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A-stable Two Derivative Mono-Implicit Runge-Kutta Methods for ODEs

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Abstract

An A-stable Two Derivative Mono Implicit Runge-Kutta (ATDMIRK) method is considered herein for the numerical solution of initial value problems (IVPs) in ordinary differential equation (ODEs). The methods are of high-order A-stable for $p = q = \{2s+1\}_{s=2}^{7}$ The p, q and s are the order of the input, output and the stages of the methods respectively. The numerical results affirm the superior accuracy of the newly develop methods compare to the existing ones.

1 Introduction

Recently, [5] presented an Extended Mono-Implicit Runge-kutta (EMIRK) method for solving ordinary differential equations (ODEs) which is build upon the existing Mono-implicit Runge-Kutta (MIRK) methods in [[8], [26], [9], [19], [25], [20]]. The

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general format of the method discussed in [5] is

$$U_r = (1 - v_r)u_n + v_r u_{n+1} + h\sum_{j=1}^{r-1} X_{rj} f(x_n + c_j h, U_j)$$
$$+ h^2 \sum_{j=1}^{r-1} \bar{X}_{rj} g(x_n + c_j h, U_j), \quad r = 1, 2, ..., s; \tag{1}$$
$$u_{n+1} = u_n + h\sum_{r=1}^{s} b_r(1) f(x_n + c_r h, U_r) + h^2 \sum_{r=1}^{s} \bar{b}_r(1) g(x_n + c_r h, U_r).$$

where $c = (c_1, c_2, \cdots, c_s)^T$, $v = (v_1, v_2, \cdots, v_s)^T$, $b = (b_1, b_2, \cdots, b_s)^T$, X is an $s \times s$ matrix, also the stage order q is not equal to output order p. It is worthy to know that the MIRK method represents a subclass of Implicit Runge-Kutta (IRK) methods designed to address the computational expenses associated with the IRK approach in [[10], [14]]. Other works related to higher derivative methods are in [[22], [13], [40], [2], [3], [33], [34], [29], [31], [18], [36], [41], [4], [32]]. In the spirit of the author in [5], we've introduced a novel class of A-stable two derivative mono implicit Runge-Kuttta (ATDMIRK) method for ODEs represented as:

$$u^{'} = Bu, \quad u(a) = u_0, \ a \leqslant x \leqslant b \tag{2}$$

where $u \in \mathbf{R}^n$, B is an $n \times n$ real matrix with its eigenvalues $\{\lambda_i\}_{i=1}^{n}$ such that $Re(\lambda_i) < 0$ and the stiff ratio $\frac{max|Re(\lambda_i)|}{min|Re(\lambda_i)|} \gg 1$, $i = 1, 2..., n$.

According to Cash [17], an important requirement to solve (2) is to have a numerical method that possesses high order and A-stability properties.

2 Formulation of the Method for ODEs

For the initial value problems (IVP)

$$u^{'} = f(x, u), \quad u'' = f_x + f_u f = g(x, u), x \in [x_0, X], \ u(x_0) = u_0 \tag{3}$$

where $f : R^s \to R^s$ and $g : R^s \to R^s$. We define our ATDMIRK method as

$$\begin{aligned} U_r = (1-v_r)u_n + v_r u_{n+1} + h\sum_{j=1}^{r-1} X_{rj} f(x_n + c_j h, U_j) \\ + h^2 \sum_{j=1}^{r-1} \bar{X}_{rj} g(x_n + c_j h, U_j), \quad c_r \in (0,1), r = 1,2,...,s; \end{aligned} \tag{4}$$

and

$$u_{n+1} = u_n + h\sum_{r=1}^{s} b_r(1) f(x_n + c_r h, U_r) + h^2 \sum_{r=1}^{s-1} \bar{b}_r(1) g(x_n + c_r h, U_r), \quad \theta = 1. \tag{5}$$

The $g(x, y)$ is the second derivative form of ODEs in (2), $c_r = (c_1, ..., c_s)^T$, is the abscissa value and $U_r = y(x_n + c_r h)$, the coefficients, $\{v_r\}_{r=1}^{s}$, $\{x_{rj}\}_{j=1,r=1}^{r-1,s}$, $\{\bar{x_{rj}}\}_{j=1,r=1}^{r-1,s}$, defined the stages, $\{b_r(\theta)\}_{r=1}^{s}$ and $\{\bar{b_r}(\theta)\}_{r=1}^{s}$, are the weight polynomials. We shall require $c_r = \sum_{j=1}^{r-1} x_{rj} + \sum_{j=1}^{r-1} \bar{x}_{rj} + v_r$ and $\theta = 1$, i.e $b_r(1) = b_r$ and $\bar{b}_r(1) = \bar{b}_r$. The second derivative side on the right hand side of equation (5)has $s - 1$ stage, this method is crafted in such a manner that stage order q equal the output order, p=$2s - 1$ (i.e. $p = q$). Equation (4) and (5) is a new class of the methods in [16]. The paper is structured thus as follows: In section 2, we present the order condition and stability analysis for the ATDMIRK methods. Section 3 is dedicated to deriving the ATDMIRK methods, and section 4,presents the numerical results. The Butcher's tableaux of the method in (4) and (5) is

$$\begin{array}{c|c|c|c} c & v & X & \bar{X} \\ \hline & & b^T(1) & \bar{b}^T(1) \end{array} = \begin{array}{c|c|c|c} c_1 & v_1 & x_{11}...x_{1s} & \bar{x}_{11}...\bar{x}_{1s} \\ \vdots & \vdots & \vdots & \vdots \\ c_s & v_s & x_{s1}...x_{ss} & \bar{x}_{1s}...\bar{x}_{ss} \\ \hline & & b_1(1)...b_s(1) & \bar{b}_1(1)...\bar{b}_s(1) \end{array} \tag{6}$$

where $c = (c_1, ..., c_s)^T$, $v = (v_1, ..., v_s)^T$, $b(1) = (b_1(1), ..., b_s(1))^T$, $\bar{b}(1) = (\bar{b}_1(1), ..., \bar{b}_s(1))^T$, X and $\bar{X}$ are the s matrix whose $(i, j)th$ components are x_{ij} and $\bar{x}_{ij}$. our aim is to maximise the order, ensuring stability, while minimising error constants and the number of functions evaluations.

3 Order Condition of the ATDMIRK Methods

The order condition for methods in (4) and (5) are obtained by Taylor's series expansion approach about x_n and equating the power of h to zero gives stage order q

$$\begin{aligned} &C = Xe + v; \\ &\frac{c^j}{j!} = \frac{Xc^{j-1}}{(j-1)!} + \frac{\bar{X}c^{j-2}}{(j-2)!} + \frac{v}{j!}; \qquad j = 2(1)q, \end{aligned} \tag{7}$$

and the method of order p

$$\begin{aligned} &b^T e = e; \\ &\frac{1}{j!} = \frac{b^T c^{j-1}}{(j-1)!} + \frac{\bar{b^T} c^{j-2}}{(j-2)!}; \qquad j = 2(1)p. \end{aligned} \tag{8}$$

4 Stability Analysis

In this section our interest is on the analysis of the stability of the method in (4) and (5). In what follows is the derivation of the stability function of the method in (4) and (5).

