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# A Note on Summability of Infinite Series

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**Abstract**

The purpose of the present paper is to get the necessary and sufficient conditions for absolute matrix summability of infinite series.

## 1 Introduction

Let $\sum a_n$ be an infinite series with its partial sums $(s_n)$. Let $C = (c_{nv})$ be a normal matrix which means a lower triangular matrix of nonzero diagonal entries, then two lower semimatrices $\bar{C} = (\bar{c}_{nv})$ and $\hat{C} = (\hat{c}_{nv})$ are defined as follows:

$$\bar{c}_{nv} = \sum_{i=v}^{n} c_{ni}, \quad n, v = 0, 1, ...$$

$$\hat{c}_{00} = \bar{c}_{00} = c_{00}, \quad \hat{c}_{nv} = \bar{c}_{nv} - \bar{c}_{n-1,v}, \quad n = 1, 2, ....$$

Let $(\varphi_n)$ be a sequence of positive numbers. The series $\sum a_n$ is said to be summable $\varphi - \mid C; \delta \mid_k$, $k \geq 1$, $\delta \geq 0$, if [1]

$$\sum_{n=1}^{\infty} \varphi_n^{\delta k+k-1} \mid C_n(s) - C_{n-1}(s) \mid^k < \infty,$$

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where

$$C_n(s) = \sum_{v=0}^{n} c_{nv} s_v, \quad n = 0, 1, ...$$

and

$$\bar{\Delta} C_n(s) = C_n(s) - C_{n-1}(s) = \sum_{v=0}^{n} \hat{c}_{nv} a_v. \tag{1}$$

By taking $\delta = 0$ in the definition of $\varphi - \mid C; \delta \mid_k$ summability method, $\varphi - |C|_k$ summability is obtained [2].

If $C$ is a normal matrix, then $C' = (c'_{nv})$ denotes the inverse of $C$, and $\hat{C} = (\hat{c}_{nv})$ is a normal matrix and it has two-sided inverse $\hat{C}' = (\hat{c}'_{nv})$ which is also normal (see [3]).

**Lemma 1.** [4] *Let $k \geq 1$, $p \geq 1$ and $\frac{1}{k} + \frac{1}{k'} = 1$, $\frac{1}{p} + \frac{1}{p'} = 1$. Let*

$$d_n = \sum_{m=0}^{\infty} h_{nm} q_m, \quad n \geq 0$$

$$v_m = \sum_{n=0}^{\infty} h_{nm} u_n, \quad m \geq 0.$$

*In order that $d \in l_p$ whenever $q \in l_k$ if and only if $v \in l_{k'}$ whenever $u \in l_{p'}$, where $l_k$ is defined by $l_k := \left\{ x = (x_j) : \sum |x_j|^k < \infty \right\}$.*

## 2 Main Result

There are many studies on matrix summability methods. Let us mention about some of the recent ones. Karakaş [5,6] , Kartal [7,8], Özarslan [9–12], Özarslan and Kartal [13, 14], Özarslan, Şakar and Kartal [15] proved theorems on the sufficient conditions to absolute matrix summability of infinite series by using different class of sequences. The object of this paper is to get the necessary and sufficient conditions to absolute matrix summability.

**Theorem 1.** *Let $k \geq 1$ and $0 \leq \delta < \frac{1}{k}$. Let $C = (c_{nv})$ and $D = (d_{nv})$ be two positive normal matrices satisfy*

$$c_{nn} - c_{n+1,n} = O(c_{nn}c_{n+1,n+1}) \tag{2}$$

$$\overline{d}_{n0} = 1, \quad n = 0, 1, ...$$

$$\sum_{v=r+2}^{n} |\hat{d}_{nv}\hat{c}'_{vr}\gamma_v| = O\left(\left|\hat{d}_{n,r+1}\gamma_{r+1}\right|\right). \tag{3}$$

