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Fixed Point Theorems in Extended Convex Quasi s-metric Spaces

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Abstract

In this work, through the convex structure, we introduce the concept of the extended convex quasi s-metric spaces. In addition, through Mann's iterative technique, we theorize the existence of a unique fixed point for two types of contraction mapping in extended convex quasi s-metric spaces.

1 Introduction

Since 1922 when introduced S. Banach in his doctoral thesis, theorem that there is a unique fixed point for each contraction over an entire metric space. We can notice the great interest in developing fixed point theory due to the diversity of its applications in many branches of mathematics, especially in metric spaces. Hundreds of scientific papers have been presented to generalize this result [2–4, 10, 11, 13, 15].

Czerwik [5] in 1993 introduced a new s-metric space with a more generalized inequality of the triangle (with a coefficient of $s \geq 1$), which is called a s-metric space. Later on, a new space, known as a quasi s-metric space, was proposed by Felhi et al. [8], which is created by taking both the s-metric and quasi metric space .

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Takahashi [14] introduced the idea of a convex structure in 1970 and named the space with a convex structure as “convex metric space”. Then many researchers studied exsists the fixed point in it, as in the papers of Goebel and Kirk [9], Shih [6] and Reich and Shafrir [12].

Chen et al. [7] introduced the convex b metric space and extend Maan's iteration algorithm. The first step in in our this work involves applying the convex suructure and extended quasi-b-metric space to interoduce a space referred to as the extended convex quasi s-metric spaces (Shortly ECQSMS). Furthermore, we consider two types of contraction mapping in ECQSMS and verify that these mappings possess a fixed point that is unique by Mann's iterative technique.

Definition 1.1. [8] Given a set $C \neq \oslash$ with a constant $s \geq 1$ and a mapping $\hbar : C \times C \longrightarrow [0, \infty)$ which is said to be a *quasi s-metric mapping*, if the following conditions are true for every $v, r, t \in C$:

(i) $\hbar(v, r) = 0$ and $\hbar(r, v) = 0 \Leftrightarrow v = r$;

(ii) $\hbar(v, r) \leq s[\hbar(v, t) + \hbar(t, r)]$.

A pair $(C, \hbar)$ is called a *quasi s-metric space* (Shortly QSMS).

Definition 1.2. A mapping $\hbar_\mu : C \times C \to [0, \infty)$ is named a *extended quasi $s-$metric* (Shortly EQSMS) on a nonempty set C if for any $n, m, v \in C$ and a mapping $\mu : C \times C \to [1, \infty)$, it satisfies the following conditions:

(i) $\hbar_\mu(n, m) = 0$ and $\hbar_\mu(m, n) = 0 \Leftrightarrow n = m$;

(ii) $\hbar_\mu(n, m) \leq \mu(n, m)[\hbar_\mu(n, v) + \hbar_\mu(v, m)]$.

The pair $(C, \hbar_\mu)$ is called *EQSMS*.

Definition 1.3. [1,8] Every sequence $\{l_n\}$ in C converges to some $\omega \in C \Leftrightarrow$

$$\lim_{n \longrightarrow \infty} \hbar_\mu(l_n, \omega) = \lim_{n \longrightarrow \infty} \hbar_\mu(\omega, l_n).$$

The left (right)-Cauchy sequence is defined as defined in the papers [1,8].

Definition 1.4. Let $(C, \hbar)$ be a EQSMS and $I = [0,1]$. Define a continuous function $F : C \times C \times I \to C$. Then F is said to be the *convex structure* on C if the following holds:

$$\hbar(v, F(n,m;\tau)) \leq \tau\hbar(v,n) + (1-\tau)\hbar(v,m) \tag{1.1}$$

for all $v, n, m \in C$ and $\tau \in I$.

2 Main Results

In this section, we begin by defining convex quasi s metric space (shortly CQSMS) and then proceed to define extended convex quasi s metric space (shortly ECQSMS).

Definition 2.1. Let $(C, \hbar)$ be a QSMS with $s \geq 1$ and $I = [0,1]$. Define a continuous function $F : C \times C \times I \to C$. Then F is said to be the *convex structure* on C if for all $r, n, m \in C$ and $\tau \in I$, the following hold:

$$\hbar(r, F(n,m;\tau)) \leq \tau\hbar(r,n) + (1-\tau)\hbar(r,m) \tag{2.1}$$

and

$$\hbar(F(n,m;\tau), r) \leq \tau\hbar(n,r) + (1-\tau)\hbar(m,r) \tag{2.2}$$

Then, we say that $(C, \hbar, F)$ is a *CQSMS*.