Theorem 4.1. *Let $R(z)$ express the stability function for ATDMIRK method. As regards a linear differential equation of the form $u^{'}(x) = \lambda u(x)$, the methods in (4) and (5) has the stability function*

$$R(z) = \frac{det[I - zX - z^2\bar{X} + zeb^T + z^2\bar{b}^T - zvb^T - z^2v\bar{b}^T]}{det[I - zX - z^2\bar{X} - zvb^T - z^2v\bar{b}^T]}, \quad z = \lambda h. \tag{9}$$

Proof. For the special problem $u^{'} = \lambda u$, the first and second stage derivatives f and $u'' = g$ respectively relates to the stage values U by $f = \lambda u$ and $g = \lambda^2 u$. To ease our prove, we take $e = (1, ..., 1)^T$ and $v = (v_1, ..., v_s)^T$. Hence, (4) reduces to the form

$$(I - zX - z^2\bar{X})U - vu_{n+1} = (e - v)u_n \tag{10}$$

and

$$(-zb^T - z^2\bar{b}^T)U + u_{n+1} = u_n. \tag{11}$$

From (10) we have

$$U = ((e-v)u_n + vu_{n+1})(I - zX - z^2\bar{X})^{-1}. \tag{12}$$

Inserting (12) into (11) gives

$$(-zb^T - z^2\bar{b}^T)\left(((e-v)u_n + vu_{n+1})(I - zX - z^2\bar{X})^{-1}\right) + u_{n+1} = u_n. \tag{13}$$

Multiplying both side of the (13) by $(I - zX - z^2\bar{X})$ gives

$$(-zb^T - z^2\bar{b}^T)\left((e-v)u_n + vu_{n+1}\right) + (I - zX - z^2\bar{X})u_{n+1} = (I - zX - z^2\bar{X})u_n. \tag{14}$$

Simplifying (14) and collecting like terms yields

$$[v(-zb^T - z^2\bar{b}^T) + (I - zX - z^2\bar{X})]u_{n+1} = [(I - zX - z^2\bar{X})(e-v)(-zb^T - z^2\bar{b}^T)]u_n. \tag{15}$$

From (15) we derive $u_{n+1} = R(z)u_n$. Thus the stability function is

$$R(z) = \frac{det[I - zX - z^2\bar{X} + zeb^T + z^2e\bar{b}^T - zvb^T - z^2v\bar{b}^T]}{det[I - zX - z^2\bar{X} - zvb^T - z^2v\bar{b}^T]} \tag{16}$$

Definition 4.1. *A Mono-Implicit Runge-Kutta Method in* (5) *is said to be A-stable if* $|\ R(z)\ |\leqslant 1\ \forall\ Re(z) \leqslant 0$.

Definition 4.2. *(c.f Butcher): Let* α *denote an angle satisfying* $\alpha(0,\pi)$ *and Let* $S(\alpha)$ *denote the set of point* $x + iu$ *in complex plane such that* $x \leqslant 0$ *and* $-\tan(\alpha)\ |\ x\ |\leqslant u \leqslant \tan(\alpha)\ |\ x\ |$.

A Runge-Kutta method in (5) with stability function $R(z)$ is A-stable if $|\ R(z)\ |\leqslant 1$ for all $z \in S(\alpha)$.

5 Construction of the ATDMIRK Methods

In this section, we will derive method (4) and (5) that exhibit order p and have stage order q respectively, where q equals p. We consider such methods because there are some strong theoretical and numerical evidences that methods with $p=q$ has the greatest potential for practical use, (see [11], [12], [34], [35]). Thus, we will restrict our discussion and investigation to such schemes. The approach adopted here in the derivation of the method in (4) and (5) is similar to that used in [34], and [35].

5.1 ATDMIRK method of order $p = 1$, $s = 1$

For example, fixing $r = 1$, and $v_1 = 0$ in (4) gives

$$U_1 = u_n. \tag{17}$$

Similarly, we obtain the output method of $p = 1$ in (5). That is

$$u_{n+1} = u_n + hf(x_n, U_1). \tag{18}$$

The tableau for (17) is

$$\begin{array}{c|c|c|c} c & v & X & \bar{X} \\ \hline & & b^T(1) & \bar{b}^T(1) \end{array} = \begin{array}{c|c|c|c} 0 & 0 & 0 & 0 \\ \hline & & 1 & 0 \end{array} \tag{19}$$

The method in (17) and (18) is an explicit Euler's method, which is not of interest in this paper but such scheme are suitable for non-stiff ODEs. The Euler's scheme has an interval of absolute stability of $[-2, 0]$.

5.2 ATDMIRK method of order $p = 3$, $s = 2$

Taking $r = 2$ in (4) and fix $v_1 = 1$ gives

$$\begin{gathered} U_1 = u_n; \quad U_2 = u_{n+1} \\ u_{n+1} = u_n + \tfrac{2h}{3} f(x_n, U_1) + \tfrac{h}{3} f(x_{n+1}, U_2) + \tfrac{h^2}{6} g(x_n, U_1). \end{gathered} \tag{20}$$

The picture of the scheme in (20) is

$$\begin{array}{c|c|c|c} c & v & X & \bar{X} \\ \hline & & b^T(1) & \bar{b}^T(1) \end{array} = \begin{array}{c|c|cc|cc} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ \hline & & \frac{2}{3} & \frac{1}{3} & \frac{1}{6} & 0 \end{array} \tag{21}$$

The algorithm in (20) is of order $p = 3$, the interval of absolute stability for ATDMIRK of order $P = 3$ is [-2,0] and such scheme is good for solving ODEs that are non-stiff numerically. Our interest in this study is implicit Runge-Kutta method. Therefore we give below some suitable methods emanating from (4) and (5) for stiff problems (2).

5.3 ATDMIRK method of order $p = q = 5$, $s = 3$

Fixing p=q=5, $s = 3$ in (7) and (8) and through the solution of subsequent system of linear equations with respect to $\{c_r\}_{r=1}^{3}$ such that $c_1 \neq c_2 \neq c_3$. The resulting tableau of the method of order $p = 5$ is

$$\begin{array}{c|c|c|c} c & v & X & \bar{X} \\ \hline & & b^T(1) & \bar{b}^T(1) \end{array} = \begin{array}{c|c|ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{3}{4} & \frac{1836}{2048} & \frac{78}{2048} & \frac{-378}{2048} & 0 & \frac{9}{2048} & \frac{27}{2048} & 0 \\ \hline & & \frac{190}{540} & \frac{-162}{540} & \frac{512}{540} & \frac{21}{540} & \frac{27}{540} & 0 \end{array} \qquad (22)$$

The stability function for ATDMIRK method in (22) is $R(z) = \frac{240+108z+18z^2+z^3}{240-132z+30z^2-3z^3}$ and plotting the stability function of (22) in boundaries locus sense shows that the method in (22) is A-stable.

