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Bounds for Weierstrass Elliptic Function and Jacobi Elliptic Integrals of First and Second Kinds

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Abstract

The Weierstrass elliptic function is presented in connection with the Jacobi elliptic integrals of first and second kinds leading to comparing coefficients appearing in the Laurent series expansion with those of Eisenstein series for the cubic polynomial in the meromorphic Weierstrass function.

It is unified in the formulation the Weierstrass elliptic function with Jacobi elliptic integral by considering motion of a unit mass particle in a cubic potential in terms of bounded and unbounded velocities and the time of flight with imaginary part in the complex function playing a major role. Numerical tools box used are the Konrad-Gauss quadrature and Runge-Kutta fourth order method.

1. Introduction

The first aim of this article is to demonstrate that there exists a functional relationship between Weisterstrass elliptic function and Jacobi elliptic integrals of first and second kinds. That Schwartizian derivative can be an integral part in the treatments of Weierstrass phase elliptic function. It is to reinvigorate the usefulness of Weierstrass elliptic function and Jacobi elliptic integrals in solving most difficult problems in the arclength of motion of body mass undergoing cubic potential well. Thus, the paper opens a new vista of approach to other areas in scientific computing.

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1.1. Preliminaries

Let $f: \mathbb{C} \to \mathbb{C}_\infty$ where $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ represents the Riemannian sphere in the meromorphic function. Given a complex number $z \in \mathbb{C}$ and a lattice Ω, the Weierstrass elliptic function is defined in the form

$$f_\Omega(z) = \frac{1}{z^2} + \sum_{\omega \in \Omega \setminus \{0\}}^{\infty} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right). \tag{1.1}$$

Later in the paper, we shall denote $\wp_\Omega(z)$ to represent $f_\Omega(z)$ in the Weierstrass elliptic function. The function $\wp_\Omega(z)$ is periodic with respect to the lattice and with order 2. It is meromorphic over the lattice.

1.2. Literature review

The first aim in this article is to answer in the affirmative that there exists [6]. In [23], it was initiated that Jacobi elliptic integrals of first and second kinds can be used to solve most difficult life time problems such as geodesy of earth meridian surface. There, it was left open as an exercise if such Jacobi elliptic integrals can be linked with Weierstrass meromorphic elliptic function. It was also mentioned that there existed yet no universally acceptable bounds for the Jacobi elliptic integrals. The above exposition motivated the present interest in this paper.

Regarding Weierstrass elliptic functions [10,25] we give information on the doubly periodic parallelogram leading to the differential equations satisfied by the Weierstrass elliptic function. The roots equation satisfied by the polynomial for the elliptic integral are discussed, [6,19,22] as well as the bound for the convergent entire meromorphic elliptic function. Using a Laurent series expansion at the point $z = 0$ for the meromorphic Weierstrass elliptic function, it is compared coefficients with the Eisenstein series and obtained values for classes of Weierstrass elliptic functions in the senses of [8,10,11,12].

Here after, we motivate our findings with information on the Schwarzian derivative which relates the Fatou set with duplication formula for the Weierstrass function.

The Jacobi elliptic integrals are derived from the Jacobi theta functions in the senses of [18,25] which have various uses in engineering and allied fields. Various theoretical conditioning bounds for the Jacobi periodic functions are described. Taking ratio of these bounds obtained in the Literatures, we obtain results for the bounds with particular reference to the gamma and hypergeometric functions relating these theorems. We realized that Jacobi elliptic integrals are a geodesic problem where ellipsoidal bounds may be necessary to compute.

1.3. The Weierstrass elliptic periodic parallelogram and the lattice

The Weierstrass elliptic function is one of the most popular researched topics in the theory of Complex functions among contemporary scientists and engineers. Yet, its interests and applications to areas such as astrodynamics, plasma physics, nuclear science and applied mathematics are still growing in the scientific world [4,6,12,13,23,24,26] with many more areas of its applications yet to be discovered.

This is more important in the theory of classical mechanics as for example [6,19,12] where it is necessary to compute Weierstrass phase functions for a motion of a body of mass under cubic potential well. The reason why Weierstrass differential equation has two independent real solutions when there are three real roots and only one, when there is one real root is explained. These roots play the role of extrema of motion and are indeed the positions the velocities vanishes.

Fundamental to this presentation is that the Weierstrass elliptic function is a complex meromorphic function with double poles and two zeros per cell. It is a doubly periodic parallelogram. The parallelogram with vertices $0, \omega_1, \omega_2, \omega_1 + \omega_2$ is called a fundamental parallelogram for the Weierstrass elliptic function.

A function $\wp(z)$ is said to be periodic with a period 2π if $\wp(z + 2\pi) = \wp(z)$.

We motivate this paper by considering the ratio of two numbers $\frac{\omega_1}{\omega_2} \neq 0$ (with $\omega_2 \neq 0$) which isnot purely real, such that $\wp(z + 2\omega_1) = p(z), \wp(z + 2\omega_2) = \wp(z)$ that is doubly periodic function for all values of z with periods $2\omega_1, 2\omega_2$. A doubly periodic function that is analytic (except at poles) and which has no singularities other than poles in a finite domain is called an elliptic function. Given two periods ω_1, ω_2 we therefore define two different fundamental periods in the form:

$$\begin{matrix} \omega_1^{/} = a\omega_1 + b\omega_2 \\ \omega_2^{/} = c\omega_1 + d\omega_2 \end{matrix}, \quad (a, b, c, d \text{ are constant}). \tag{1.2}$$

Let us denote a lattice as $\Omega = 2m\omega_1 + 2n\omega_2$, where $\Omega \in \mathbb{C}$. Any period, defined by

$\omega_1^/$ and $\omega_2^/$ such that $2\omega = 2m^/\omega_1^/ + 2n^/\omega_2^/$ is also a period of an old lattice. The area of parallelogram defined by two complex numbers z_1 and z_2 is given by

$$A = |Im(z_1.z_2)|, \text{ where } Im(z_1.z_2) = (ad - bc)\, Im(\omega_1\omega_2).$$

The new period also generates the original lattice if $det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \pm 1$ or $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, Z)$.

Hence it follows that the parameters ω_1 and ω_2 are the generators of the parallelogram.

We say that the transformation [11,16] of $(\omega_1, \omega_2) \to (\omega_1^/, \omega_2^/)$ is a unimodular transformation, if $ad - bc = 1$.

If the above holds, we then say that the lattice (ω_1, ω_2) is similar to $\left(1, \frac{\omega_2}{\omega_1}\right) \approx (1, \tau)$ and that, equation (1.2) is equivalent to the form:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \tau \\ 1 \end{pmatrix} = \begin{pmatrix} a\tau + b \\ c\tau + d \end{pmatrix} = \begin{pmatrix} \frac{a\tau+b}{c\tau+d} \\ 1 \end{pmatrix}. \tag{1.3}$$

Four types of lattices [17] are defined according to appearances of Ω below:

A lattice Ω is a square if $\tau(\Omega) = i$; Ω is triangular if $\tau(\Omega) = e^{2\pi i/3}$; Ω is rectangular if $\tau(\Omega)$ is pure imaginary; and Ω is a rhombus if $|\tau(\Omega)| = 1$.

This paper is motivated by the following standard theorems of Weierstrass elliptic function.

Let $f(z) \in \mathbb{C}$ be a complex function in the class of Weierstrass elliptic function $\wp(z)$.

Theorem 1 [25]**.** *The sum of residues over an irreducible set of poles of an elliptic function vanishes.*

Theorem 2 [25]**.** *The sum of the locations of irreducible set of poles* (*weighted by their multiplicity*) *is congruent to the sum of locations of an irreducible set of roots* (*also weighted by their multiplicity*).

That $\frac{f^/(z)}{f(z)}$ is elliptic and thus, meromorphicis, by definition. The sum of residues at poles of $f(z)$ inside the parallelogram is zero. Using the fact that

$$\oint_C \frac{f^/(z)}{f(z)} = 0 \Rightarrow \Sigma \operatorname{res}\left(\frac{f^/}{f}\right) = 0.$$

This means that $\frac{f^/}{f}$ has poles at poles and zeros of f and nowhere else as enunciated by the Rouche's theorem in complex analysis. The number of zeros of $v(f, D) = \frac{1}{2\pi i}\int_{\partial D} \frac{f^/(z)}{f(z)} dz$ is based on argument principle.