Example 2.1. Let $C = R$ and for any $a, b \in C$, $l > 1$, define the function such as

$$\begin{aligned} \hbar(a,b) &= |a-b+2|^l \quad \text{if} \quad a = 1 \neq b \\ &= |a-b|^l \quad \text{if} \quad o.w. \end{aligned}$$

To prove that $(C, \hbar, F)$ is a CQSMS with $s = 2^{l-1}$, we will use the fact

$$(u+v)^l \leq 2^{l-1}(u^l + v^l)$$

holds, for any $u, v \in [0, \infty)$ and $l \geq 1$. Indeed, $(C, \hbar)$ is a QSMS. However, $(C, \hbar)$ is not s-metric space because $\hbar$ does not satisfy symmetric condition.

$$\hbar(1,3) = |1-3+2|^l = 1 \neq \hbar(3,1) = |1-3|^l = 2^l.$$

Let $F : C \times C \times \left\{\frac{1}{2}\right\} \to C$ be the mapping defined as

$$F(a,b;\tau) = \frac{a+b}{2}$$

for any $a, b \in C$. To prove that F is convex structure on C. For any $a, b, k \in C$, for $a = 1 \neq b$, we have

$$\begin{aligned}
\hbar(1, F(b,k;\frac{1}{2})) &= \left|1 - \frac{b+k}{2} + 2\right|^l \\
&\leq 2^{l-1}\left[\left|\frac{3-b}{2}\right|^l + \left|\frac{3-k}{2}\right|^l\right] \\
&\leq 2^{-1}[|3-b|^l + |3-k|^l] \\
&= \tau\hbar(1,b) + (1-\tau)\hbar(1,k),
\end{aligned}$$

and for otherwise, we get

$$\begin{aligned}
\hbar(a, F(b,k;\frac{1}{2})) &= \left|a - \frac{b+k}{2}\right|^l \\
&\leq 2^{l-1}\left[\left|\frac{a-b}{2}\right|^l + \left|\frac{a-k}{2}\right|^l\right] \\
&\leq 2^{-1}\left[|a-b|^l + |a-k|^l\right] \\
&= \tau\hbar(a,b) + (1-\tau)\hbar(a,k).
\end{aligned}$$

By the same argument we can prove that $\hbar(F(b,k;\frac{1}{2}), a) \leq \tau\hbar(b,a) + (1-\tau)\hbar(k,a)$. Hence, $(C, \hbar, F)$ is a CQSMS with $s = 2^{l-1}$.

Definition 2.2. Let $(C, \hbar_\mu)$ be a EQSMS and $I = [0,1]$. Define a continuous function $F : C \times C \times I \to C$. Then F is said to be the *convex structure* on C if for all $s, n, m \in C$ and $\tau \in I$, the following hold:

$$\hbar_\mu(s, F(n,m;\tau)) \leq \tau\hbar_\mu(s,n) + (1-\tau)\hbar_\mu(s,m) \tag{2.3}$$

and

$$\hbar_\mu(F(n,m;\tau),s) \leq \tau\hbar_\mu(n,s) + (1-\tau)\hbar_\mu(m,s). \tag{2.4}$$

Then, we say that $(C,\hbar_\mu,F)$ is an *ECQSMS space.*

Let $(C,\hbar_\mu,F)$ is a ECQSMS and $H : C \to C$ be a mapping, a sequence x_n where

$$x_{n+1} = F(x_n, Hx_n;\tau_n), \quad n \in N$$

is called *Mann's iteration sequence* for H.

Theorem 2.2. *Let $(C,\hbar_\mu,F)$ be a complete ECQSMS with a a contraction mapping $\mu : C \times C \to (1,\infty)$ and $H : C \to C$; that is, there exists $\alpha \in [0,1)$ such that*

$$\hbar_\mu(Hs,Hr) \leq \alpha\hbar_\mu(s,r),$$

for all $s,r \in C$. Let us choose $v_0 \in C$ such that $\hbar_\mu(v_0,Hv_0) = N < \infty$ and define $v_n = F(v_{n-1},Hv_{n-1};\tau_{n-1})$, where $0 \leq \tau_{n-1} < 1$ and $n \in \mathbb{N}$. If $[\tau_{n-1}+\alpha(1-\tau_{n-1})] \leq \frac{1}{(\mu(v_{n-1},v_n))^2}$, then H has a unique fixed point in C.