Note that $ATDMIRK$ method for order $p = q = 5$, $s = 3$ is represented by $ATDMIRK5$.

5.4 ATDMIRK method of order $p = q = 7$, $s = 4$

Similarly, setting $p = q = 7$, $s = 4$ in (7) and (8) and resolving the ensuing system of linear equations in terms of $\{c_r\}_{r=1}^{4}$ such that $c_1 \neq c_2 \neq c_3 \neq c_4$. The resulting tableau of ATDMIRK method of order $p = 7$ is;

$$\begin{array}{c|c|c|c} c & v & X & \bar{X} \\ \hline & & b^T(1) & \bar{b}^T(1) \end{array} = \begin{array}{c|c|cccc|cccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{96}{192} & \frac{18}{192} & \frac{-18}{192} & 0 & 0 & \frac{1}{192} & \frac{1}{192} & \frac{-8}{192} & 0 \\ \frac{3}{4} & \frac{5724}{8192} & \frac{474}{8192} & \frac{-918}{8192} & \frac{864}{8192} & 0 & \frac{27}{8192} & \frac{45}{8192} & \frac{-144}{8192} & 0 \\ \hline & & \frac{706}{3780} & \frac{162}{3780} & \frac{864}{3780} & \frac{2048}{3780} & \frac{39}{3780} & \frac{9}{3780} & \frac{-288}{3780} & 0 \end{array} \tag{23}$$

The stability function associated with the method in (23) is $R(z) = \frac{40320+18720z+3840z^2+444z^3+30z^4+z^5}{3(13440-7200z+1760z^2-252z^3+22z^4-z^5)}$ and the method in (23) is A-stable has showed in the stability plot in Figure 1. Again, the $ATDMIRK$ method of order $p = q = 7$, $s = 4$ implies $ATDMIRK7$.

5.5 ATDMIRK method of order $p = q = 9$, $s = 5$

Setting $s = 5$, $c = (0, 1, \frac{1}{4}, \frac{3}{4}, \frac{4}{5})^T$ in (7) and (8) yield the $ATDMIRK$ methods of order 9 with the modified Butcher tableaux of the resulting coefficients given below.

$$\begin{array}{c|c|c|c} c & v & X & \bar{X} \\ \hline & & b^T(1) & \bar{b}^T(1) \end{array} \tag{24}$$

where,

$v = (0, 1, \frac{3296}{26624}, \frac{23328}{26624}, \frac{603538560}{685546875})^T$,

$$X = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \frac{2238}{26624} & \frac{-414}{26624} & \frac{2640}{26624} & \frac{-1104}{26624} & \mathbf{0} \\ \frac{414}{26624} & \frac{-2238}{26624} & \frac{1104}{26624} & \frac{-2640}{26624} & \mathbf{0} \\ \frac{10314620}{685546875} & \frac{-57551360}{685546875} & \frac{27484160}{685546875} & \frac{-35348480}{685546875} & \mathbf{0} \end{pmatrix},$$

$$\bar{X} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \frac{63}{26624} & \frac{15}{26624} & \frac{-348}{26624} & \frac{36}{26624} & \mathbf{0} \\ \frac{15}{26624} & \frac{63}{26624} & \frac{36}{26624} & \frac{-348}{26624} & \mathbf{0} \\ \frac{374136}{685546875} & \frac{1611744}{685546875} & \frac{903168}{685546875} & \frac{-8091648}{685546875} & \mathbf{0} \end{pmatrix},$$

$b(1)^T = (\frac{4000865}{32931360}, \frac{-1355200}{32931360}, \frac{10885120}{32931360}, \frac{-117708800}{32931360}, \frac{137109375}{32931360})^T,$
$\bar{b}(1)^T = (\frac{140844}{32931360}, \frac{142296}{32931360}, \frac{287232}{32931360}, \frac{-6876672}{32931360}, 0)^T.$
The stability function is
$R(z) = \frac{77414400+36126720z+7714560z^2+983040z^3+81540z^4+4492z^5+159z^6+3z^7}{77414400-41287680z+10295040z^2-1570560z^3+160260z^4-11128z^5+501z^6-12z^7}.$
The stability plot for the method of order $p = 9$ in Figure 1 shows method in (24) is A-stable

5.6 ATDMIRK method of order $p = q = 11$, $s = 6$

Having satisfied the order and stage order conditions and configuring $c = (0, 1, \frac{1}{2}, \frac{1}{4}, \frac{2}{3}, \frac{9}{10})^T$, solving a system of equations which yields the methods of order and stage order $p = q = 11$ with specified coefficients.

$$\begin{array}{c|c|c|c} c & v & X & \bar{X} \\ \hline & & b^T(1) & \bar{b}^T(1) \end{array} \tag{25}$$

where,
$v = (0, 1, \frac{1074870000}{4769280000}, \frac{3008330000}{15073280000}, \frac{5276340000}{22918393125}, \frac{410414278950000}{575000000000000})^T,$

$$X = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \frac{312008625}{4769280000} & \frac{-106396000}{4769280000} & \frac{-670464000}{4769280000} & \frac{457179136}{4769280000} & \frac{1317442239}{4769280000} & \mathbf{0} \\ \frac{1007872875}{15073280000} & \frac{-298812000}{15073280000} & \frac{-3328128000}{15073280000} & \frac{-50331648}{15073280000} & \frac{3429388773}{15073280000} & \mathbf{0} \\ \frac{1490963625}{22918393125} & \frac{-521326000}{22918393125} & \frac{-1292544000}{22918393125} & \frac{2210922496}{22918393125} & \frac{8114572629}{22918393125} & \mathbf{0} \\ \frac{14103587201625}{575000000000000} & \frac{-27353095164000}{575000000000000} & \frac{22288566528000}{575000000000000} & \frac{24480395034624}{575000000000000} & \frac{73566267449751}{575000000000000} & \mathbf{0} \end{pmatrix},$$

$$\bar{X} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \frac{7468875}{4769280000} & \frac{2949000}{4769280000} & \frac{-267840000}{4769280000} & \frac{-90439680}{4769280000} & \frac{-144374805}{4769280000} & \mathbf{0} \\ \frac{23864625}{15073280000} & \frac{8313000}{15073280000} & \frac{-635040000}{15073280000} & \frac{-350453760}{15073280000} & \frac{-390609135}{15073280000} & \mathbf{0} \\ \frac{35717625}{22918393125} & \frac{14424000}{22918393125} & \frac{-1213920000}{22918393125} & \frac{-429588480}{22918393125} & \frac{-746674605}{22918393125} & \mathbf{0} \\ \frac{343278799875}{575000000000000} & \frac{550967373000}{575000000000000} & \frac{-7728395328000}{575000000000000} & \frac{-3647341854720}{575000000000000} & \frac{-806113623645}{575000000000000} & \mathbf{0} \end{pmatrix},$$