This number of zeros of f in D counting multiplicity is also the winding number about 0 leading to argument principle. We noted that this can be accurately computed using appropriate quadrature rule along the boundary of the Contour. In particular, the function $\frac{f^/}{f}$ is $C^/$ near the boundary ∂D which is holomorphic on $D\backslash\{0\}$.

In what follows, let it be that f has zeros at $z = \hat{z}$ whose multiplicity is μ. Then $f(z) = (z - \hat{z})^{\mu} p(z)$ where p is analytic and, $p(\hat{z}) \neq 0$ so that $f(z) = \mu(z - \hat{z})^{\mu-1} p(z) + (z - \hat{z})p^/(z)$. This implies that the ratio $\frac{f^/}{f}$ is elliptic as $z \to \hat{z}$.

Following this direction, the representation of Weierstrass elliptic function earlier defined in equation (1.1) which was derived from an entire function Whittaker and Watson [25] is expressed in the form:

$$\wp(z) = \frac{1}{z^2} + \sum_{(m,n)\in\Omega}^{\infty} \left\{\frac{1}{(z + 2m\omega_1 + 2n\omega_2)^2} - \frac{1}{(2m\omega_1 + 2n\omega_2)^2}\right\}. \tag{1.4}$$

The parameters m and n appearing in equation (1.4) are integers, hence the doubly periodic Weierstrass function with fundamental periods $2\omega_1$ and $2\omega_2$ are defined as

$$\wp(z + 2\omega_1) = \wp(z); \quad \wp(z + 2\omega_2) = \wp(z).$$

The complex number z may be replaced by $z = z + \omega_1$ known as a shifted Weierstrass function and this has the representation

$$\wp(z + \omega_1) = \frac{1}{(z + \omega_1)^2} + \sum_{\omega\cup\omega^*}^{\infty} \left\{\frac{1}{(z + \omega_1 - \omega)^2} - \frac{1}{\omega^2}\right\}. \tag{1.5}$$

The congruence of two points $z_1, z^/$ is defined as $z^/ = z\, mod(2\omega_1, 2\omega_2)$. The translation of a mesh with rotation when singularity of the integers type are present at the boundary is referred to as a 'Cell' and are the values attained by an elliptic function.

Therefore, the sum of residues of $\wp(z)$ is given by

$$\frac{1}{2\pi i}\int_C f(z)dz = \frac{1}{2\pi i}\left\{\int_t^{t+2\omega_1} + \int_{t+2\omega_1}^{t+2\omega_2} + \int_{t+2\omega_2}^{t+2\omega_1+2\omega_2} + \int_{t+2\omega_2}^{t}\right\} f(z)dz. \quad (1.6)$$

Thus the difference between the number of zeros and number of poles of $f(z) - C$ for $f(z) \in \wp(z)$ which lies in a given Cell is defined by the equation

$$\frac{1}{2\pi i}\int_C \frac{f^{/}(z)}{f(z) - C}dz = \sum s_p - \sum r_p. \quad (1.7)$$

It is understood that $\sum s_p$ = number of poles; $\sum r_p$ = number of zeros.

By Liouville's theorem, we have the following well known results:

(i) If f has no poles in the interior of parallelogram, then f is constant.

(ii) If f has no poles or zeros on the boundary of parallelogram, and $\{a_i\}$ are the singular points of f in the parallelogram where f has order μ_i at a_i, then, $\sum\mu_i = 0$.

In passing, we noted that $\Omega = 2m\omega_1 + 2n\omega_2$, and the differential equation satisfied by the Weierstrass elliptic function can be expressed as

$$\wp^{/}(z) = \frac{d}{dz}\wp = -\frac{2}{z^3} - 2\sum_{\substack{(m,n)\in\Omega \\ \neq(0,0)}}\left(z - \Omega_{m,n}\right)^{-3} = -2\sum_{m,n}\left(z - \Omega_{m,n}\right)^{-3} = -\wp(z) \quad (1.8)$$

$$\wp^{/}(z + 2\omega_1) = \wp^{/}(z). \quad (1.9)$$

This is a periodic elliptic function with period $2\omega_1$. Hence, we again have that

$\int \wp^{/}(z + 2\omega_1)\,dz = \int \wp^{/}(z)dz$ and $\wp(z) + A = \wp(z + 2\omega)$ for some constant A.

It also holds that $\wp(-z) = \wp(z)$, and, that $\wp^{/}(-z) = -\wp(z)$. This means that the Weierstrass function is even whilst its first order derivative function is odd.

2. Materials and Methods

This section details introduction, theoretical foundation and in-depth knowledge of materials and procedures necessary for the research. It is expected that the reader will be able to follow the presentation equipped with the basics necessary to understanding the paper after reading this part.

The study in this paper rests principally on the modification of Weierstrass elliptic function based on the derivatives of the Laurent series expansion and the coefficients of polynomial $\wp(z)$. Its variants are the Fatou sets and Scwartzian derivatives. The duplication formula of Weierstrass elliptic function draws an important recurrence in the iteration for $\wp(z)$.

In this paper, we will give a numerical example illustrating the functional relationship between Jacobi elliptic integrals with Weierstrass elliptic functions using a unit mass particle in a cubic potential problem. The elliptic curve is an important tool for analysis that defines a surface with genus $g = \frac{1}{2}(n-s)(s-1)$, where $n < s$, and n, s are co-prime. Inspired in this direction, we introduce the Laurent series expansion for $Q(z)$ as follows:

$$Q(z) = \frac{1}{z^2} + c_1 z^2 + c_2 z^4 + \ldots + c_n z^{2n} + O(Z^3) \tag{2.1}$$

We take the Laurent series for the Weierstrass elliptic function at the point $z = 0$ [8,10,25] such that

$$\frac{1}{(z+\omega)^2} - \frac{1}{\omega^2} = \frac{1}{\omega^2}\sum_{j=1}^{\infty}(j+1)\left(\frac{-z}{\omega}\right)^j. \tag{2.2}$$

The following properties of $\wp(z)$ [10,16] hold true:

$$\wp(z) = \frac{1}{z^2} + \sum_{j=1,(m,n)\neq(0,0)}^{\infty} \frac{(j+1)(-1)^j}{(m\omega_1 + n\omega_2)^{j+2}} z^j = \frac{1}{z^2} + \sum_{j=1}^{\infty} c_j z^j \tag{2.3}$$

where the term c_j appearing in equation (2.3) is given by

$$c_j = (j+1)(-1)^j \sum_{(m,n)\neq(0,0)} \frac{1}{(2m\omega_1 + 2n\omega_2)^{j+2}}.$$

Because the fact that $\wp$ is even, it means therefore that only even indexed coefficients do not vanish. Hence, $c_{2j+1} = 0$. The following results hold true for Weierstrass function

$$\wp^{/}(Z) = -\frac{2}{z^3} + \frac{g_2}{10} z + \frac{g_3}{7} z^3 + O(z^5) \tag{2.4}$$

$$\wp^{/^2}(z) = \frac{4}{z^6} - \frac{2g_2}{5}\frac{1}{z^2} - \frac{4g_3}{7} + O(z^2) \tag{2.5}$$

$$\wp^3(z) = \frac{1}{z^6} + \frac{3g_2}{20}\frac{1}{z^2} + \frac{3g_3}{28} + O(z^2). \tag{2.6}$$

The terms g_2 and g_3 appearing above in the equations are expressed in the forms:

$$g_2 = 60 \sum_{\{m,n\}\neq\{0,0\}}^{\infty} \frac{1}{(2m\omega_1 + 2n\omega_2)^4},$$

$$g_3 = 140 \sum_{\{m,n\}\neq\{0,0\}}^{\infty} \frac{1}{(2m\omega_1 + 2n\omega_2)^6}.$$

This gives the relationship of Weierstrass differential equation and its coefficients

$$\wp^{/2}(z) = 4\wp^3(z) + g_2\wp(z) + g_3 + O(z^2). \tag{2.7}$$

We define the inverse transform of Weierstrass elliptic function $\wp$ in the senses of [19,22] as

$$z(y) = \int_y^{\infty} \frac{1}{\sqrt{4t^3 - g_2 t - g_3}} dt. \tag{2.8}$$

Verification of convergence for Weierstrass meromorphic function is a consequence of equation (1.4) by substituting $\Omega = 2m\omega_1 + 2\omega_2$. This is initiated by letting $n \geq 3$ for the elliptic function $g_n(z) = \sum_{\omega\in\Omega}^{\infty}(z-\omega)^{-n}$ of order n with respect to Ω. By further setting $|z| < R$ and $|\omega| < 2R$, it follows from [11,16] that $\left|2 - \frac{z}{\omega}\right| \leq 2 + \left|\frac{-z}{\omega}\right| \leq \frac{5}{2}, \left|\frac{z}{\omega} - 1\right|^2 \geq \frac{1}{4}$. Then, it follows that

$$\left|\frac{1}{(z-\omega)^2} - \frac{1}{\omega}\right| = \left|\frac{z(2\omega - z)}{\omega^2(z-\omega)^2}\right| = \left|\frac{\omega z\left(2 - \frac{z}{\omega}\right)}{\omega^4\left(\frac{z}{\omega^2} - 1\right)^2}\right| \leq \frac{10|z|}{|\omega|^3} \leq \frac{10R}{|\omega|^3} = \frac{CR}{|\omega|^3} \tag{2.9}$$

for a constant C.

