = 0. \tag{27}$$

The initial condition described by A is the value of the Galerkin's approximation y_G at $t = 0$. Problem (27) is forced by a harmonic function with parameter values $\delta = 0.002$ and $\mu = 1.01$. However, by Urabe's method applied to Galerkin's procedure, Van Dooren has carried out computation of the Galerkin's approximation of order $p = 9$ to a periodic solution having the same period as the forcing term with a precision of the coefficients of $\frac{1}{10^{12}}$:

$$y_G = \sum_{i=0}^{5} \alpha_{2i+1} \cos(2i+1)\mu t, \tag{28}$$

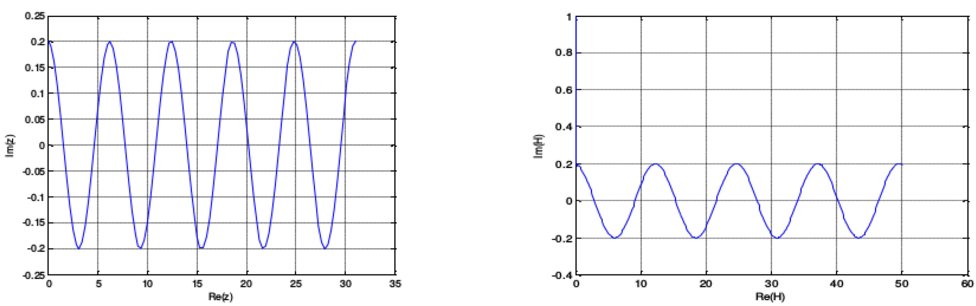
in a simplified form, (28) becomes

$$\begin{aligned} y_G &= \alpha_1 \cos \mu t + \alpha_3 \cos(3\mu t) + \alpha_5 \cos(5\mu t) + \alpha_7 \cos(7\mu t) \\ &\quad + \alpha_9 \cos(9\mu t) + \alpha_{11} \cos(11\mu t), \end{aligned} \tag{29}$$

where

$$\begin{aligned} &\alpha_1 = 0.200179477536, \ \alpha_3 = 0.246946143 \times 10^{-3}, \ \alpha_5 = 0.304014 \times 10^{-6} \\ &\alpha_7 = 0.374 \times 10^{-9}, \ \alpha_9 = 0.460964452 \times 10^{-12}, \ \alpha_{11} = 0.5676 \times 10^{-15}. \end{aligned} \tag{30}$$

Using step size $h = \frac{\pi}{2^3}$ at $t = 40\pi$, we apply the new method on problem (27) and obtained the following behaviours



Analytical Solution Numerical Solution

Figure 6. Numerical simulation of undamped duffing problem comparing with its analytical solution.

4. Conclusion

PAA is an interesting method for the development of numerical schemes that must possess p-stability property. In view of the foregoing, we present a class of symmetric p-stable Obrechkoff methods which possess minimum phase-lag error for the solutions of IVPs that are periodic in nature. However, we desired higher order p-stable method, but due to the limitation encountered in the use of PAA, we therefore remark that numerical algorithms obtained by directly applying PAA will often be limited to order $(p \leq 4)$, see also the works in [7, 8, 10]. In particular, the new results in this article enjoy considerable order of accuracy and stronger p-stability property which are illustrated in Table 1 and Figures 1, 4, 5, 6, respectively.

Acknowledgements

The authors are indebted to Professors M. N. O. Ikhile and R. I. Okuonghae (Department of Mathematics, University of Benin) for their detailed comments and helpful suggestions.

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