$b(1)^T = (\frac{79104873575}{929674746000}, \frac{13812708000}{929674746000}, \frac{47758183200}{929674746000}, \frac{127821152256}{929674746000}, \frac{481490328969}{929674746000}, \frac{179687500000}{929674746000})^T,$
$\bar{b}(1)^T = (\frac{1909805625}{929674746000}, \frac{476985600}{929674746000}, \frac{-52587662400}{929674746000}, \frac{-21603778560}{929674746000}, \frac{-15717367575}{929674746000}, 0)^T.$
The stability function is
$R(z) = \frac{38320128000+18347212800z+4159572480z^2+590150400z^3+58240560z^4+4190520z^5+222800z^6+8584z^7+222z^8+3z^9)}{38320128000-19972915200z+4972423680z^2-782503680z^3+86679600z^4-7104120z^5+438640z^6-20180z^7+652z^8-12z^9)}.$
The boundary locus plot in Figure 1 reveals that the scheme in (25) is A-stable

5.7 ATDMIRK method of order $p = q = 13$, $s = 7$

Fixing $s = 7, c = (0, 1, \frac{1}{2}, \frac{1}{4}, \frac{2}{3}, \frac{3}{4}, \frac{1}{3})^T$ in (7) and (8) solving the arising system of equations yields the coefficients below

$$\begin{array}{c|c|c|c} \text{c} & \text{v} & \text{X} & \bar{X} \\ \hline & & b^T(1) & \bar{b}^T(1) \end{array}, \tag{26}$$

where,

$v = (0, 1, \frac{5840046000}{6815232000}, \frac{52941374000}{64618496000}, \frac{758858868000}{884251393875}, \frac{55460862000}{64618496000}, \frac{735932455875}{884251393875})^T$,

$$X = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \frac{74857125}{6815232000} & \frac{-395852000}{6815232000} & \frac{-4679424000}{6815232000} & \frac{-90701824}{6815232000} & \frac{-23096957301}{6815232000} & \frac{25755648000}{6815232000} & \mathbf{0} \\ \frac{880924125}{64618496000} & \frac{-3604572000}{64618496000} & \frac{-44686080000}{64618496000} & \frac{-6852231168}{64618496000} & \frac{-203051382957}{64618496000} & \frac{220526592000}{64618496000} & \mathbf{0} \\ \frac{9627843875}{884251393875} & \frac{-51414754000}{884251393875} & \frac{-541762560000}{884251393875} & \frac{-11484594176}{884251393875} & \frac{-2950714018449}{884251393875} & \frac{3376390144000}{884251393875} & \mathbf{0} \\ \frac{703144125}{64618496000} & \frac{-3757404000}{64618496000} & \frac{-39543552000}{64618496000} & \frac{-838385664}{64618496000} & \frac{-212693848461}{64618496000} & \frac{249133056000}{64618496000} & \mathbf{0} \\ \frac{11264555750}{884251393875} & \frac{-50047359250}{884251393875} & \frac{-632772864000}{884251393875} & \frac{-33494171648}{884251393875} & \frac{-2846622264102}{884251393875} & \frac{3110490112000}{884251393875} & \mathbf{0} \end{pmatrix},$$

$$\bar{X} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \frac{1597575}{6815232000} & \frac{7420200}{6815232000} & \frac{-388108800}{6815232000} & \frac{-41287680}{6815232000} & \frac{-1756412505}{6815232000} & \frac{-1051852800}{6815232000} & \mathbf{0} \\ \frac{18478575}{64618496000} & \frac{67900200}{64618496000} & \frac{-2421964800}{64618496000} & \frac{-659189760}{64618496000} & \frac{-14659799985}{64618496000} & \frac{-9163468800}{64618496000} & \mathbf{0} \\ \frac{205531275}{884251393875} & \frac{963326400}{884251393875} & \frac{-48511353600}{884251393875} & \frac{-5297848320}{884251393875} & \frac{-232821644895}{884251393875} & \frac{-137449881600}{884251393875} & \mathbf{0} \\ \frac{15010575}{64618496000} & \frac{70396200}{64618496000} & \frac{-3541708800}{64618496000} & \frac{-386887680}{64618496000} & \frac{-16955625105}{64618496000} & \frac{-10072780800}{64618496000} & \mathbf{0} \\ \frac{237833400}{884251393875} & \frac{941517150}{884251393875} & \frac{-36423561600}{884251393875} & \frac{-7185653760}{884251393875} & \frac{-208037893860}{884251393875} & \frac{-128685465600}{884251393875} & \mathbf{0} \end{pmatrix},$$

$b(1)^T = (\frac{170040625}{2432430000}, \frac{150954250}{2432430000}, \frac{4679424000}{2432430000}, \frac{-4190273536}{2432430000}, \frac{15236944911}{2432430000}, \frac{-21474672640}{2432430000}, \frac{7860012390}{2432430000})^T$,

$\bar{b}(1)^T = (\frac{3283125}{2432430000}, \frac{-2539500}{2432430000}, \frac{856656000}{2432430000}, \frac{-261611520}{2432430000}, \frac{1756412505}{2432430000}, \frac{748953600}{2432430000}, 0)^T$.

The computed stability function is $R(z) = \frac{N(z)}{D(z)}$, where

$N(z) = 28694111846400 + 13979182694400z + 3282757632000z^2 + 493594214400z^3 + 53205707520z^4 + 4358135040z^5 + 280119000z^6 + 14346480z^7 + 585560z^8 + 18656z^9 + 435z^{10} + 6z^{11}$,

$D(z) = 28694111846400 - 14714929152000z + 3650630860800z^2 - 581924044800z^3 + 66714520320z^4 - 5828390400z^5 + 400447320z^6 - 21955560z^7 + 959860z^8 - 32740z^9 + 816z^{10} - 12z^{11}$.

The stability function plot for the method discussed in (26) exhibits A-stability.