Therefore, $g_n(z)$ converges since $\frac{1}{|z-\omega|} \leq \frac{2}{|\omega|} \Rightarrow \frac{1}{|z-\omega|^p} < \frac{2^p}{|\omega|^p}$ as explained by comparison and P-series tests.

Additionally, we expand the function in the power series

$$\left\{\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right\} = \frac{2}{\omega^2}z + \frac{3}{\omega^4}z^2 + \frac{4}{\omega^5}z^3 + \frac{5}{\omega^6}z^6 + \ldots + \frac{n+1}{\omega^{n+2}}z^{n+1};\ n = 0,1,\ldots \tag{2.10}$$

By comparing coefficients in equation (2.10) with equation (2.1), we have that:

$$c_1 = 3g_4,\ c_2 = 5g_6,\ c_n = (2n+1)g_{n+2}.$$

Thus for $n \geq 2$ the series

$$g_{2n} = g_{2n}(\omega) = \sum_{\omega \in \Omega} \frac{1}{\omega^{2n}}. \tag{2.11}$$

To synchronize the Eisenstein series g_{2n} with the Laurent Series we adopt the approaches described in [10,12] so that equations (2.10) and (2.1) are in the form:

$$\wp(z) = \frac{1}{z^2} + 3g_4 z^2 + 5g_6 z^4 + 7g_8 z^6 + \ldots + (2n+1)g_{2n+2} z^{2n} + \cdots \tag{2.12}$$

$$\wp(z)^2 = \frac{1}{z^4} + 6g_4 + 10g_6 z^2 + (9g_4^2 + 14g_8)z^4 + (30g_4 g_6 + 18g_{10})z^6 + \cdots \tag{2.13}$$

$$\wp^{/}(z) = -\frac{2}{z^3} + 6g_4 z + 20g_6 z^3 + 42g_8 z^5 + 72g_{10} z^7 + 110g_{12} z^9 + \cdots \tag{2.14}$$

$$\wp^{/}(z)^2 = \frac{4}{z^6} - 24g_4 \frac{1}{z^2} - 80g_6 + (36g_4^2 - 168g_8)z^2 + \cdots \tag{2.15}$$

$$\wp(z)^3 = \frac{1}{z^6} + 9g_4 \frac{1}{z^2} + 15g_6 + (27g_4^2 + 21g_8)z^2 + (90g_4 g_6 + 27g_{10})z^4 + \cdots \tag{2.16}$$

From the cubic equation for the Weierstrass function given as $f(x) = ax^3 + bx^2 + cx + d$ (where a, b, c, d are coefficients of $f(x)$), we are able to write that

$$\wp^{/}(z)^2 = a\wp(z)^3 + b\wp(z)^2 + c\wp(z) + d. \tag{2.17}$$

Again, multiply equation (2.16) by a and compare the principal parts of $\wp^{/}(z)^2$ and a term $a.\wp(z)^3$ in equation (2.17). By further setting $a = 4$, see, e.g., [25]; one then have that

$$\wp^{/}(z)^2 - 4\wp(z)^3 = -\big(24g_4 + 4(9g_4)\big)\frac{1}{z^2} - \big(80g_6 + 4(15g_6)\big) + h(z), \tag{2.18}$$

where $h(z)$ is analytic and vanishes at $z = 0$.

If we further multiply equation (2.12) by c and compare the principal part with equation (2.16), we have that $c = -60g_4$.

In this case,

$$\wp^{/}(z)^2 - 4\wp(z)^3 - 60g_4\wp(z) = -\big(80g_6 + 4(15g_6)\big) + h(z) \tag{2.19}$$

The $h(z)$ is holomorphic and vanishes at $z = 0$. So, $d = -140g_6$, $g_2 = 60g_4$ and $g_3 = 140g_6$.

The addition formula [10] for the single variable Weierstrass elliptic function is

$$\wp(z+u)=\frac{1}{4}\left[\frac{\wp^{/}(z)-\wp^{/}(u)}{\wp(z)-\wp(u)}\right]^{2}-\wp(z)-\wp(u). \tag{2.20}$$

We define an (n,s) curve in the sense of [1] as an algebraic curve by the equation

$$y^{n}=x^{s}+a_{s-1}x^{s-1}+a_{s-2}x^{n-2}+\ldots+a_{1}x+a_{0}. \tag{2.21}$$

The polynomial $p(x)=4x^{3}-g_{2}x-g_{3}$ can be plotted against the discriminant function $\Delta=g_{2}^{3}-27g_{3}^{2}$, where $y=\frac{1}{2}\wp^{/}(z), x=\wp(z)$. For this, we omit here.

The following relations are derived from the identity [12]:

$$4x^{3}-g_{2}x-g_{3}=(x-e_{1})(x-e_{2})(x-e_{3}).$$

Its roots are invariants in the form:

$$\begin{aligned} &e_{1}+e_{2}+e_{3}=0\\ &g_{2}=-4(e_{1}e_{2}+e_{1}e_{3}+e_{2}e_{3})\\ &g_{3}=4e_{1}e_{2}e_{3}. \end{aligned} \tag{2.22}$$

The Schwartzian derivative [16] of $\wp$ at z is described by the equation

$$S_{\wp}(z)=\frac{\wp^{///}(z)}{\wp^{/}(z)}-\frac{3}{2}\left(\frac{\wp^{//}(z)}{\wp^{/}(z)}\right)^{2}. \tag{2.23}$$

The duplication formula [9,25] which relates the Fatou set with the Weierstrass function is expressed in the form

$$\wp_{k+1}=-2p_{k}+\frac{\left(6\wp_{k}^{2}-\frac{1}{2}g_{2}\right)^{2}}{4\left(4\wp_{k}^{2}-g_{2}\wp_{k}-g_{3}\right)};\ \ k=0,1,2,\ldots \tag{2.24}$$

$z_{0}=\frac{z}{2^{k}};\ \wp_{k}=\wp_{k}(z)$ - complex, $k=0,1,2,\ldots$

$$\wp_{0}=\frac{1}{z_{0}^{2}}+\frac{g_{2}}{20}z_{0}^{2}+\frac{g_{3}}{28}z_{0}^{4} \tag{2.25}$$

$$\wp_{1}=-2\wp_{0}+\frac{\left(6\wp_{0}^{2}-\frac{1}{2}g_{2}\right)^{2}}{4\left(4\wp_{0}^{2}-g_{2}\wp_{0}-g_{3}\right)}. \tag{2.26}$$

In the limit, as $k\to\infty$, the term $\wp_{k}(z)\to\wp(z)$. The error estimate occurring in the computation is

$$E_{k}=\frac{|g_{2}|^{2}}{2450*2^{8k}}|z|^{9}\left|\wp^{/}(z)\right|;\quad \text{with } \left|\wp^{/}\right|=\left|(4\wp^{3}-g_{2}\wp-g_{3})^{\frac{1}{2}}\right|.$$

3. Results

This section gives a detailed presentation of our findings. We will present our numerical results in the form of tables.

3.1. The functional relationships between Weierstrass elliptic functions and Jacobi elliptic integrals

As a follow up to the discussion we recollect that

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Omega \setminus \{0,0\}} \left\{ \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right\}$$

has two poles and two zeros per cell which has been analyzed in the previous section. It is meromorphic, periodic with respect to Ω of order 2.