Proof. The following inequality holds for any $n \in \mathbb{N}$,

$$\hbar_\mu(v_n,v_{n+1}) = \hbar_\mu(v_n,F(v_n,Hv_n;\tau_n)) \leq (1-\tau_n)\hbar_\mu(v_n,Hv_n)$$

and

$$\begin{aligned}
&\hbar_\mu(v_n,Hv_n)\\
&\leq \mu(v_n,Hv_n)\hbar_\mu(v_n,Hv_{n-1}) + \mu(v_n,Hv_n)\hbar_\mu(Hv_{n-1},Hv_n)\\
&\leq \mu(v_n,Hv_n)\hbar_\mu(F(v_{n-1},Hv_{n-1};\tau_{n-1}),Hv_{n-1}) + \mu(v_n,Hv_n)\alpha\hbar_\mu(v_{n-1},v_n)\\
&\leq \mu(v_n,Hv_n)[\tau_{n-1}\hbar_\mu(v_{n-1},Hv_{n-1}) + \alpha(1-\tau_{n-1})\hbar_\mu(v_{n-1},Hv_{n-1})]\\
&= \mu(v_{n-1},v_n)[\tau_{n-1}+\alpha(1-\tau_{n-1})]\hbar_\mu(v_{n-1},Hv_{n-1}).
\end{aligned}$$

Since $[\tau_{n-1} + \alpha(1 - \tau_{n-1})] \leq \frac{1}{(\mu(v_{n-1}, v_n))^2}$ for any $n \in \mathbb{N}$, we get

$$\hbar_\mu(v_n, Hv_n) \leq \mu(v_{n-1}, v_n)[\tau_{n-1} + \alpha(1 - \tau_{n-1})]\hbar_\mu(v_{n-1}, Hv_{n-1}) \tag{2.5}$$

$$< \frac{1}{[\mu(v_{n-1}, v_n)]}\hbar_\mu(v_{n-1}, Hv_{n-1}). \tag{2.6}$$

That means that,$\hbar_\mu(v_n, Hv_n)$ is a non-negative real decreasing sequence. Hence, there exists $\eta \geq 0$ such that

$$\lim_{n \longrightarrow \infty} \hbar_\mu(v_n, Hv_n) = \eta.$$

Suppose that $\eta > 0$, letting $n \to \infty$ in inequality (2.5), we get

$$\eta \leq \frac{1}{[\mu(v_{n-1}, v_n)]}\eta < \eta,$$

and this contraction. Then $\eta = 0$ and we have

$$\hbar_\mu(v_n, v_{n+1}) \leq (1 - \tau_n)\hbar_\mu(v_n, Hv_n) < \hbar_\mu(v_n, Hv_n),$$

that means that

$$\lim_{n \longrightarrow \infty} \hbar_\mu(v_n, v_{n+1}) = 0.$$

By the same procedure, we can write

$$\hbar_\mu(v_{n+1}, v_n) < \hbar_\mu(v_n, Hv_n),$$

that means that

$$\lim_{n \longrightarrow \infty} \hbar_\mu(v_{n+1}, v_n) = 0.$$

Now, we provide proof that $\{v_n\}$ is a Cauchy sequence. For any $n, k \in \mathbb{N}$, we have

$\hbar_\mu(v_n, v_{n+k+1})$
$= \hbar_\mu(v_n, F(v_{n+k}, Hv_{n+k}, \tau_{n+k}))$
$\leq \tau_{n+k}\hbar_\mu(v_n, v_{n+k}) + (1-\tau_{n+k})\hbar_\mu(v_n, Hv_{n+k})$
$\leq \tau_{n+k}\mu(v_n, v_{n+k}[\hbar_\mu(v_n, Hv_{n+k}) + \hbar_\mu(Hv_{n+k}, v_{n+k})] + (1-\tau_{n+k})\hbar_\mu(v_n, Hv_{n+k})$
$= [\tau_{n+k}\mu(v_n, v_{n+k}) + 1 - \tau_{n+k}]\hbar_\mu(v_n, Hv_{n+k}) + \tau_{n+k}\mu(v_n, v_{n+k})\hbar_\mu(Hv_{n+k}, v_{n+k})$
$\leq [\tau_{n+k}\mu(v_n, v_{n+k}) + 1]\hbar_\mu(v_n, Hv_{n+k}) + \tau_{n+k}\mu(v_n, v_{n+k})\hbar_\mu(Hv_{n+k}, v_{n+k})$
$\leq \mu(v_n, Hv_{n+k})[\tau_{n+k}\mu(v_n, v_{n+k}) + 1][\hbar_\mu(v_n, Hv_n) + \hbar_\mu(Hv_n, Hv_{n+k})]$
$\leq \mu(v_n, Hv_{n+k})[\tau_{n+k}\mu(v_n, v_{n+k} + 1]\alpha\hbar_\mu(v_n, v_{n+k})$
$\leq \mu(v_n, Hv_{n+k})[\tau_{n+k}\mu(v_n, v_{n+k}) + 1]\mu(v_n, Hv_{n+k-1})[\tau_{n+k-1}\mu(v_n, v_{n+k-1}) + 1]\alpha^2$
$\hbar_\mu(v_n, v_{n+k-1})$
$\leq ....$
$\leq \mu(v_n, Hv_{n+k})[\tau_{n+k}\mu(v_n, v_{n+k}) + 1]\mu(v_n, Hv_{n+k-1})[\tau_{n+k-1}\mu(v_n, v_{n+k-1}) + 1]....$
$\mu(v_n, Hv_{n+k-(k-1)})[\tau_{n+k-(k-1)}\mu(v_n, v_{n+k-(k-1)}) + 1]\alpha^k\hbar_\mu(v_n, v_n).$