*Then $\sum a_n\gamma_n$ is summable $\varphi - \mid D; \delta \mid$ whenever $\sum a_n$ is summable $\varphi - \mid C; \delta \mid_k$ if and only if*

$$\left\{\varphi_v^{-\delta-\frac{1}{k'}}\left(\frac{d_{vv}}{c_{vv}}|\gamma_v| + \sum_{n=v+1}^{\infty}\left(\frac{\left|\Delta_v\left(\hat{d}_{nv}\gamma_v\right)\right|}{c_{vv}} + \left|\hat{d}_{n,v+1}\gamma_{v+1}\right|\right) + \sum_{n=v+2}^{\infty}\left|\hat{d}_{n,v+1}\gamma_{v+1}\right|\right)\right\} \in l_{k'}.$$

## Proof of Theorem 1

Let $(I_n)$ and $(U_n)$ denote $C$-transform and $D$-transform of the series $\sum a_n$ and $\sum a_n\gamma_n$, respectively. Then we get

$$x_n = \bar{\Delta}I_n = \sum_{v=0}^{n}\hat{c}_{nv}a_v \quad \text{and} \quad y_n = \bar{\Delta}U_n = \sum_{v=0}^{n}\hat{d}_{nv}a_v\gamma_v$$

by (1). Also, using the above equalities, we get $a_v = \sum_{r=0}^{v}\hat{c}'_{vr}x_r$ and $y_n = \sum_{v=0}^{n}\hat{d}_{nv}\gamma_v\sum_{r=0}^{v}\hat{c}'_{vr}x_r$. Then, we obtain

$$\begin{aligned} y_n &= \sum_{v=1}^{n}\hat{d}_{nv}\gamma_v\sum_{r=0}^{v}\hat{c}'_{vr}x_r \\ &= \sum_{v=1}^{n}\hat{d}_{nv}\gamma_v\hat{c}'_{vv}x_v + \sum_{v=1}^{n}\hat{d}_{nv}\gamma_v\hat{c}'_{v,v-1}x_{v-1} + \sum_{v=1}^{n}\hat{d}_{nv}\gamma_v\sum_{r=0}^{v-2}\hat{c}'_{vr}x_r \\ &= \hat{d}_{nn}\gamma_n\hat{c}'_{nn}x_n + \sum_{v=1}^{n-1}\left(\hat{d}_{nv}\gamma_v\hat{c}'_{vv} + \hat{d}_{n,v+1}\gamma_{v+1}\hat{c}'_{v+1,v}\right)x_v \\ &+ \sum_{r=0}^{n-2}x_r\sum_{v=r+2}^{n}\hat{d}_{nv}\gamma_v\hat{c}'_{vr}. \end{aligned} \tag{4}$$

For $\delta_{nv}$(Kronecker delta), by using the equality $\sum_{k=v}^{n}\hat{c}'_{nk}\hat{c}_{kv}=\delta_{nv}$, we get

$$\begin{aligned}
\hat{d}_{nv}\gamma_v\hat{c}'_{vv}+\hat{d}_{n,v+1}\gamma_{v+1}\hat{c}'_{v+1,v} &= \frac{\hat{d}_{nv}\gamma_v}{\hat{c}_{vv}}+\hat{d}_{n,v+1}\gamma_{v+1}\left(-\frac{\hat{c}_{v+1,v}}{\hat{c}_{vv}\hat{c}_{v+1,v+1}}\right)\\
&= \frac{\hat{d}_{nv}\gamma_v}{c_{vv}}-\hat{d}_{n,v+1}\gamma_{v+1}\frac{(\bar{c}_{v+1,v}-\bar{c}_{vv})}{c_{vv}c_{v+1,v+1}}\\
&= \frac{\hat{d}_{nv}\gamma_v}{c_{vv}}-\hat{d}_{n,v+1}\gamma_{v+1}\frac{(c_{v+1,v+1}+c_{v+1,v}-c_{vv})}{c_{vv}c_{v+1,v+1}}\\
&= \frac{\Delta_v(\hat{d}_{nv}\gamma_v)}{c_{vv}}+\hat{d}_{n,v+1}\gamma_{v+1}\frac{(c_{vv}-c_{v+1,v})}{c_{vv}c_{v+1,v+1}}.
\end{aligned}$$