In the realm of further exposition of $\wp(z)$ to areas of applications, we relate these properties given below as follows:

$$\wp^{/}(z) = -2 \sum_{\omega \in \Omega \setminus \{0,0\}} \frac{1}{(z-\omega)^2}; \tag{3.1}$$

$$\wp^{/}(z)^2 = 4\wp(z)^2 - g_2\wp(z) - g_3 \tag{3.2}$$

$$g_2(\Omega) = 60 \sum_{\omega \in \Omega \setminus \{0\}} \omega^{-3}; \;\; g_3 = 140 \sum_{\omega \in \Omega \setminus \{0\}} \omega^{-6}. \tag{3.3}$$

The zeros of $\wp^{/}(z)^2 = 4\wp(z)^2 - g_2\wp(z) - g_3$ are e_1, e_2, e_3 distinct values.

The quarter period of Weierstrass elliptic functions $h_i^2 = (e_i - e_j)(e_i - e_k) = 3e_i^2 - \frac{g_2}{4}$ is defined as $\wp_\Omega\left(\frac{\omega_i}{4}\right) = e_i + h_i$. By a well-known theorem for a real rectangular lattice Ω, then $e_1 + h_1 > e_1 > e_3 > e_1 - h_1 > e_2$.

Therefore, the addition formula [10] for n-variable u complex numbers for the Weierstrass elliptic function as compared to equation (2.20) is given by

$$(-1)^{\frac{(k-1)(k-2)}{2}} \sigma\left(\sum_{k=1}^{n} \mu^{(k)} \prod_{k<j} \frac{\delta(\mu^{(k)} - \mu^{(j)})}{\Pi_k \delta(\mu^{(k)})^n}\right) = \prod_{k=1}^{n-1} \begin{vmatrix} 1 & \wp(\mu^{(1)}) & \wp^{/}(\mu^{(1)}) & \dots & \wp^{n-1}(\mu^{(1)}) \\ 1 & \wp(\mu^{(2)}) & \wp^{/}(\mu^{(2)}) & \dots & \wp^{n-1}(\mu^{(2)}) \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 1 & \wp(\mu^{(n)}) & \wp^{/}(\mu^{(n)}) & \dots & \wp^{n-1}(\mu^{(n)}) \end{vmatrix} \tag{3.4}$$

The Jacobi theta function [25] is defined by the equation

$$g(z,q)=\sum_{j=-\infty}^{\infty}(-1)^j q^{j^2}e^{2ijz} \tag{3.5}$$

where for instance, $q=e^{\pi i\tau}$ and $|q|<1$.

By defining a positive constant B, then we have that

$$\left|q^{j^2}e^{2jiz}\right|\le |q|^{j^2}e^{2jB}; \qquad j=1,2,\ldots. \tag{3.6}$$

The series $g(z,q)$ is convergent and analytic by the Ratio test and uniformly continuous and bounded for all $|z|<B$.

Therefore, we write that

$$\sum_{j=-\infty}^{\infty}(-1)^j q^{j^2}e^{2ijz}=1+\sum_{1}^{\infty}(-1)^j q^{j^2}e^{2ijz}+\sum_{j=-\infty}^{-1}(-1)^j q^{j^2}e^{2ijz}$$

$$=1+\sum_{j=1}^{\infty}(-1)^j\left[q^{j^2}(e^{2ijz}+e^{-2ijz})\right]$$

$$=1+2\sum_{j=1}^{\infty}(-1)^j q^{j^2}\cos 2jz. \tag{3.7}$$

Hence it holds that

$$g(z,q)=g(z+\pi,q). \tag{3.8}$$

Similarly, writing $g(z+\tau\pi,q)=\sum_{j=-\infty}^{\infty}(-1)^j q^{j^2}e^{(z+\tau\pi)2ji}$, where $q=e^{\pi i\tau}$, then we have that

$$g(z+\tau\pi,q)=\sum_{j=-\infty}^{\infty}(-1)^j q^{j^2}q^{2j}e^{2jiz}=\sum_{j=-\infty}^{\infty}(-1)^j q^{(j^2+2j+1)}q^{-1}e^{2jiz}. \tag{3.9}$$

Continuing, after some analysis, it follows that

$$g(z+\tau\pi,q)=-q^{-1}e^{-2iz}\sum_{j=-\infty}^{\infty}(-1)^{j+1}q^{(j+1)^2}e^{2iz(j+1)}=-q^{-1}e^{-2iz}g(z,q). \tag{3.10}$$

Thus equations (3.7) and (3.10) are doubly periodic with a multiplier for equation (3.7) and a term $-q^{-1}e^{-2iz}$ for equation (3.10).

By writing $g_4(z,q)$ as $g(z,q)$, the other three theta functions of Jacobi are given in the form:

$$g_3(z,q) = g(z+\frac{\pi}{2},q) = 1 + 2\sum_{j=1}^{\infty} q^{j^2} \cos 2jz \tag{3.11}$$

$$g_1(z,q) = -ie^{\left(iz+\frac{\pi i\tau}{4}\right)} g(z+\frac{\pi\tau}{2},q) = 2\sum_{j=0}^{\infty}(-1)^j q^{\left(j+\frac{1}{2}\right)^2} \sin(2j+1)z \tag{3.12}$$

$$g_2(z,q) = g_1(z+\frac{\pi}{2},q) = 2\sum_{j=0}^{\infty} q^{\left(j+\frac{1}{2}\right)^2} \cos(2j+1)z \tag{3.13}$$

Because of definition of q and since $z = u + iv$, then using ideas of [22] we have that

$$q^{2j-1} = e^{-(2j-1)\pi v}[\cos(2j-1)\,\pi u + i\sin(2j-1)\,\pi u].$$

Wherefrom, in particular, the last expression given above, we have that:

$$Re(-ig_1(z,q)) = -\sum_{j=1}^{\infty}(1)^j q^{(j-1/2)^2}[e^{(2j-1)jv} - e^{-(2j-1)jv}]\cos(2j-1)\,\pi u,$$

$$Im(-ig_1(z,q)) = \sum_{j=1}^{\infty}(-1)^j q^{(j-1/2)^2}\left[e^{(2j-1)\pi v} + e^{-(2j-1)^{\pi v}}\right]\sin(2j-1)\,\pi u.$$

Similarly,

$$\mathrm{Re}\big(g_2(z,q)\big) = \sum_{j=1}^{\infty} q^{(j-1/2)^2}\left[e^{(2j-1)\pi v} - e^{-(2j-1)\pi v}\right]\cos(2j-1)\,\pi u,$$

$$Im\big(g_2(z,q)\big) = -\sum_{j=1}^{\infty} q^{(j-1/2)^2}\left[e^{(2j-1)jv} - e^{-(2j-1)jv}\right]\sin(2j-1)\,\pi u.$$

Thus from the above analysis, it holds that $g_2(z,q), g_3(z,q)$ and $g_4(z,q)$ are all even functions whilst $g_1(z,q)$ is an odd function.

The differential equations satisfied by the Jacobi elliptic functions in terms of theta functions are described [25] in the form:

$$\frac{d}{dz}\left[\frac{g_1(z,q)}{g_4(z,q)}\right] = [g_4(0,q)]^2 . \frac{g_2(z,q)g_3(z,q)}{g_4(z,q)g_4(z,q)}. \tag{3.14}$$

Thus if

$$\eta = \frac{g_1(z,q)}{g_4(z,q)}, \tag{3.15}$$

then, we have that

$$\left(\frac{d\eta}{dz}\right)^2 = [g_2(0,q) - \eta^2 g_3(0,q)][g_3^2(0,q) - \eta^2 g_2^2(0,q)]. \tag{3.16}$$

If we further set as

$$y = \frac{g_3(0,q)}{g_2(0,q)}\eta, \tag{3.17}$$

$$x = z(g_3(0,q))^2, \tag{3.18}$$

$$\left(\frac{dy}{dx}\right)^2 = (1-y^2)(1-k^2y), \tag{3.19}$$

$$k^{\frac{1}{2}} = \frac{g_2(0,q)}{g_3(0,q)}, \tag{3.20}$$

then, as in Weierstrass elliptic function we have that:

$$x = \int_0^y \frac{du}{(1-u^2)^{\frac{1}{2}}(1-r^2u^2)^{\frac{1}{2}}} = sn^{-1}y, \quad (r \in (0,1)). \tag{3.21}$$

Equation (3.21) defines Jacobi elliptic sine function $y = snx$ which is a uniformly convergent series for equations (3.18), (3.20) and (3.21).

