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K^{th}-order Differential Subordination Results of Analytic Functions in the Complex Plane

Abbas Kareem Wanas[1,*] and Madan Mohan Soren[2]

[1] Department of Mathematics, College of Science, University of Al-Qadisiyah,
Al Diwaniyah 58001, Al-Qadisiyah, Iraq
e-mail: abbas.kareem.w@qu.edu.iq
ORCID ID: https://orcid.org/0000-0001-5838-7365

[2] Department of Mathematics, Berhampur University, Bhanja Bihar 760007,
Ganjam, Odisha, India
e-mail: mms.math@buodisha.edu.in
ORCID ID: https://orcid.org/0000-0001-7344-8082

Abstract

In recent years, there have been many interesting usages for differential subordinations of analytic functions in Geometric Function Theory of Complex Analysis. The concept of the first and second-order differential subordination have been pioneered by Miller and Mocanu. In 2011, the third-order differential subordination were defined to give a new generalization to the concept of differential subordination. While the fourth-order differential subordination has been introduced in 2020. In the present article, we introduce new concept that is the K^{th}-order differential subordination of analytic functions in the open unit disk U.

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*Corresponding author

1 Definitions and Main Results

Denote $\mathcal{H}(U)$ the family of analytic functions in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. For a positive integer number $n \in \mathbb{N}$ and $a \in \mathbb{C}$, we indicate by

$$\mathcal{H}[a, n] = \left\{ f \in \mathcal{H}(U) : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots , z \in U \right\}.$$

Also, let $\mathcal{A}$ be the subclass of $\mathcal{H}(U)$ consisting of functions of the form:

$$f(z) := z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in U). \tag{1.1}$$

Now we recall the principle of subordination between analytic functions, let the functions f and g be analytic in U, we say that the function f is subordinate to g, if there exists a Schwarz function w analytic in U with $w(0) = 0$ and $|w(z)| < 1$ $(z \in U)$ such that $f(z) = g(w(z))$. This subordination is indicated by $f \prec g$ or $f(z) \prec g(z)$ $(z \in U)$. Furthermore, if the function g is univalent in U, then we have the following equivalent (see [10]), $f(z) \prec g(z) \iff f(0) = g(0) \text{and} f(U) \subset g(U)$.

The theory of differential subordination in $\mathbb{C}$ is the generalization of differential inequality in $\mathbb{R}$. Many of the significant works on differential subordination have been pioneered by Miller and Mocanu, and their monograph [10] compiled their great efforts in introducing and developing the same. In recent years, various authors have successfully applied the theory of first and second order differential subordination to address many important problems in this field for example (see [5, 6, 9, 12, 14, 18–21]).

The concept of the third-order differential subordination in the open unit disk U was introduced by Antonino and Miller [1] in 2011. They extended the theory of second-order differential subordination in U introduced by Miller and Mocanu [10] to the third-order case that satisfy the third-order differential subordination $\{\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z) : z \in U\} \subset \Omega$. Recently, the several authors have considered the applications of these results to third-order differential subordination for analytic functions in U for example (see [1,3,4,7,8,13,15–17]). In

2020, Atshan et al. [2] extended the theory of third-order differential subordination in U introduced by Antonino and Miller [1] to the fourth-order case. They determined properties of functions p that satisfy the fourth-order differential subordination $\{\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z), z^4p''''(z); z) : z \in U\} \subset \Omega$. Now, we investigate the problem of determining properties of functions p that satisfy the following K^{th}-order differential subordination:

$$\{\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z), \ldots, z^{k-1}p^{(k-1)}(z), z^kp^{(k)}(z); z) : z \in U\} \subset \Omega, \quad k = 2, 3, \ldots.$$

Definition 1.1. *[10] Let $\mathcal{Q}$ denote the family of functions $q(z)$ that are analytic and univalent on the set $\overline{U} \setminus E(q(w))$, where*

$$E(q(w)) = \left\{\zeta \in \partial U : \lim_{z\to\zeta} q(w) = \infty\right\}$$

is such that $\min|q'(\zeta)| = \rho > 0$ for $\zeta \in \partial U \setminus E(q)$. Further let the subfamily of $\mathcal{Q}$ for which $q(0) = a$ be denoted by $\mathcal{Q}(a)$ and $\mathcal{Q}(1) = \mathcal{Q}_1$.