Hence, $\lim_{n\longrightarrow\infty} \hbar_\mu(v_n, v_{n+k+1}) = 0$ for all k.

Therefore, the sequence $\{v_n\}$ is a right-Cauchy in C. By the same process, we can prove that $\{v_n\}$ is a left-Cauchy sequence. Hence, $\{v_n\}$ is a Cauchy sequence in the ECQSMS C.

Since $(C, \hbar_\mu)$ is complete, the sequence v_n converges to some $v^* \in C$, that is

$$\lim_{n\longrightarrow\infty} \hbar_\mu(v_n, v^*) = \lim_{n\longrightarrow\infty} \hbar_\mu(v^*, v_n) = 0.$$

The continuity of H yields,

$$\lim_{n\longrightarrow\infty} \hbar_\mu(Hv_n, Hv^*) = \lim_{n\longrightarrow\infty} \hbar_\mu(Hv_{n+1}, Hv^*) = 0.$$

Using the fact that the limit is unique, we get $Hv^* = v^*$. Therefore, v^* is a fixed point of H. □

Theorem 2.3. *Let $(C, \hbar_\mu, F)$ be a complete ECQSMS with a mapping $\mu : C\times C \to [1,\infty)$ and a contraction mapping $H : C \to C$; that is, there exists $k \in (0, \frac{1}{2})$ such that*

$$\hbar_\mu(Hv, Hu) \leq k[\hbar_\mu(u, Hu) + \hbar_\mu(v, Hv)],$$

for all $u, v \in C$. Let us choose $v_0 \in C$ such that $\hbar_\mu(v_0, Hv_0) = N < \infty$ and define $v_n = F(v_{n-1}, Hv_{n-1}; \tau_{n-1})$, where $0 < \tau_{n-1} \leq \frac{1}{4[\mu(r,s)]^2}$ and $n \in \mathbb{N}$. If for any $r, s \in C$, $k \in [0, \frac{1}{4[\mu(r,s)]^2}]$, then H has a unique fixed point in C.

Proof. We have for any $n \in \mathbb{N}$, there holds

$$\hbar_\mu(v_n, v_{n+1}) = \hbar_\mu(v_n, F(v_n, Hv_n; \tau_n)) \leq (1-\tau_n)\hbar_\mu(v_n, Hv_n) \leq \hbar_\mu(v_n, Hv_n)$$

and

$$\begin{aligned}
&\hbar_\mu(v_n, Hv_n)\\
&= \hbar_\mu(F(v_{n-1}, Hv_{n-1}; \tau_{n-1}), Hv_n)\\
&\leq \tau_{n-1}\hbar_\mu(v_{n-1}, Hv_n) + (1-\tau_{n-1})\hbar_\mu(Hv_{n-1}, Hv_n)\\
&\leq \tau_{n-1}\hbar_\mu(v_{n-1}, Hv_n) + \hbar_\mu(Hv_{n-1}, Hv_n)\\
&\leq \tau_{n-1}\hbar_\mu(v_{n-1}, Hv_n) + k\hbar_\mu(v_{n-1}, Hv_{n-1}) + k\hbar_\mu(v_n, Hv_n)\\
&\leq \tau_{n-1}\mu(v_{n-1}, Hv_n)[\hbar_\mu(v_{n-1}, Hv_{n-1}) + \hbar_\mu(v_n, Hv_n)] + k\hbar_\mu(v_{n-1}, Hv_{n-1})\\
&+ k\hbar_\mu(v_n, Hv_n)\\
&\leq \tau_{n-1}\mu(v_{n-1}, Hv_n)\hbar_\mu(v_{n-1}, Hv_{n-1}) + k\tau_{n-1}\mu(v_{n-1}, Hv_n)\hbar_\mu(v_{n-1}, Hv_{n-1})\\
&+ k\tau_{n-1}\mu(v_{n-1}, Hv_n)\hbar_\mu(v_n, Hv_n) + k\hbar_\mu(v_{n-1}, Hv_{n-1}) + k\hbar_\mu(v_n, Hv_n),
\end{aligned}$$