The values for Jacobi cosine cn and dn are given by the equations [23]:

$$\frac{dy}{dx} = \frac{d}{dx}snx = \frac{1}{dx/dy} = \sqrt{(1-y^2)(1-r^2y^2)} = \sqrt{1-sn^2x}.\sqrt{1-r^2sn^2} = cnxdnx \quad (3.21)$$

$$\frac{d}{dx}cnx = \frac{d}{dn}\sqrt{1-sn^2} = \frac{1}{2}(1-sn^2x)^{-\frac{1}{2}}\frac{d}{dn}(-sn^2x) = sndnx \tag{3.22}$$

$$\frac{d}{dx}(dnx) = \frac{d}{dx}\sqrt{1-r^2sn^2} = \frac{1}{2}(1-r^2sn^2)^{-\frac{1}{2}}.(-2r^2(snxcnxdnx)) = -r^2snxcnx. \tag{3.23}$$

The observation here is that sn is an odd function of x whilst $cn(-x) = cn(x)$ and $dn(-x) = dn(x)$.

We now give the periods of Jacobi elliptic function of the First and Second kind integrals

$$F(k) = K = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-r^2t^2)}} = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-r^2 \sin^2\theta}} \qquad (\text{where } t = \sin\theta) \quad (3.24)$$

$$E(k) = K^{/} = \int_0^1 \frac{dt}{\sqrt{(1-t^2)\left(1-r^{/^2}t^2\right)}} = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-r^{/^2}\sin^2\theta}}, \qquad t = \sin\theta. \quad (3.25)$$

The parameters K and $K^{/}$ are known as modulus and complementary modulus respectively and $K^{/} = \sqrt{1-r^2}, r \in (0,1)$.

The periods of sn are $4K$ and $2iK^{/}$ while that of dnx are $2K$ and $4iK^{/}$.

Thus equation (3.24) in the form hypergeometric function is

$$K = \frac{\pi}{2} 2F_1\left(\frac{1}{2}, \frac{1}{2}, 1, r^2\right). \quad (3.26)$$

3.2. The half periods of Jacobi and Weierstrass elliptic functions with their bounds

We give the half-period functions by the scaling equivalent of the Jacobi class.

Setting as:

$$u = \sqrt{e_1 - e_3}x, \qquad k^2 = \frac{e_2 - e_3}{e_1 - e_3}, \quad (3.27)$$

then the Jacobi sine integral and its allied functions are expressed in the forms:

$$sn(u,k) = \frac{\sqrt{e_1 - e_3}}{W_3(x)}; \;\; cn(u,k) = \frac{W_1(x)}{W_3(x)}; \;\; dn(u,k) = \frac{W_2(x)}{W_3(x)}. \quad (3.28)$$

An alternative approach for representing equation (3.2) is expressed in the form

$$sn(u,k) = \frac{\sqrt{e_1 - e_3}}{\sqrt{\wp(x) - e_3}}, \; cn(u,k) = \sqrt{\frac{\wp(x) - e_1}{\wp(x) - e_3}}, \; dn(u,k) = \sqrt{\frac{\wp(x) - e_2}{\wp(x) - e_3}}. \quad (3.29)$$

These functions are periodic with periods $4K$ and $4iK$ whose argument is u. K and $K^{/}$ are the real and imaginary quarter periods as described earlier above.

One major difference between Jacobi and Weierstrass elliptic functions is that Jacobi has two simple poles per cell and is considered as a solution to the differential equation

$$\frac{d^2x}{dt^2} = A + Bx + Cx^2 + Dx^3. \tag{3.30}$$

On the other hand, the Weierstrass elliptic function has one double pole and is a solution to

$$\frac{d^2x}{dt^2} = A + Bx + Cx^2. \tag{3.31}$$

After all these given earlier and due to definition of $\wp\left(\frac{\omega}{2}\right)$, and since $g_j(z) = g_j(z,\tau)$ is periodic it follows from [22] that

$$\wp(z) = e_1 + \left(\frac{g_1^{/}(0)}{g_1(z)}.\frac{g_2(z)}{g_2(0)}\right)^2 = e_2 + \left(\frac{g_1^{/}(0)}{g_1(z)}.\frac{g_3(z)}{g_3(0)}\right)^2 = e_3 + \left(\frac{g_1^{/}(0)}{g_1(z)}.\frac{g_4(z)}{g_4(0)}\right)^2. \tag{3.32}$$

Furthermore, we express the difference between two roots in the form:

$$e_1 - e_2 = \left(\frac{g_1^{/}(0)g_4(0)}{g_2(0)g_3(0)}\right)^2 = \pi^2 g_4(0)^4.$$

$$e_1 - e_3 = \left(\frac{g_1^{/}(0)g_3(0)}{g_2(0)g_4(0)}\right)^2 = \pi^2 g_3(0)^4.$$

$$e_2 - e_3 = \left(\frac{g_1^{/}(0)g_2(0)}{g_3(0)g_4(0)}\right)^2 = \pi^2 g_2(0)^4. \tag{3.33}$$

Evaluating the second identity in equation (3.16) at the point $z = \frac{1}{2}$ implies that $e_1 - e_2 = \left(\frac{g_1^{/}(0)}{g_1\left(\frac{1}{2}\right)}; \frac{g_3\left(\frac{1}{2}\right)}{g_3(0)}\right)^2$. Thus $g_1\left(\frac{1}{2}\right), g_3\left(\frac{1}{2}\right)$, can be checked from the Table of Jacobi elliptic function.

3.3. The role of Gamma and Hypergeometric functions

The main roles of Gamma and Hypergeometric functions to the elliptic Jacobi integrals are explained as follows:

Fact 1.

F is the classical hypergeometric function defined in the form

$$\Gamma(a,b,c,x) = 2F_1(a,b,c,x) = \sum_{n=0}^{\infty} \frac{(a,n)(b,n)}{(c,n)n!} x^n, \quad |x < 1| \tag{3.34}$$

where,

$$(a,n) = (a)_n = a(a+1).(a+2)\ldots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)} \qquad (3.35)$$

is the Pochammer symbol or shifted factorial, $(a,0) = 1; a \neq 0$. Note that the Euler's Gamma function written $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$ is a well-known fact for analysis in this direction.

Following [5,13] the generalized hypergeometric function is

$${}_mF_n(a_1, a_2, \ldots, a_m; b_n, x) = \sum_{k=0}^{\infty} T(k) \qquad (3.36)$$

where, $T(k) = \frac{(a_1)_k \ldots (a_m)_k}{(b_1)_k \ldots (b_n)_k k!} x^k$.

A much stronger version of hypergeometric functions can be found [21,26].

In passing, we noted that the asymptotic Sterling series for hypergeometric series is the expression

$$\log\Gamma(k) = \left(k - \frac{1}{2}\right)\log(k) - k + \log\left(\frac{2\pi}{2}\right) + \sum_{n=1}^{N-1} \frac{B_{2n}}{2n(2n-1)k^{2n-1}} + R(n,k) \quad (3.37)$$

where

$$\Gamma(k)\Gamma(1-k) = \frac{\pi}{\sin(\pi k)},$$

B_{2k} is the Bernoulli numbers and is given by the equation

$$B_{2k} = (-1)^{k+1} 2(2k)!\, \zeta(2k)(2\pi); \qquad (3.38)$$

$\zeta(2k) = \sum_{j=1}^{\infty} j^{-2k}$, and $j^{-2k} = j^2 . j^{-(2k+2)}$.

Theorem [3,24]. *Let* $K = K(r) = \int_0^{\frac{\pi}{2}} (1 - r^2 \sin^2 t)^{\frac{-1}{2}} dt, K^{/} = K^{/}(r) = K(r^{/})$;

$$\zeta = \zeta(r) = \int_0^{\frac{\pi}{2}} (1 - r^2 \sin^2 t)^{\frac{1}{2}} dt; \quad \zeta^{/} = \zeta^{/}(r) = \zeta(r^{/})$$

where $r^{/} = \sqrt{1-r^2}, K(0) = \zeta(0) = \frac{\pi}{2}; K(1) = \infty, \zeta(1) = 1$.