Definition 1.2. *[10] Let $\psi : \mathbb{C}^{K+1} \times U \times \overline{U} \longrightarrow \mathbb{C}$ and the function $h(z)$ be univalent in U. If the function $p(z)$ is analytic in U and satisfies the following k^{th}-order differential subordination:*

$$\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z), \ldots, z^{k-1}p^{(k-1)}(z), z^kp^{(k)}(z); z) \prec h(z), \tag{1.2}$$

then $p(z)$ is called a solution of the differential subordination. A univalent function $q(z)$ is called a dominant of the solutions of the differential subordination, or more simply a dominant if $p(z) \prec q(z)$ for all $p(z)$ satisfying (1.2). A dominant $\tilde{q}(z)$ that satisfies $\tilde{q}(z) \prec q(z)$ for all dominants of (1.2) is said to be the best dominant of (1.2).

Lemma 1.3. *Let $z_0 \in U$ with $r_0 = |z_0|$. For $n \in \mathbb{N} \setminus \{k-2\}$, let $f(z) = a_nz^n + a_{n+1}z^{n+1} + \cdots$ be continuous on $\overline{U}_{r_0}$ and analytic in $U_{r_0} \cup \{z_0\}$, with $f(z) \neq 0$. If*

$$|f(z_0)| = \max\left\{|f(z)| : z \in \overline{U}_{r_0}\right\}, \tag{1.3}$$

$$|f'(z_0)| = \max\left\{|f'(z)| : z \in \overline{U}_{r_0}\right\}, \tag{1.4}$$

and

$$|f^{(l)}(z_0)| = \max\left\{|f^{(l)}(z)| : z \in \overline{U}_{r_0}\right\} \quad (l \geq 2) \tag{1.5}$$

then there exists a $\tau \geq n$ *such that*

$$\frac{z_0 f'(z_0)}{f(z_0)} = \tau, \tag{1.6}$$

$$Re\left(\frac{z_0 f''(z_0)}{f'(z_0)} + 1\right) \geq \tau, \tag{1.7}$$

$$Re\left(\frac{z_0(z_0(z_0 f'(z_0))')'}{z_0 f'(z_0)}\right) \geq \tau^2, \tag{1.8}$$

$$Re\left(\frac{z_0(z_0(z_0(z_0 f'(z_0))')')'}{z_0 f'(z_0)}\right) \geq \tau^3, \tag{1.9}$$

then $(k-1)$ *times differentiating to get*

$$Re\left(\frac{z_0(\cdots(z_0(z_0(z_0 f'(z_0))')')'\cdots)'}{z_0 f'(z_0)}\right) \geq \tau^{k-2}. \tag{1.10}$$

Proof. We only need to show that relation (1.10) holds true. This Lemma 1.3 can be verified by the method of induction. The relations (1.6), (1.7) were due to Miller and Mocanu [10] and (1.8) were proved in [11], (1.9) were also proved in [2]. Thus, the proof of the Lemma is completed. □

Lemma 1.4. *Let* $p(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots$ *be analytic in* U, *with* $p(z) \neq a$ $z_0 \in U$ *and* $\tau \geq n$, *and* $q \in \mathcal{Q}(a)$, *with* $= |z_0|$. *If there exists points* $z_0 = r_0 e^{i\theta_0} \in U$ *and* $w_0 \in \partial U \setminus E(q)$ *such that* $p(z_0) = q(w_0)$, $p(\overline{U}_{r_0}) \subset q(U)$,

$$Re\frac{w_0 q''(w_0)}{q'(w_0)} \geq 0, \text{ and } Re\frac{zp'(z)}{q'(w)} \leq \tau, \tag{1.11}$$

$$Re\frac{w_0^2 q'''(w_0)}{q'(w_0)} \geq 0, \text{ and } Re\frac{z^2 p''(z)}{q'(w)} \leq \tau^2, \tag{1.12}$$

$$Re\frac{w_0^{k-2} q^{(k-1)}(w_0)}{q'(w_0)} \geq 0, \text{ and } Re\frac{z^{k-2} p^{(k-2)}(z)}{q'(w)} \leq \tau^{k-2}, \tag{1.13}$$

when $z \in \overline{U}_{r_0}$ *and* $w \in \partial U \setminus E(q)$, *then there exists a real constant* $\tau \geq n \geq 3$ *such that*

$$z_0p'(z_0) = \tau w_0 q'(w_0), \tag{1.14}$$

$$Re\left(\frac{z_0p''(z_0)}{p'(z_0)}+1\right) = \tau Re\frac{w_0q''(w_0)}{q'(w_0)} + k \geq \tau\left[Re\frac{w_0q''(w_0)}{q'(w_0)}+1\right], \tag{1.15}$$

$$Re\left(\frac{z_0^2p'''(z_0)}{p'(z_0)}+1\right) \geq \tau^2\left[Re\frac{w_0^2q'''(w_0)}{q'(w_0)}\right] + (k^2-3k+3) \geq \tau^2\left[Re\frac{w_0^2q'''(w_0)}{q'(w_0)}\right]+1, \tag{1.16}$$