5.8 ATDMIRK method of order $p = q = 15$, $s = 8$

Here,we determine the coefficients for the $ATDMIRK$ method attaining an order of $p = 15$, fixing $s = 8$, $c = (0, 1, \frac{1}{10}, \frac{1}{4}, \frac{2}{3}, \frac{3}{4}, \frac{9}{10}, \frac{1}{3})^T$ in (7) and (8) and solving the arising system of equations yields the coefficients below,

$$\begin{array}{c|c|c|c} \text{c} & \text{v} & \text{X} & \bar{X} \\ \hline & & b^T(1) & \bar{b}^T(1) \end{array}, \tag{27}$$

where,

$$v = \left(0, 1, \frac{1933180295835505421589430000}{59674968258485125000000000000}, \frac{1226892264535530561150000}{16894909669484510576640000}, \frac{27766140198085704629254950000}{28089956041270569414972241875}, \frac{16705297173723869635470000}{16894909669484510576640000}, \frac{59256150774442681531833270000}{59674968258485125000000000000}, \frac{75483848204941349569943137 50}{56179912082541138829944483750}\right)^T,$$

$$X = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{2113639111753991180707973625}{59674968258485125000000000000} & \frac{-170804947564430045642412000}{59674968258485125000000000000} & \frac{2876028790301430227050781250}{59674968258485125000000000000} & \frac{123521217173671434271064064}{59674968258485125000000000000} & \frac{5348562548611926011031832311}{59674968258485125000000000000} & \frac{-5568472663058384001957888000}{59674968258485125000000000000} & \frac{-688157527205197727050781250}{59674968258485125000000000000} & 0 \\ \frac{618561927774215287715125}{16894909669484510576640000} & \frac{-100154042350042853172000}{16894909669484510576640000} & \frac{1839796720586297607421875}{16894909669484510576640000} & \frac{1217839158085485405044736}{16894909669484510576640000} & \frac{3039999041851301053598139}{16894909669484510576640000} & \frac{-3217615423837861810176000}{16894909669484510576640000} & \frac{-401592229273797607421875}{16894909669484510576640000} & 0 \\ \frac{19785564308436236223299750}{28089956041270569414972241875} & \frac{-1024895778034060601136286000}{28089956041270569414972241875} & \frac{56831481674009808349609375}{28089956041270569414972241875} & \frac{56578273085663686468435968}{28089956041270569414972241875} & \frac{-4994670958655816253599550468}{28089956041270569414972241875} & \frac{-219779254026818920562688000}{28089956041270569414972241875} & \frac{-2933352165590072308349609375}{28089956041270569414972241875} & 0 \\ \frac{11612181847732352830125}{16894909669484510576640000} & \frac{-615952934869226304532000}{16894909669484510576640000} & \frac{33329909691033935546875}{16894909669484510576640000} & \frac{33091899511020111691776}{16894909669484510576640000} & \frac{-2324239041802213333763901}{16894909669484510576640000} & \frac{587507194889700470784000}{16894909669484510576640000} & \frac{-1759464130878533935546875}{16894909669484510576640000} & 0 \\ \frac{26070368124614959274117625}{59674968258485125000000000000} & \frac{-2094361080217177071320748000}{59674968258485125000000000000} & \frac{74387516516160227050781250}{59674968258485125000000000000} & \frac{72446952751239921785634816}{59674968258485125000000000000} & \frac{-4303598249806083831701682441}{59674968258485125000000000000} & \frac{3358993112654444490129408000}{59674968258485125000000000000} & \frac{-2682617961829267727050781250}{59674968258485125000000000000} & 0 \\ \frac{1974145845613233200368698875}{56179912082541138829944483750} & \frac{-535614487251701928109994500}{56179912082541138829944483750} & \frac{5943427694935117059326171875}{56179912082541138829944483750} & \frac{7358815841988628214459006976}{56179912082541138829944483750} & \frac{14751107462448326951582152149}{56179912082541138829944483750} & \frac{-16207721713361887618646016000}{56179912082541138829944483750} & \frac{-2105908104018804559326171875}{56179912082541138829944483750} & 0 \end{pmatrix},$$

$$\bar{X} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{22167574768711068608149875}{59674968258485125000000000000} & \frac{3019339569807022390929000}{59674968258485125000000000000} & \frac{-152579597707438268554687500}{59674968258485125000000000000} & \frac{-71984501779825403607121920}{59674968258485125000000000000} & \frac{284615025595654734872683155}{59674968258485125000000000000} & \frac{295338613438234798129152000}{59674968258485125000000000000} & \frac{33517909767580075195312500}{59674968258485125000000000000} & 0 \\ \frac{6986416798102417050375}{16894909669484510576640000} & \frac{1725109789258866465000}{16894909669484510576640000} & \frac{-11675260784465332031250}{16894909669484510576640000} & \frac{-73956442342710615367680}{16894909669484510576640000} & \frac{165595278828272468506695}{16894909669484510576640000} & \frac{167132232324090528768000}{16894909669484510576640000} & \frac{18405587385065917968750}{16894909669484510576640000} & 0 \\ \frac{313200351842645189695500}{28089956041270569414972241875} & \frac{10745802336049153031295000}{28089956041270569414972241875} & \frac{1783801876228220214843750}{28089956041270569414972241875} & \frac{4434928597131406016348160}{28089956041270569414972241875} & \frac{-160345769471577470969050590}{28089956041270569414972241875} & \frac{-311708030007491458338816000}{28089956041270569414972241875} & \frac{-46381488859052248535156250}{28089956041270569414972241875} & 0 \\ \frac{184055471695504875375}{16894909669484510576640000} & \frac{6450616227904524585000}{16894909669484510576640000} & \frac{1051404080866699218750}{16894909669484510576640000} & \frac{2609124184664960901120}{16894909669484510576640000} & \frac{-86934324761714164967505}{16894909669484510576640000} & \frac{-199369158420774730752000}{16894909669484510576640000} & \frac{-28219973921477050781250}{16894909669484510576640000} & 0 \\ \frac{417038046323927873173875}{59674968258485125000000000000} & \frac{21032880092726662463121000}{59674968258485125000000000000} & \frac{2428972463197731445312500}{59674968258485125000000000000} & \frac{5940738708251044172267520}{59674968258485125000000000000} & \frac{-169775011199876229319613805}{59674968258485125000000000000} & \frac{-304283679441491143360512000}{59674968258485125000000000000} & \frac{-182486759800015924804687500}{59674968258485125000000000000} & 0 \\ \frac{22937098805700462950419125}{56179912082541138829944483750} & \frac{8759587161524837501272500}{56179912082541138829944483750} & \frac{-17282876829623723144531250}{56179912082541138829944483750} & \frac{-84903929471521182999674880}{56179912082541138829944483750} & \frac{844002454328241203166325245}{56179912082541138829944483750} & \frac{807573277222756492750848000}{56179912082541138829944483750} & \frac{84432899610850495605468750}{56179912082541138829944483750} & 0 \end{pmatrix},$$

$$b(1)^T = \left(\frac{60696313251569279375}{2188353942563058270000}, \frac{120596656363033077500}{2188353942563058270000}, \frac{156324893927001953125}{2188353942563058270000}, \frac{-184733577533055172608}{2188353942563058270000}, \frac{-2517592097430127276767}{2188353942563058270000}, \frac{3094390057177898680320}{2188353942563058270000}, \frac{451121793572998046875}{2188353942563058270000}, \frac{1007549903233739682180}{2188353942563058270000}\right)^T$$

$$\bar{b}(1)^T = \left(\frac{377985276119894625}{2188353942563058270000}, \frac{-1742803220431728000}{2188353942563058270000}, \frac{-10284562507324218750}{2188353942563058270000}, \frac{-54010865921356431360}{2188353942563058270000}, \frac{-165951379887253804935}{2188353942563058270000}, \frac{-136522241578237132800}{2188353942563058270000}, \frac{-11992305476074218750}{2188353942563058270000}, 0\right)^T,$$