Then

(i) $For\, r \in (0,1)$,

$$r'^2 = \frac{r'\, exp(K(r)) - 4}{exp\left(\frac{\pi}{2}\right) - 4} < 2\frac{\sqrt{1-r}}{2-r} \tag{3.39}$$

(ii) $For\, r \in (0,1)$

$$K(r) < log\left(1 + \left(\frac{4}{r'}\right)\right) - \left(log\, 5 - \frac{\pi}{2}\right)(1-r). \tag{3.40}$$

Then the ratio of $\frac{\zeta'(r)[K(r)-\zeta(r)]}{\left[r^2\left(2\, log\left(\frac{4}{r'}\right)-1\right)\right]}$ increases from $(0,1)$ onto $\frac{\pi}{\left[(4\, log\, 16-1), \frac{\pi}{4}\right]}$, it follows that the function $\frac{[\zeta(r)\zeta'(r)-K(r)\zeta'(r)+r^2K(r)K'(r)]}{r^2K'(r)}$ decreases from $(0,1)$ onto $\left(1, \frac{\pi}{2}\right)$. Besides, we also mentioned that because of [20, Theorem 2.8] the function defined by$f(r) = K'(r) + log\left(\frac{r}{1+r}\right)$ is a decreasing function from $(0,1)$ onto $\left(\frac{\pi}{2} - log\, 2, log\, 4\right)$. In addition. it holds that the following inequality is valid

$$\frac{\pi}{2} - log\, 2 + log\left(1 + \frac{1}{r}\right) < K'(r) < \frac{\pi}{2} - log\, 2 + \left(log\, 8 - \frac{\pi}{2}\right)(1-r) + log\left(1 + \frac{1}{r}\right),$$

$$\text{for } r \in (0,1). \tag{3.41}$$

Using [20] we may compute possible upper bounds for the Jacobi elliptic integral in the form:

Problem 1.

$$\frac{log\left(1 + \frac{4}{\sqrt{1-r^2}}\right) - \left(log\, 5 - \frac{\pi}{5}\right) + \left(\frac{\pi}{8} - \frac{2}{5}\right)r^2 + \alpha^* r^4}{log\left(1 + \frac{4}{\sqrt{1-r^2}}\right) - \left(log\, 5 - \frac{\pi}{5}\right) + \left(\frac{\pi}{8} - \frac{2}{5}\right)r^2 + \beta * r^4}; \quad r \in (0,1) \tag{3.42}$$

Where $\alpha^* = \frac{9\pi}{128} - \frac{11}{50}$; $\beta^* = \frac{2}{5} + log\, 5 - \frac{5\pi}{8}$.

Fact 2 [20].

Similarly, by further setting as:

$$\theta = \pi \frac{(17-5\pi)}{32} = 0.126845,$$

$$\lambda = \frac{8}{5} - log\, 4 = 0.213705,$$

$$\alpha = \frac{85}{8}\pi - \frac{185}{32}\pi^2 + \frac{25}{32}\pi^3 = 0.544425,$$

$$\beta = (8 - 10\,log\,2)\pi - \frac{85}{32}\pi^2 + \frac{25}{32}\pi^3 = 1.364397,$$

$$\delta = \frac{128}{5} - 32\,log\,2 - \frac{17}{2}\pi + \frac{5\pi^2}{2} = 1.3897,$$

$$\xi = \frac{128}{5} + 32\,log\,2 + \left(\frac{47}{8} - 10\,log\,2\right)\pi + \frac{5}{8}\pi^2 = 0.569791,$$

and letting:

$$f(r) = \frac{\pi}{2}[16 - 5\,log(1 - r^2)](\theta r^2 + K(r))(16 + (5\pi - 16)r^2), \tag{3.43}$$

and

$$g(r) = [\lambda r^2 + K(r)][16 + (5\pi - 16)r^2] - \frac{\pi}{2}[16 - 5\,log(1 - r^2)], \tag{3.44}$$

we define $\frac{f(r)}{g(r)}$ as a rational function for which the function is either monotone increasing or monotone decreasing. We take $r \in \left[\frac{1}{4}, \frac{1}{2}\right]$. Other similar bound for equation (3.21) expressed in the form $K(r) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - r^2 \sin^2\theta}}$ is

$$K(r) = log\left(1 + \frac{4}{\sqrt{1 - r^2}}\right) - \left[log\,5 - \frac{\pi}{2}\right](1 - r). \tag{3.45}$$

The asymptotic formulas [26] for the Jacobi elliptic functions are

$$F\left(\frac{1}{2}, \frac{1}{2}, 1, x\right) = \frac{ln(16/t)}{\pi} + \frac{t}{4\pi}(ln(16/t) - 2) + O(t^2\,ln\,t), \tag{3.46}$$

and

$$F\left(\frac{1}{2}, \frac{1}{2}, 2, x\right) = \frac{4}{\pi} - \frac{t}{\pi}(ln(16/t) - 3) + O(t^2\,ln\,t) \tag{3.47}$$

as $t \to 0^+$, for $t = 1 - x$.

In the limit it can be derived that

$$1 + \left(\frac{3\pi}{8} - 1\right)r'^2 < \frac{K(r)}{ln\left(\frac{e^{4/3}}{r'}\right)} < \frac{21\pi}{64} + \frac{3\pi}{64}r'^2;\ \ r \in (0,1). \tag{3.48}$$

The concavity of Jacobi elliptic function is expressed by the equation $r \to Q_1(x) = \frac{K(r)}{ln\left(\frac{c}{\sqrt{1-x}}\right)}$ which is strictly concave on $(0,1)$ if and only if $c = e^{4/3}$.

Besides the above ideas, we introduce also the bounds obtained for the Jacobi elliptic integrals using inverse hyperbolic function in the form:

[5]: $\frac{\pi}{2}\left(\frac{arthr}{r}\right)^{\frac{1}{2}} < K(r) < \frac{\pi}{2}\left(\frac{arthr}{r}\right)$;

[4]: $\frac{\pi}{2}\left(\frac{arthr}{r}\right)^{\alpha} < K(r) < \frac{\pi}{2}\left(\frac{arthr}{r}\right)^{\beta}; r \in (0,1), \alpha = \frac{3}{4}, \beta = 1$

[7]: $\frac{\pi}{2}\left(\frac{arthr}{r}\right)^{\frac{3}{4}+\alpha^* r} < K(r) < \frac{\pi}{2}\left(\frac{arthr}{r}\right)^{\frac{3}{4}+\beta^* r}$, where for optimal values, $\alpha^* = 0, \beta = \frac{1}{4}$, $r \in (0,1)$ hold good.

However, we noted that the term $(1 + r^2 \sin^2 \theta)^{\frac{1}{2}}$ can be approximated [15] by the Taylor Series expansion given as

$$(1 + r^2 \sin^2 \theta)^{\frac{1}{2}} = 1 + \frac{1}{2}r^2 \sin^2 \theta - \frac{1}{8}r^4 \sin^4 \theta + \frac{r^6}{16}\sin^6 \theta, \tag{3.49}$$

where $\sin^2 \theta = \frac{1}{2}(1 - \cos 2\theta)$, etc.

If we substitute these values into equation (3.49) for $r \in [0,1]$ we then have that

$$(1 + r^2 \sin^2)^{\frac{1}{2}} = \left[1 + \frac{r^2}{4} - \frac{3}{64}r^4 + \ldots\right] + \left[-\frac{1}{4}r^2 + \frac{1}{16}r^4 + \ldots\right]\cos\theta - \frac{r^4}{64}\cos 4\theta + \ldots \tag{3.50}$$

3.4. Numerical example

The Jacobi elliptic integrals of first and second kind cannot be evaluated in exact form as they can only be evaluated in the form of elementary functions.

We integrate equations (3.24) and (3.25) for the Jacobi elliptic integrals using Konrod-Gauss quadrature method. The Konrod-Gauss quadrature transforms the given range of integration from $[0, \frac{\pi}{2}]$ to $[-1,1]$ while carrying out the operations.

In the tabular forms, we presented results in Tables 1-3 representing the approximate values of the Jacobi elliptic integrals of first and second kinds, computed using the Konrod–Gauss quadrature method. The range of integration is from 0 to $\frac{\pi}{2}$ while the modulus k varies from 0to 1 using a step length of 0.1. Table 1 below shows results computed for the respective modulus k and complementary modulus $k^/$ in the Jacobi elliptic integrals of the first and second kinds.