then, we get

$$\begin{aligned}
&[1 - k\tau_{n-1}\mu(v_{n-1}, Hv_n) - k]\hbar_\mu(v_n, Hv_n)\\
&\leq [k\tau_{n-1}\mu(v_{n-1}, Hv_n) + \tau_{n-1}\mu(v_{n-1}, Hv_n) + k]\hbar_\mu(v_{n-1}, Hv_{n-1})
\end{aligned}$$

then $\hbar_\mu(v_n, Hv_n) \leq \frac{[k\tau_{n-1}\mu(v_{n-1}, Hv_n) + \tau_{n-1}\mu(v_{n-1}, Hv_n) + k]}{[1 - k\tau_{n-1}\mu(v_{n-1}, Hv_n) - k]}\hbar_\mu(v_{n-1}, Hv_{n-1})$ and since

$$\frac{[k\tau_{n-1}\mu(v_{n-1}, Hv_n) + \tau_{n-1}\mu(v_{n-1}, Hv_n) + k]}{[1 - k\tau_{n-1}\mu(v_{n-1}, Hv_n) - k]} \leq \frac{9}{11},$$

then

$$\hbar_\mu(v_n, Hv_n) < \frac{9}{11}\hbar_\mu(v_{n-1}, Hv_{n-1}) \tag{2.7}$$

which enables us to deduce that $\{\hbar_\mu(v_n, Hv_n)\}$ is decreasing sequence of non-negative reals. Hence, there exists $\rho \geq 0$ such that $\lim_{n\longrightarrow\infty} \hbar_\mu(v_n, Hv_n) = \rho$. Letting $n \to \infty$ in (2.7), we get that $\rho \leq \frac{9}{11}\rho < \rho$, a contradiction. Hence, we obtain that $\rho = 0$; i.e.

$$\lim_{n\longrightarrow\infty} \hbar_\mu(v_n, Hv_n) = 0.$$

Similarly, we can see that

$$\lim_{n\longrightarrow\infty} \hbar_\mu(Hv_n, v_n) = 0.$$

Now, we show that $\{v_n\}$ is Cauchy sequence. For any $n, k \in \mathbb{N}$, we have

$$\begin{aligned}
&\hbar_\mu(v_n, v_{n+k+1})\\
&= \hbar_\mu(v_n, F(v_{n+k}, Hv_{n+k}, \tau_{n+k}))\\
&\leq \tau_{n+k}\hbar_\mu(v_n, v_{n+k}) + (1 - \tau_{n+k})\hbar_\mu(v_n, Hv_{n+k})\\
&\leq \tau_{n+k}\mu(v_n, v_{n+k}[\hbar_\mu(v_n, Hv_{n+k}) + \hbar_\mu(Hv_{n+k}, v_{n+k})] + (1 - \tau_{n+k})\hbar_\mu(v_n, Hv_{n+k})\\
&= [\tau_{n+k}\mu(v_n, v_{n+k}) + 1 - \tau_{n+k}]\hbar_\mu(v_n, Hv_{n+k}) + \tau_{n+k}\mu(v_n, v_{n+k})\hbar_\mu(Hv_{n+k}, v_{n+k})\\
&\leq [\tau_{n+k}\mu(v_n, v_{n+k}) + 1]\hbar_\mu(v_n, Hv_{n+k}) + \tau_{n+k}\mu(v_n, v_{n+k})\hbar_\mu(Hv_{n+k}, v_{n+k}).
\end{aligned}$$

Hence, $\lim_{n\longrightarrow\infty} \hbar_\mu(v_n, v_{n+k+1}) = 0$ for all k.