The computed stability function is $R(z) = \frac{N(z)}{D(z)}$, where,

$N(z) = 5021469573120000000 + 2454940680192000000z + 576525814824960000z^2 + 86307487580160000z^3 + 9214233273907200z^4 + 743350564454400z^5 + 46826970785280z^6 + 2345843781120z^7 + 94199495880z^8 + 3027645200z^9 + 76879640z^{10} + 1493520z^{11} + 20601z^{12} + 162z^{13}$,

$D(z) = 5021469573120000000 - 2566528892928000000z + 632319921192960000z^2 - 99659582668800000z^3 + 11240771954227200z^4 - 961159491532800z^5 + 64345025364480z^6 - 3431359196160z^7 + 146713177800z^8 - 5012683960z^9 + 134802220z^{10} - 2757324z^{11} + 39744z^{12} - 324z^{13}$.

The plot of the boundary locus for $R(z)$ indicates that the method in (27) is thus A-stable.

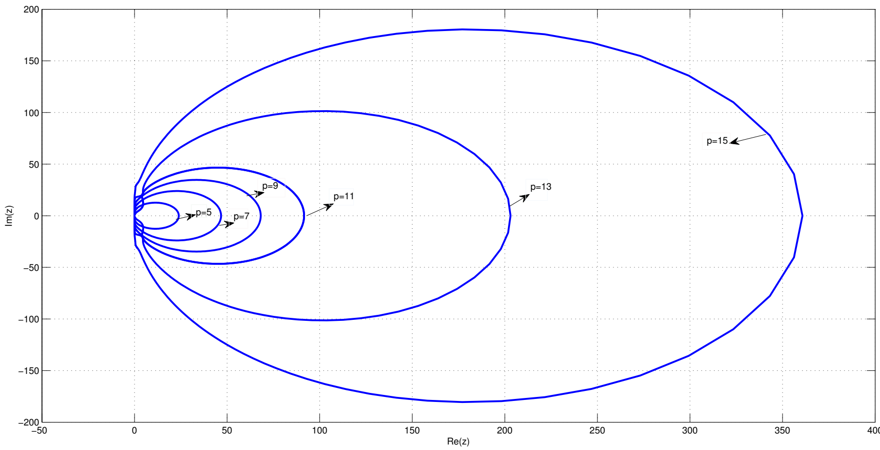

Figure 1: Stablilty plot for $ATDMIRK5$, $ATDMIRK7$, $ATDMIRK9$, $ATDMIRK11$, $ATDMIRK13$ and $ATDMIRK15$.

The Figure 1 shows the stability plot for A-stable Mono-implicit Runge-Kutta methods for p=5(2)15, the stability region includes the exterior region beyond the closed curve. The root of the stability function $\mid R(z) \mid \leqslant 1$. It is evident in Table 1

Table 1: Properties of ATDMIRK

s	$Errorconstant$	$order$	$zero\ stability$	$stability\ properties\ of\ ATDMIRK\ method$
3	$\frac{1}{14400}$	5	$zero\ stable$	$A-stable$
4	$\frac{1}{16934400}$	7	$zero\ stable$	$A-stable$
5	$\frac{1}{20575296000}$	9	$zero\ stable$	$A-stable$
6	$\frac{1}{53111697408000}$	11	$zero\ stable$	$A-stable$
7	$\frac{661}{62041260869812224000}$	13	$zero\ stable$	$A-stable$
8	$\frac{9587}{469031932175780413440000}$	15	$zero\ stable$	$A-stable$

that the ATDMIRK method has relatively small error constant and by Definition 4.1 the method in (4) and (5) are A-stable for order $5, 7, 9, 11, 13$ and 15. Figure 2. shows that the ATDMIRK scheme in (5) has small error constant compare to the methods in Cash [15], Okor and Nwachukwu [43] for the same order p=5, 7 and 9. This confirm theoretically the possibility of having more accurate solution on stiff problem in (2).

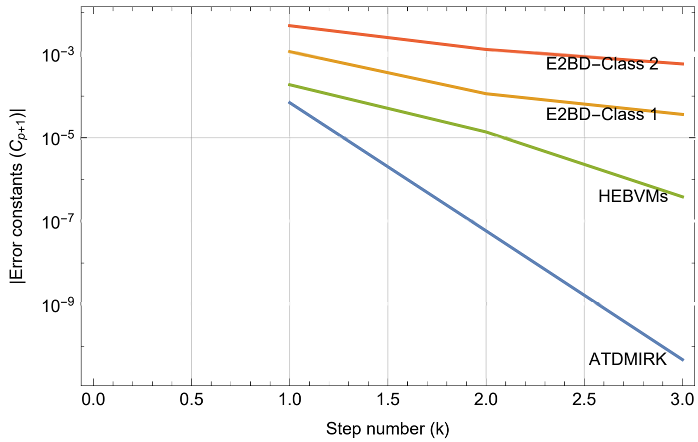

Figure 2: The Semi-Log plot of the absolute values of the error constants plotted against order $p = 5$,7 and 9 of the ATDMIRK, HEBVMs [43] and E2BD [15]

6 Implementation Procedures

This section present an illustration for the implementation of ATDMIRK method for $s = 3$ and 4 with order $p = 5$ and $p = 7$ respectively, as a block method by following the approached in Jator [24], Akinfenwa [42] and Okor and Nwachukwu [43]. The $5th$ and $7th$ order are denoted by ATDMIRK5 and ATDMIRK7.