Table 1. It shows the Jacobi elliptic integral computed for the first and second kinds with modulus$k$ from Equations (3.24 and 3.25), range of integration is $[0,\frac{\pi}{2}]$.

k	$F(k)$	$E(k)$	$k^{/}$	$F(k^{/})$	$E(k^{/})$
0.0	1.5708	1.5708	1.0	1.5708	1.5708
0.1	1.5583	1.5398	0.9949	1.5583	1.5398
0.2	1.5346	1.4963	0.9802	1.5346	1.4963
0.3	1.5006	1.4414	0.9550	1.5006	1.4414
0.4	1.4571	1.3774	0.9189	1.4571	1.3774
0.5	1.4047	1.3050	0.8716	1.4047	1.3050
0.6	1.3436	1.2244	0.8127	1.3436	1.2244
0.7	1.2745	1.1353	0.7420	1.2745	1.1353
0.8	1.1980	1.0372	0.6593	1.1980	1.0372
0.9	1.1154	0.9293	0.5642	1.1154	0.9293
1.0	1.0284	0.8111	0.4560	1.0284	0.8111

Table 2. Computed results for the inverse transform of Weierstrass elliptic function from Equation (2.8) using the Konrod–Gauss method equation (2.8) and Inverse Jacobi elliptic integral of first and second kinds using Runge-Kutta method of fourth order from Equation (3.24 and 3.25). The range of integration is from 0 to $\frac{\pi}{2}$.

Step length k	The inverse transform of Weierstrass elliptic function using Konrod-Gauss method equation (2.8)		Inverse Jacobi elliptic integral of first and second kinds using Runge-Kutta method of fourth order method equation (3.24 and 3.25)	
	y	$sn(y,k)$	$F\left(\frac{\pi}{2},k\right)$	$E\left(\frac{\pi}{2},k\right)$
0.0	0.0000	0.000000	1.5708	1.5708
0.1	0.0115750	0.011575	1.5708	1.5625
0.2	0.024467	0.024467	1.5708	1.5492
0.3	0.039042	0.039042	1.5708	1.5315
0.4	0.055745	0.055745	1.5708	1.5102
0.5	0.075013	0.075013	1.5708	1.4863
0.6	0.097335	0.097335	1.5708	1.4613
0.7	0.123263	0.123263	1.5708	1.4370
0.8	0.153430	0.153430	1.5708	1.4150
0.9	0.188593	0.188593	1.5708	1.3971
1.0	0.229848	0.229848	1.5708	1.3856

We compute the inverse Jacobi elliptic integral for different values of modulus \(k\) denoted as \(\text{{sn}^{-1}(x,k)\) and its inverse function given by \(\text{{sn}}(y,k)\) where \(x\) and \(y\) are related as

\[x = \int_ 0^ y \frac{{dt}}{{\sqrt{1 - k^2(t)}}}\].

Then we use the Konrod Gauss quadrature method to approximate the values of \(\text{{sn}}(y,k)\) for different values of \(k\) by dividing the interval \([0,\frac{\pi}{2}]\) into smaller subintervals and applying numerical integration.

Table 3. Iterated Fatou set for the Weirstrass Function in Equation (2.24); values of g2=1,g3=1,step size=0.1 for 10 iterations.

k	z_0	p_0	p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8	p_9	p_{10}
0	0.5	8.5	3.9616	3.3683	3.2873	3.2759	3.2743	3.2743	3.2741	3.2741	3.2741	3.2741
1	0.25	34.5	2.1681	1.3375	1.0065	0.9218	0.9016	0.9016	0.8971	0.8960	0.8959	0.8959
2	0.125	138.5	1.3046	0.5652	0.2569	0.1663	0.1465	0.1465	0.1423	0.1413	0.1413	0.1413
3	0.0625	554.5	0.6863	0.1397	0.0256	0.0087	0.0053	0.0053	0.0047	0.0046	0.0046	0.0046
4	0.0312	2218.5	0.3103	0.0192	0.0004	0.0001	0.0000	0.0001	0.0001	0.0001	0.0001	0.0001
5	0.0156	8874.5	0.1404	0.0020	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
6	0.0078	35502.5	0.0632	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
7	0.0039	142010	0.0282	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
8	0.0019	568042	0.0127	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
9	0.0010	2272190	0.0056	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
10	0.0005	9088762	0.0024	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Table 3 showed the numerical values computed with Coquereaux and Lautrup iterative method [9] from Equation (2.24) for the Weierstrass elliptic function. The method depends mainly on the initial values of $z_0, \wp_0, g$ and g_3. The value of $\wp_k(z)$ vanishes as $k \to \infty$ and it exists for $\wp_0(z)$ as $k \to \infty$.

3.5. The analysis of a unit mass particle under cubic potential

We consider the numerical example taken from [6]:

$$V(x) = \frac{3}{2}x - 2x^3. \tag{3.51}$$

The turning points are $x_1 = 0, x_2 = -\frac{\sqrt{3}}{2}, x_3 = \frac{\sqrt{3}}{2}$. The first part of the potential describes the linear potential of the velocity while the second part describes the small deviation of an harmonicity of the potential. We noted that the cubic term does not give information on the potential function as symmetric about the ordinate axis [14]. We will describe [19] the point at which the cubic potential well escapes to infinity. We also will give the attractive region for the well potential, the unstable region as well. We do know that the particle trapped in the attractive region does move with an oscillatory motion with asymmetric amplitude.

The motion of a unit particle expressed in equation (3.51) following closely [6] has a point of inflexion at 0, minimum at $x=-\frac{1}{2}$, and $V(-1/2)=6$. Again for $x=\frac{1}{2}$, the minimum $V\left(\frac{1}{2}\right)=\frac{1}{2}$, maximum $V^{//}\left(\frac{1}{2}\right)=-6$.

Let E be the total energy of motion of a unit mass of the particle, we then express the given problem in the form Weierstrass elliptic phase

$$\dot{x}^2 = 2E - 3x + 4x^3 = 4x^3 - g_2 x - g_3 \tag{3.52}$$

where $g_2=3, g_3=-2E$. The $\dot{x}=\frac{dx}{dt}$ is the velocity of particle. The total energy $E=-\frac{1}{2}\cos\vartheta$ in the form Weierstrass phase is continuous along the path in the complex ϑ plane.

The turning points are the roots of the cubic equation and they give the energy solution. That is $E=-\frac{1}{2}\cos\theta=V(x)$. Thus the differential equation satisfied by the Weierstrass elliptic function is $\left(y^{/}\right)^2=4x^3-3g_2-g_3$. The roots are expressed as$x_1(\theta)=\cos\left(\frac{\theta}{3}\right)$, $x_2(\theta)=-\cos\left(\left(\frac{\pi+\theta}{3}\right)\right)$, $x_3(\theta)=-\cos\left(\left(\frac{\pi-\theta}{3}\right)\right)$. The $x_1+x_2+x_3=0$. Hence this is expressed as $\dot{x}^2=4(x-x_1)(x-x_2)(x-x_3)$. The discriminant $\Delta$is given by $\Delta=g_2^3-27g_3^2$. For the $g_2=3, g_3=\cos\theta$. After a little simplification of algebra gives that

$$\Delta = 27\sin^2\theta = \begin{cases} -27\sinh^2\alpha \le 0\left(|E| \ge \dfrac{1}{2}\right) \\ 27\sin^2\alpha \ge 0\left(|E| \le \dfrac{1}{2}\right) \end{cases}. \tag{3.53}$$

The orbits of the moving particle are in three parts, namely, region I, region II, and region III. Particularly the orbits in region III described by the particle is $\bar{x}(t)=p(t+\omega_3^{III}, g_2, g_3)=p(t-i\omega, g_2, g_3)=-p(it+\omega, g_2, |g|_3)$. Continuing after a while, one obtains the bounded orbit solution for cubic well potential function [6] as

$$\bar{x}_{III}(0) = -1 + \frac{3}{2}\coth^2\left(-i\frac{\pi}{2}\right) = -1 \text{ as } t\to\infty,\ \bar{x}_{III}(t) \to -1+\frac{3}{2}=\frac{1}{2}.$$

The unbounded separatrix solution is

$$\hat{x}_{III}(t) = -1 + \frac{3}{2}\coth^2\left(\frac{\sqrt{3}}{2}t\right), \hat{x}_{III}(0)=\infty; \hat{x}_{III}(\infty) = -1+\frac{3}{2}=\frac{1}{2}.$$

The time of flight for scattering solutions and period of the oscillating solutions in the sense of [19] are linked by the distance between two consecutive poles on the real axis $T_{scattering} = 2\omega_1$, and period of oscillating solutions is the distance between two consecutive appearances of the same root $T_{oscillating} = 2\omega_1$.