If we write the above equality in (4), we obtain

$$\begin{aligned}
y_n &= \frac{d_{nn}\gamma_n}{c_{nn}}x_n+\sum_{v=1}^{n-1}\frac{\Delta_v(\hat{d}_{nv}\gamma_v)}{c_{vv}}x_v+\sum_{v=1}^{n-1}\hat{d}_{n,v+1}\gamma_{v+1}\frac{(c_{vv}-c_{v+1,v})}{c_{vv}c_{v+1,v+1}}x_v\\
&+ \sum_{r=0}^{n-2}x_r\sum_{v=r+2}^{n}\hat{d}_{nv}\gamma_v\hat{c}'_{vr}.
\end{aligned}$$

Let $X_v=\varphi_v^{\delta+1-\frac{1}{k}}x_v$, then let us define the sequence $(h_{nv})$ as in the following form

$$h_{nv}=\begin{cases}
\varphi_v^{-\delta+\frac{1}{k}-1}\left(\frac{\Delta_v(\hat{d}_{nv}\gamma_v)}{c_{vv}}+\hat{d}_{n,v+1}\gamma_{v+1}\frac{(c_{vv}-c_{v+1,v})}{c_{vv}c_{v+1,v+1}}+\sum_{r=v+2}^{n}\hat{d}_{nr}\gamma_r\hat{c}'_{rv}\right), & 1\le v\le n-2\\
\varphi_v^{-\delta+\frac{1}{k}-1}\left(\frac{\Delta_v(\hat{d}_{nv}\gamma_v)}{c_{vv}}+\hat{d}_{n,v+1}\gamma_{v+1}\frac{(c_{vv}-c_{v+1,v})}{c_{vv}c_{v+1,v+1}}\right) & v=n-1\\
\varphi_v^{-\delta+\frac{1}{k}-1}\frac{\hat{d}_{nv}\gamma_v}{c_{vv}} & v=n\\
0, & v>n
\end{cases}$$

Thus we can write $y_n=\sum_{v=1}^{\infty}h_{nv}X_v$. It can be clearly seen that the necessary and sufficient condition for the series $\sum a_n\gamma_n$ to be summable $\varphi-\mid D;\delta\mid$ whenever $\sum a_n$ is summable $\varphi-\mid C;\delta\mid_k$ is

$$\sum|y_n|<\infty \quad \text{whenever} \quad \sum|X_n|^k<\infty. \tag{5}$$

Also by Lemma 1, (5) holds if and only if $\sum_{v=1}^{\infty}\left|\sum_{n=v}^{\infty}h_{nv}u_n\right|^{k'}<\infty$ whenever $u_n=O(1)$. By (2), (3), we can say that $\sum_{v=1}^{\infty}\left|\sum_{n=v}^{\infty}h_{nv}u_n\right|^{k'}<\infty$ whenever

$u_n = O(1)$ if and only if

$$\left\{ \varphi_v^{-\delta-\frac{1}{k'}} \left( \frac{d_{vv}}{c_{vv}} |\gamma_v| + \sum_{n=v+1}^{\infty} \left( \frac{\left|\Delta_v \left(\hat{d}_{nv}\gamma_v\right)\right|}{c_{vv}} + \left|\hat{d}_{n,v+1}\gamma_{v+1}\right| \right) + \sum_{n=v+2}^{\infty} \left|\hat{d}_{n,v+1}\gamma_{v+1}\right| \right) \right\} \in l_{k'}$$

holds. Hence the proof is completed.

For $\delta = 0$, the necessary and sufficient conditions for $\varphi- \mid D \mid$ summability of the series $\sum a_n \gamma_n$ whenever the series $\sum a_n$ is summable $\varphi- \mid C \mid_k$ can be achieved.

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