The $5th$ order of ATDMIRK

$$\begin{aligned} u_{n+3/4} &= \tfrac{212}{2048}u_n + \tfrac{1836}{2048}u_{n+1} + \tfrac{78}{2048}hf_n - \tfrac{378}{2048}hf_{n+1} + \tfrac{9}{2048}h^2g_n + \tfrac{27}{2048}h^2g_{n+1}, \\ u_{n+1} &= u_n + \tfrac{190}{540}hf_n - \tfrac{162}{540}hf_{n+1} + \tfrac{512}{540}hf_{n+\frac{3}{4}} + \tfrac{21}{540}h^2g_n + \tfrac{27}{540}h^2g_{n+1}, \end{aligned}$$

The $7th$ order of ATDMIRK

$$\begin{aligned}
u_{n+1/2} &= \tfrac{96}{192}u_n + \tfrac{96}{192}u_{n+1} + \tfrac{18}{192}hf_n - \tfrac{18}{192}hf_{n+1} + \tfrac{1}{192}h^2g_n + \tfrac{1}{192}h^2g_{n+1} \\
&-\tfrac{8}{192}h^2g_{n+\frac{1}{2}}, \\
u_{n+3/4} &= \tfrac{2468}{8192}u_n + \tfrac{5724}{8192}u_{n+1} + \tfrac{474}{8192}hf_n - \tfrac{918}{8192}hf_{n+1} + \tfrac{864}{8192}hf_{n+\frac{1}{2}} + \tfrac{27}{8192}h^2g_n \\
&+\tfrac{45}{8192}h^2g_{n+1} - \tfrac{144}{8192}h^2g_{n+\frac{1}{2}}, \\
u_{n+1} &= u_n + \tfrac{706}{3780}hf_n + \tfrac{162}{3780}hf_{n+1} + \tfrac{864}{3780}hf_{n+\frac{1}{2}} + \tfrac{2048}{3780}hf_{n+\frac{3}{4}} + \tfrac{39}{3780}h^2g_n \\
&+\tfrac{9}{3780}h^2g_{n+1} - \tfrac{288}{3780}h^2g_{n+\frac{1}{2}},
\end{aligned}$$

The main output obtain from (5) and input method derive from (4) form a composite scheme of the same order, by this the main output and input method are combined as a one blocked method

$$A_1U_{\phi+1} = A_0U_\phi + h(B_0F_\phi + B_1F_{\phi+1}) + h^2(C_0G_\phi + C_1G_{\phi+1}). \qquad (28)$$

where $U_\phi = (u_{n-\frac{3}{4}}, u_{n-\frac{1}{2}}, u_n)^T$, $U_{\phi+1} = (u_{n+\frac{1}{2}}, u_{n+\frac{3}{4}}, u_{n+1})^T$, $F_\phi = (f_{n-\frac{3}{4}}, f_{n-\frac{1}{2}}, f_n)^T$, $F_{\phi+1} = (f_{n+\frac{1}{2}}, f_{n+\frac{3}{4}}, f_{n+1})^T$, $G_\phi = (g_{n-\frac{3}{4}}, g_{n-\frac{1}{2}}, g_n)^T$, $G_{\phi+1} = (g_{n+\frac{1}{2}}, g_{n+\frac{3}{4}}, g_{n+1})^T$ which simultaneously generate block solution values.

The summary of the implementation procedure for order 7, $s = 4$ is as follows:

Let the partition $\Pi_N : a = x_0 < x_1 < ... < x_n < x_{n+1} = b, h = x_{n+1} - x_n, n = 0(1)N-1$

Step1: input value of N, for $s = 4$, $h = \frac{b-a}{N}$, the number of block $\tau = \frac{N}{3}$. using (28) $n = 0, \phi = 0$, the solution value of $\left(u_{\frac{1}{2}}, u_{\frac{3}{4}}, u_1\right)^T$ are generated concurrently over the sub-interval $[x_0, x_1]$, where u_0 is provided by the problem (2).

Step2: $n = 1, \phi = 1$, the solution $\left(u_{\frac{3}{2}}, u_{\frac{7}{4}}, u_2\right)^T$ are obtained over the sub-interval $[x_1, x_2]$, since u_1 is generated from the previous block.

Step3: the iteration is process for $n = 2, ...N-2$ and $\phi = 2, ..., \tau$ to generate solution of (2) on sub-intervals $[x_2, x_3]...[x_{N-1}, x_N]$.

By this, the accumulated error is negligible in the numerical solution, as the solution are generated concurrently,([21], [29]). When dealing with non-linear problems, a modified Newton-Raphson method is studied while the linear problems is a direct approach of Gaussian elimination using partial pivoting.

7 Numerical Experiment

Numerical results showing the implementation and accuracy of the constructed ATDMIRK5 and ATDMIRK7 in (22) and (23) respectively are presented. The order of ATDMIRK5 and ATDMIRK7 are $p = 5$ and $p = 7$ respectively, see Section 5 of this article. The implementation is done in fixed step size mode for accuracy purpose. Our objective is contrast the outcome of our methods with the results obtained from some existing methods. Computational experiments are done by applying the ATDMIRK5 methods and ATDMIRK7 methods to the problems that follows:

Problem 1:. Consider the system of differential equations [30],

$$\begin{cases} u_1'(x) = -21u_1 + 19u_2 - 20u_3, & u_1(x) = \frac{1}{2}(e^{-2x} + e^{-40x}(cos(40x) + sin(40x))), \\ u_2'(x) = 19u_1 - 21u_2 + 20u_3, & u_2(x) = \frac{1}{2}(e^{-2x} - e^{-40x}(cos(40x) + sin(40x))), \\ u_3'(x) = 40u_1 - 40u_2 - 40u_3, & u_2(x) = -e^{-40x}(cos(40x) + sin(40x))), \\ x \in [0,1], u(0) = [1, 0, -1]^T & \end{cases}$$

We have solved this problem at $h = 0.05, 0.025, 0.0125$ and 0.00625.

Table 2 show that the new methods $ATDMIRK5$ and $ATDMIRK7$ outperforms the existing method in [30], [31], [7] in accuracy and is highly compatible with the integrating IVPs in ODEs.

Table 2: Numerical results for problem 1

h	$ATDMIRK5$	$ATDMIRK7$	$GSDLMM3$ [30]	$IMEXSDLMM$ [31]	$SDBDF6$ [31]	$Amodio6$ [7]
0.05	$5.9e-003$	$6.58e-005$	$3.0e-002$	$6.20e-002$	$6.22e-002$	$5.70e-002$
	$(-)$	$(-)$	$(-)$	$(-)$	$(-)$	$(-)$
0.025	$3.89e-004$	$4.20e-007$	$3.55e-003$	$9.20e-002$	$9.28e-002$	$8.70e-003$
	(3.95)	(6.04)	(3.07)	(2.75)	(2.74)	(2.70)
0.0125	$9.98e-006$	$2.68e-009$	$2.26e-004$	$5.61e-004$	$7.22e-004$	$4.9e-004$
	(5.28)	(7.28)	(3.97)	(4.03)	(4.02)	(4.20)
0.00625	$1.82e-007$	$1.42e-011$	$5.86e-006$	$1.09e-005$	$1.20e-005$	$1.20e-005$
	(5.77)	(7.5)	(5.27)	(5.68)	(5.57)	(5.80)