But the position of the particle has to be real [19] this makes the unbounded motion of the point particle moving in a cubic potential V given by

$$y = \wp(i(x - x_0); -g_2, -E). \tag{3.54}$$

The bounded one is given by

$$y = \wp(i(x - x_0) + \omega_1; -g_2, -E).$$

Hence the time of flight of the unbounded motion of a particle of a unit mass in a cubic potential is guided by the imaginary period of $\wp(z, -g_2, -E)$ implying that $T_{scattering} = T_{oscillating} = -2i\omega_2$.

Furthermore, the Jacobi elliptic functions $cn(z|k), sn(z|k)$, and $dn(z|k)$ are linked to the cubic potential function [6]. However, we give the bounded and unbounded solutions here. By substituting $\bar{x}(t) = x_3 + \alpha y^2(kt)$ into $(y^/)^2 = 4x^3 - g_2 x - g_3$ in equation (2.7) gives $\alpha k^2 (y^/)^2 = [(x_1 - x_3) - \alpha y^2][(x_2 - x_3) - \alpha y^2]$. Continuing after a while, one then obtains the bounded separatrix solution

$$\bar{x}(t) = -1 + \frac{3}{2} tanh^2\left(\frac{\sqrt{3}}{2} t\right) \tag{3.55}$$

while the unbounded solution where the particle entering the escape region with possibly increasing speed away from the origin is expressed in the form

$$\hat{x}(t) = \frac{1}{2} + \frac{3}{2} \frac{sec\,h^2\left(\frac{\sqrt{3}}{2} t\right)}{2\,tanh^2\left(\frac{\sqrt{3}}{2} t\right)} = -1 + \frac{3}{2} coth^2\left(\frac{\sqrt{3}}{2} t\right), \quad (0 \le t \le 2\pi). \tag{3.56}$$

The above hold in the third quadrant for the roots (x_1, x_2, x_3), for a unit mass particle of a motion governed by the equation (3.51).

We demonstrate below with practical realization the use of Jacobi elliptic integral of first kind $F(\vartheta|k)$ with Konrod–Gauss quadraturemethod to compute the Weierstrass phase function given by $g(x) = \int_0^{\frac{\pi}{2}} \sqrt{(V(x) - E)}\, dx$, E is the energy of the system, ϑ is the amplitude and k is the modulus. Letting $E = 0$ for the sake of simplicity we have that

$$g(x) = \int_0^{\frac{\pi}{2}} \sqrt{\left(\left(\frac{3}{2}\right)x - 2x^3\right)}\,dx.$$

The discriminant for the Weierstrass function is $\Delta = 4\wp^3 + 27q^2$. Thus $\wp$ and q are the coefficients of the cubic potential $V(x) = \wp x - qx^3$. Thus $\wp = \frac{3}{2}, q = 2$. Using these values in the discriminant function, we have

$$\Delta = 4\left(\frac{3}{2}\right)^3 + 27(2)^2 = \frac{27}{2} + 108 = \frac{27}{2} + \frac{216}{2} = \frac{243}{2}.$$

The modulus k is computed as

$$k = \frac{\Delta}{4\wp^3} = \frac{243/2}{4(3/2)^3} = \frac{243/2}{27/2} = 9.$$

We then express the Weierstrass phase function $g(x) = F(\vartheta(x)|k)$. The $g(x)$ is given by

$$g(x) = \int_0^{\frac{\pi}{2}} \sqrt{\left(\left(\frac{3}{2}\right)x - 2x^3\right)}\,dx. \tag{3.57}$$

Results are presented in Table 6 for problem given in Equation (3.57).

Table 4 shows numerical results computed for the Jacobi elliptic integral of first kind Runge-Kutta fourth order method.

Table 4. Results computed from Equation (3.57) with the Jacobi Elliptic Integral of First kind

k	Jacobi elliptic integral computed using Konrod-Gauss quadrature
0.0	0.0000
0.1	0.010504
0.2	0.042102
0.3	0.095730
0.4	0.173554
0.5	0.279566
0.6	0.418055
0.7	0.593890
0.8	0.812098
0.9	1.078623
1.0	1.400000

4. Discussion of Results

We discussed the elements of Weierstrass elliptic functions. Using Laurent Series expansion, a class of Weierstrass elliptic function was obtained by comparing coefficients of Eisenstein series for a polynomial of degree three.

The (n,s)–curves and genus were described in the form defining an (n,s) curve as an algebraic curve with equation

$$y^n = x^s + a_{s-1}x^{s-1} + a_{s-2}x^{n-2} + \ldots + a_1 x + a_0, \quad (s > n).$$

Using Jacobi theta function an elliptic integral of Jacobi of first and second kinds [23] were obtained which led to the optimization problem of hypergeometric function. We unified the Weierstrass elliptic function and Jacobi elliptic integrals by computing their respective integrals using Kond-Rod–Gauss quadrature formula and the Runge-Kutta fourth order method as in presented in Tables 1-2.

Various theoretical bounds for the Jacobi elliptic integrals are given which are amenable to the ellipsoidal problems in complicated geodesy and astrodynamics though not fully exhibited here in the paper. It is mentioned that inverse hyperbolic functions can be used also to bind the Jacobi elliptic integral. It was also discussed that the term appearing in the integration of this Jacobi integral can be expanded in the form Taylor series before integration can be carried out. This may be useful in the task of using interval arithmetic operations in providing the upper and lower bounds of Jacobi elliptic integrals. This we hope to research further in the subsequent papers. It is also observed that Runge-Kutta Fourth order numerical method or any of its higher order variants can be applied to the Jacobi elliptic integrals as well we illustrated in Tables 2 and 4. This we demonstrated in Table 2. We realized that Jacobi elliptic integrals are a geodesic problem arising from computing the meridian distance on the earth surface [2,15]. We took a sample problem from [6] for the analysis of a well cubic potential function. The point where the particle entered the escape region with increasing velocity away from its origin was detailed. We took note of the major role played by the imaginary part in the Weierstrass elliptic function.

5. Conclusion

The paper presented Weierstrass elliptic function and the differential equations satisfied by the cubic equation. Roots of the cubic equation and the accompany discriminant equation satisfied in the Weierstrass elliptic function have been presented.

Using Laurent Series expansion, we described the equations for the Weierstrass elliptic function by comparing the coefficients of Eisenstein series with resulting cubic polynomial. We gave information on the (n,s)–curves and genus which helps in describing the tori of the Weierstrass meromorphic function.

Further using Jacobi theta function, we relate the Jacobi sine integral to obtain the Jacobi elliptic integrals of first and second kinds. We synchronized the Jacobi integrals with Weierstrass elliptic function to obtain the Jacobi sine integrals. The hypergeometric function accompanying the Jacobi elliptic integrals is described with Pochammer symbol or shifted factorial function. The asymptotic Sterlin's formula regarding hypergeometric function is given.

Various theoretical bounds for the Jacobi elliptic integrals are described. It is established that a ratio of these bounds could help in further providing information on the Jacobi elliptic integrals. We also mentioned that the Jacobi integrals can be expressed in the forms of Taylor series expansion where interval arithmetic could be applied. We followed [6] to demonstrate with a numerical example by considering a motion of a unit mass particle in a cubic potential and a detailed analysis of the time of flight is discussed for the bounded and unbounded solutions where the imaginary part plays a major role. Equations for the bounded and unbounded solutions to the cubic potential function in terms of their velocities are then given.

We demonstrated with huge success numerical examples with these methods using Konrod-Gauss Quadrature method in computing respectively the Jacobi elliptic integrals of first and second kinds as well as the inverse Jacobi elliptic integral as presented in Tables 1 and 2. In Table 3 we computed the Fatou set for the Weierstrass elliptic function by implementing the iteration due to [9] as represented in Equation (2.24). We used the Runge-Kutta forth order method to compute the Weierstrass phase function for the unit mass particle under cubic potential well function with Jacobi elliptic integral of first kind. This is reported in Table 4. We implemented our computations importing numpy as np from scipy.integrate. As a closing remark, it is further suggested that Jacobi elliptic integrals can be expanded in Taylor Series expansion where interval arithmetic can be used in bounding the integral problem.

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