or

$$Re\frac{z_0^2p'''(z_0)}{p'(z_0)} \geq \tau^2\left[Re\frac{w_0^2q'''(w_0)}{q'(w_0)}\right], \tag{1.17}$$

$$Re\left(\frac{z_0^3p^{(4)}(z_0)}{p'(z_0)}\right) \geq \tau^3 Re\left(\frac{w_0^3q^{(4)}(w_0)}{q'(w_0)}\right), \tag{1.18}$$

then $(k-1)$ *times differentiating to get*

$$Re\frac{z_0^{k-2}p^{(k-1)}(z_0)}{p'(z_0)} \geq \tau^{k-2}\left[Re\frac{w_0^{k-2}q^{(k-1)}(w_0)}{q'(w_0)}\right]. \tag{1.19}$$

Proof. We only need to show that relation (1.19) holds true. This Lemma 1.4 can be verified by the method of induction. The relations (1.14), (1.15) were due to Miller and Mocanu [10] and (1.17) were proved in [11], (1.18) were also proved in [2]. Thus, the proof of the Lemma is completed. □

Definition 1.5. *Let* Ω *be a set in* $\mathbb{C}$, $q \in \mathcal{Q}$ *and* $n \in \mathbb{N}\setminus\{k-2\}$. *The class of admissible functions* $\Psi_n[\Omega, q]$ *consists of those functions* $\psi : \mathbb{C}^{K+1}\times U \times \overline{U} \longrightarrow \mathbb{C}$ *that satisfy the following admissibility condition:*

$$\psi(r_1, r_2, r_3, \ldots, r_k, r_{k+1}; z) \notin \Omega \quad (z \in U),$$

whenever

$$r_1 = q(\xi), \quad r_2 = \tau \xi q'(\xi),$$
$$Re\left(\frac{r_3}{r_2}+1\right) \geq \tau Re\left(\frac{\xi q''(\xi)}{q'(\xi)}+1\right),$$
$$Re\left(\frac{r_4}{r_2}+1\right) \geq \tau^2 Re\left(\frac{\xi^2 q'''(\xi)}{q'(\xi)}+1\right),$$
$$\vdots$$
$$Re\left(\frac{r_k}{r_2}+1\right) \geq \tau^{k-2} Re\left(\frac{\xi^{k-2} q^{(k-1)}(\xi)}{q'(\xi)}+1\right),$$
$$Re\left(\frac{r_{k+1}}{r_2}+1\right) \geq \tau^{k-1} Re\left(\frac{\xi^{k-1} q^{(k)}(\xi)}{q'(\xi)}+1\right),$$

where $z \in U,\ \xi \in \partial U \setminus E(q),$ *and* $\tau \geq n.$

Theorem 1.6. *Let* $p \in \mathcal{H}[a,n]$ *with* $n \in \mathbb{N} \setminus \{k-2\}$. *Also, let* $q \in \mathcal{Q}(a)$ *and satisfy the following conditions:*

$$Re\left(\frac{\xi^{k-2} q^{(k-1)}(\xi)}{q'(\xi)}\right) \geq 0, \quad \left|\frac{\xi^{k-2} p^{(k-1)}(\xi)}{q'(\xi)}\right| \leq \tau^{k-2}, \tag{1.20}$$

where $z \in U,\ \xi \in \partial U \setminus E(q)$ *and* $\tau \geq n$. *If* Ω *is a set in* $\mathbb{C}$, $\psi \in \Psi_n[\Omega, q]$ *and*

$$\psi(p(z), zp'(z), z^2p''(z), z^3p^{(3)}(z), \ldots, z^{k-1}p^{(k-1)}(z), z^kp^{(k)}(z); z) \in \Omega, \tag{1.21}$$

then

$$p(z) \prec q(z).$$

Proof. If we assume that $p(z) \not\prec q(z)$, then there exist point $z_0 = r_0 e^{i\theta_0} \in U$ and $s_0 \in \partial U \setminus E(q)$ such that $p(z_0) = q(s_0)$ and $p(\overline{U}_{r_0}) \subset q(U)$. From (1.20), we see that the conditions (1.11), (1.12) and (1.13) of Lemma 1.4 are satisfied when $z \in \overline{U}_{r_0}$ and $s_0 \in \partial U \setminus E(q)$. Since all the conditions of that Lemma are satisfied, conclusions (1.14),(1.15),(1.17), (1.18) and (1.19) follow. Using these results of Lemma 1.5 leads to

$$\psi(p(z), zp'(z), z^2p''(z), z^3p^{(3)}(z), \ldots, z^{k-1}p^{(k-1)}(z), z^kp^{(k)}(z); z) \notin \Omega.$$

Since this contradicts (1.21), we must have $p(z) \prec q(z)$. $\square$

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