Therefore, the sequence $\{v_n\}$ is a right-Cauchy in C. By the same process, we can prove that $\{v_n\}$ is a left-Cauchy sequence. Hence, $\{v_n\}$ is a Cauchy sequence in the ECQSMS C.

Since $(C, \hbar_\mu)$ is complete, the sequence v_n converges to some $v^* \in C$, that is $\lim_{n\longrightarrow\infty} \hbar_\mu(v_n, v^*) = \lim_{n\longrightarrow\infty} \hbar_\mu(v^*, v_n) = 0$.

The continuity of H yields,

$$\lim_{n\longrightarrow\infty} \hbar_\mu(Hv_n, Hv^*) = \lim_{n\longrightarrow\infty} \hbar_\mu(Hv_{n+1}, Hv^*) = 0.$$

Using the fact that the limit is unique, we get $Hv^* = v^*$. Therefore, v^* is a fixed point of H. □

References

[1] Aydi, H., Bilgili, N., & Karapinar, E. (2015). Common fixed point results from quasi-metric spaces to G-metric spaces. *J. Egypt. Math. Soc., 23*, 356-361. https://doi.org/10.1016/j.joems.2014.06.009

[2] Chen, L., Gao, L., & Chen, D. (2017). Fixed point theorems of mean nonexpansive set-valued mappings in Banach spaces. *J. Fixed Point Theory Appl., 19*, 2129-2143. https://doi.org/10.1007/s11784-017-0401-9

[3] Chen, L., Gao, L., & Zhao, Y. (2017). A new iterative scheme for finding attractive points of (α, β)-generalized hybrid set-valued mappings. *J. Nonlinear Sci. Appl., 10*, 1228-1237. https://doi.org/10.22436/jnsa.010.03.31

[4] Chen, L., Zou, J., Zhao, Y., & Zhang, M. (2019). Iterative approximation of common attractive points of (α, β)-generalized hybrid set-valued mappings. *J. Fixed Point Theory Appl., 21*, 58. https://doi.org/10.1007/s11784-019-0692-0

[5] Czerwik, S. (1993). Contraction mappings in b-metric spaces. *Acta Mathematica et Informatica Universitatis Ostraviensis, 1*(1), 5-11.

[6] Ding, X.P. (1988). Iteration processes for nonlinear mappings in convex metric spaces. *J. Math. Anal. Appl., 132*, 114-122. https://doi.org/10.1016/0022-247X(88)90047-9

[7] Chen, L., Li, C., Kaczmarek, R., & Zhao, Y. (2020). Several fixed point theorems in convex b-metric spaces and applications. *Mathematics, 8*(2), 242. https://doi.org/10.3390/math8020242

[8] Felhi, A., Sahmim, S., & Aydi, H. (2016). Ulam-Hyers stability and well-posedness of fixed point problems for $\alpha - \lambda$-contractions on quasi b-metric spaces. *Fixed Point Theory Appl., 2016*(1), 20 pp. https://doi.org/10.1186/s13663-015-0491-2

[9] Goebel, K., & Kirk, W.A. (1983). Iteration processes for nonexpansive mappings. *Contemp. Math., 21*, 115-123. https://doi.org/10.1090/conm/021/729507

[10] Mihet, D. (2004). A Banach contraction theorem in fuzzy metric spaces. *Fuzzy Sets Syst., 144*, 431-439. https://doi.org/10.1016/S0165-0114(03)00305-1

[11] Ran, A.C.M., & Reurings, M.C.B. (2004). A fixed point theorem in partially ordered sets and some applications to matrix equations. *Proc. Am. Math. Soc., 132*, 1435-1443. https://doi.org/10.1090/S0002-9939-03-07220-4

[12] Reich, S., & Shafrir, I. (1990). Nonexpansive iterations in hyperbolic spaces. *Nonlinear Anal., 19*, 537-558. https://doi.org/10.1016/0362-546X(90)90058-O

[13] Ruzhansky, M., Cho, Y.J., Agarwal, P., & Area, I. (2017). *Advances in real and complex analysis with applications*. Springer. https://doi.org/10.1007/978-981-10-4337-6

[14] Takahashi, W. (1970). A convexity in metric space and nonexpansive mappings. I. *Kodai Math. Semin. Rep., 22*, 142-149. https://doi.org/10.2996/kmj/1138846111

[15] Wang, H.Q., Song, X.H., Chen, L.L., & Xie, W. (2019). A secret sharing scheme for quantum gray and color images based on encryption. *Int. J. Theor. Phys., 58*, 1626-1650. https://doi.org/10.1007/s10773-019-04057-z