Problem 2: A stiff system which has also been solved by Cash [15],

$$\begin{cases} u_1' = -u_1 - 30u_2 + 30e^{-x}, & u_1(0) = 1, \\ u_2' = 30u_1 - u_2 - 30e^{-x}, & u_2(0) = 1, \\ \text{The exact solution is} & \\ u_1(x) = e^{-x}, u_2(x) = e^{-x} & \end{cases}$$

The problem in 2 is integrated with $h = 0.01$ and $h = 0.09$ for the purpose of

Table 3: Numerical results for problem 2, $h = 0.01$. $error\ u_i = \mid u_i - u(x_i) \mid)i = 1, 2.$

x	u_i	Error in $ATDMIRK5(21)$ $p=5$	Error in $ECBBDF1001$[42] $p=5$
1.0	u_1	$5.11e-016$	$1.28e015$
	u_2	$7.06e-016$	$1.17e-014$
10.0	u_1	$1.15e-019$	$1.08e019$
	u_2	$2.26e-019$	$1.62e-018$
20.0	u_1	$1.09e-023$	$7.25e023$
	u_2	$9.43e-024$	$5.29e-023$

comparison. Thus, results for $h = 0.01$ and $h = 0.09$ are tabulated at different

Table 4: Absolute error for for problem 2, $h = 0.09$

x	u_i	$ATDMIRK5(21)$ $p=5$	$ATDMIRK7(22)$ $p=7$	$E2BD$ Class 2 [15] $p=8$	$HEBVM5$ [43] $p=9$
4.5	u_1	$2.7e-13$	$2.7e-017$	$0.1e010$	$0.4e016$
	u_2	$1.1e-13$	$1.4e-017$	$0.1e-010$	$0.4e016$
9.0	u_1	$2.1e-15$	$4.7e-019$	$0.1e012$	$0.7e018$
	u_2	$2.6e-15$	$2.7e-019$	$0.1e-012$	$0.5e018$
13.5	u_1	$3.3e-17$	$9.8e-021$	$0.8e011$	$0.9e020$
	u_2	$2.7e-17$	$8.2e-021$	$0.6e-011$	$0.6e020$
18	u_1	$3.2e-19$	$8.9e-023$	$0.1e011$	$0.1e021$
	u_2	$2.7e-19$	$1.1e-022$	$0.1e-011$	$0.1e021$

values of x to show the performance of the methods. Table 3 and 4, reveals that the newly derived schemes in (22) and (23) are better in terms of accuracy than the $ECBBDF5$ [42] ,$E2BD$ [15] and $HEBVM5$ [43]

.

Problem 3: Non-linear stiff system [30],

$$\begin{cases} u_1' = -1002u_1 + 1000u_2^2, & u_1(0) = 1, \\ u_2' = u_1 - u_2(1 + u_2), & u_2(0) = 1, \\ u_1(x) = e^{-2x}, u_2(x) = e^{-x} & \end{cases}$$

In like manner, the numerical results in Table 5 and 6 show that the new methods $ATDMIRK5$ and $ATDMIRK7$ exhibit the capability to yield accurate and stable results, comparable to existing methods presented herein

Problem 4: Consider stiff equation [42], [43],

$$\begin{cases} u_1' = -2000u_1 + 1000u_2 + 1, & u_1(0) = 0, \\ u_2' = u_1 - u_2, & u_2(0) = 0, \\ \text{The eigenvalues of the Jacobian are 2000.5 and } -0.5\text{. The theoretical solution is} & \\ u_1(x) = -4.97 \times 10^{-4} e^{-2000.5x} - 5.034 \times 10^4 e^{0.5x} + 0.001 & \\ u_2(x) = -2.5 \times 10^7 e^{-2000.5x} - 1.007 \times 10^{-3} e^{-0.5x} + 0.001 & \end{cases}$$

Table 5: Comparison of results at t = 1 and maximum absolute error, for problem 3

$Methods$	$order$	N	h	u_1 $(Max \mid\mid u_i - u(x_i) \mid\mid)$	u_2 $(Max \mid\mid u_i - u(x_i) \mid\mid)$
$ATDMIRK5$	5	125	0.008	$5.66e016$	$1.11e016$
$ATDMIRK7$	7	125	0.008	$1.11e016$	0.00
$SDGBDF5$ [29]	5	125	0.008	$1.80e015$	$6.11e-016$
$Ehigieetal(BVM3)$ [21]	5	125	0.008	$3.88e014$	$3.10e014$
$Jator$ [24]	6	125	0.008	$1.63e014$	0.00
$GSDLMM5$ [30]	5	125	0.008	$6.88e015$	$3.33e015$

Table 6: Comparison of results for problem 3, Error u_i=$(\mid u_i - u(x_i) \mid)$, $i = 1, 2$

$Methods$	x	h	N	$Erroru_1$	$Erroru_2$
$ATDMIRK5$ $p = 5$	10	0.01	1000	$1.3896e-023$	$1.4501e-019$
$ECBBDF_5$ $p = 6$	10	0.01	1000	$2.87e-022$	$2.93e-019$
$HEBVM3$ $p = 7$	10	0.01	1000	$2.3988e023$	$2.6427e-019$
$BBDFs$ $p = 8$	10	0.01	1000	$6.6466e020$	$2.3988e-017$

Table 7: The numerical solution of problem 4 at $x = 5$ and 10, $h = 0.1$. $error \quad u_i =\mid u_i - u(x_i) \mid, i = 1, 2.$

h	x	$ATDMIRK5(21)$ $p=5$	$ECBBDF51001$[42] $p=6$	$HEBVM51001$[43] $p=9$
		$errory_1$	$errory_1$	$errory_1$
		$errory_2$	$errory_2$	$errory_2$
0.1	5	$3.146697e-007$	$3.163426e004$	$1.702501e-007$
		$6.793685e-007$	$6.610743e-007$	$3.678791e007$
0.1	10	$2.297511e-008$	$2.005234e004$	$1.378729e-008$
		$5.005828e-008$	$1.373470e-007$	$3.005916e008$

The numerical results in Table 7, reveals that the $ATDMIRK5$ in (22) performs better than the method $ECBBDF5$ in [42] and compete in terms of accuracy with order $p = 9$ of $HEBVM5$ in [43].

8 Conclusion

In this paper, a family of A-stable Two Derivative Mono-Implicit Runge-Kutta method is proposed for the numerical solution of IVPs in ODEs. The plot in Figure 1 and the stability analysis in Section 4 shows that ATDMIRK methods possess zero- and A-stability properties. Figure 2 contain error constant of ATDMIRK method, ECBBDF5 [42] and HEBVM5 [43]. The ATDMIRK method possess smaller error constant than the compared method in [42] and [43] to ODEs system theoretically. The numerical results in Table 2-7 distinctly reveal how the proposed methods outperform some established methods in the literature. Moreso, the result displayed in Table 2-7, shows that the new schemes has the qualities of handling ODEs equations with some eigenvalue of Jacobian lying close to imaginary axis.

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