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High Order Continuous Extended Linear Multistep Methods for Approximating System of ODEs

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Abstract

A class of high-order continuous extended linear multistep methods (HOCELMs) is proposed for solving systems of ordinary differential equations (ODEs). These continuous schemes are obtained through multistep collocation at various points to create a single block method with higher dimensions. This class of schemes consists of A-stable methods with a maximum order of $p \leq 14$, capable of yielding moderately accurate results for equations with several eigenvalues of the Jacobians located close to the imaginary axis. The results obtained from numerical experiments indicate that these schemes show great promise and competitiveness when compared to existing methods in the literature.

1 Introduction

For many years, significant interest has been devoted to the development of efficient schemes with good stability for solving stiff problem of the form:

$$y' = Ay; \quad y(a) = y_0, \quad a \leq t \leq b, \tag{1}$$

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where $y \in \mathbb{R}^v$, the matrix A is an $v \times v$ with its eigenvalues $\{\lambda_i\}_{i=1}^{v}$ such that Re $(\lambda_i) < 0$ and the stiffness ratio is $\frac{\max|Re(\lambda_i)|}{\min|Re(\lambda_i)|}$. Although the use of ODEs in (1) is often encountered in atmospheric convention, control theory and biological system.

According to Cash [1], a fundamental requirement to integrate (1) is to have a numerical scheme equipped with high order and good stability properties. Meanwhile, Dahlquist [2] order and stability criteria limit the possibility of deriving high order A-stable linear multistep methods (LMMs). This allows the backward differentiation formula (BDF) in Curtis and Hirschfelder [3] to be the most used code for solving (1).

This pessimistic result enabled Bickart and Rubin [4] to modify the traditional LMM to a different class of schemes. On this note, Enright [5] proposed a special class of the Obrechkoff schemes [6], which is A-stable for $k \leq 2$ and of order $p \leq 4$. Some other works related to multi-derivative schemes include: [7–10]. The use of hybrid schemes to bypass the Dahlquist Barrier theorem for LMMs has also been considered; see [11–19]. However, it was noted in Gupta [20] that the algorithm procedure for the hybrid scheme is more computationally intensive due to the occurrence of an off-step function in the method, which requires more predictors during implementation.

A different approach known as the boundary value method (BVM), which overcomes the results in Dahlquist, has been considered and fully documented in [21–26]. The BVM implementation technique (also known as a one-block scheme) simultaneously computes an approximation of the block solution of (1) and (2) [27–31]. The scheme has been known to overcome the problems associated with the conventional step-by-step implementation [32]. This approach is self-starting and allows a change of stepsize in the implementation. Cash [33] further noted that the boundary value approach improves the A-stable algorithm of BDF to order 4.

By adopting the boundary value approach, we propose a continuous extended Enright's method through the multistep collocation procedure. The newly derived block scheme will generate a block of numerical solutions $(y_1, y_2, \cdots, y_N)^T$ simultaneously. We note that block schemes came into existence through Milne [34] as an alternate way of generating initial solutions for predictor-corrector schemes (see, [35–41]). In the spirit of Conte and de Boor [42], computer algorithms associated with predictor-corrector schemes are more complicated, especially when introducing subroutines in starting solution values for the methods, thus yielding longer computer time and more computational work. On this premise, the general use of the proposed self-starting schemes cannot be overemphasized [43].

The article is organized as follows: In Section 2, we obtain a continuous approximation for the theoretical solution $y(t)$, which is used to generate the constituent LMMs in the block schemes for solving (1). Section 3 is devoted to the properties of the proposed HOCELMs, while the implementation strategy of the continuous schemes is given in Section 4. The numerical results obtained with the new schemes are reported in Section 5, along with the existing methods.

2 General Derivation and Some Analysis of the Continuous Scheme

Consider the first order initial value problems of the form

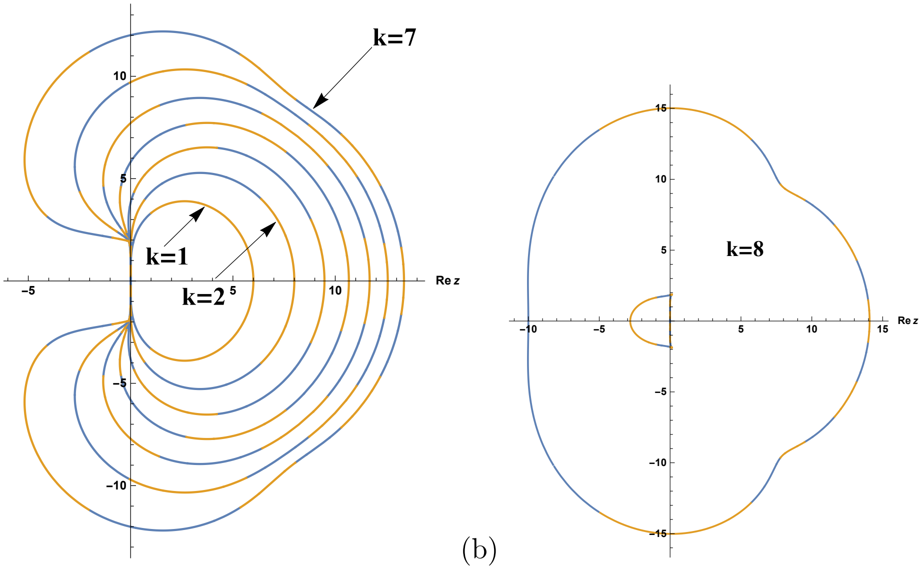
$$y' = f(t, y), \quad t_0 \leq t \leq T, \quad y(t_0) = y_0, \tag{2}$$

over the discrete interval $t_n = t_0 + nh$, $n = 1(1)N$, step size $h = \frac{T-t_0}{N}$. The second derivative k-step Enright method for numerical solution of the continuous problem in (2) is of the form

$$y_{n+k} - y_{n+k-1} = h\sum_{j=0}^{k} \alpha_j f_{n+j} + h^2 \beta_k f'_{n+k}, \tag{3}$$

where $y_{n+j} \approx y(t_n + jh)$, $f_{n+j} \equiv f(t_n + jh, y_n + jh)$ and $f'_{n+k} = \frac{df(t,y(t))}{dt} |_{y=y_{n+j}}^{t=t_{n+j}}$, $j = 0, 1, \cdots, k$. The scheme in (3) is A$-$stable for $k = 1, 2$, A$(\alpha)-$ stable for

$k = 3(1)7$ and instability set in from $k \geq 8$ (see, Fig. 1).



(a) (b)

Figure 1: Stability plot of the (a): stable and (b): unstable SDLMM of Enright (3) of order $p = k + 2$.

By adopting the approach of Cash [7], we extended the works of Enright in (3) to the form

$$y_{n+i} - y_{n+i-1} = h \sum_{j=0}^{k} \alpha_{i,j} f_{n+j} + h^2 \sum_{j=i-1}^{i} \beta_{i,j} f'_{n+j}, \quad i = 1, 2, \cdots, k. \tag{4}$$

We observe that for $i = k$ in (4), it becomes a scheme that is zero-stable for $k \geq 2$, and in the spirit of Brugnano and Trigiante [23], it is only suitable for approximating the solution of non-stiff systems in (2). This is because the absolute stability regions of this scheme are all bounded and tend to be smaller as k increases. On this note, we consider the continuous scheme with $i = 1, 2, \cdots, k$ in (4), which produces a block scheme having an improved A-stability property up to order $p = 14$. To obtain the continuous methods, the theoretical solution $y(t)$

is approximated by a continuous $Y(t)$ of the form

$$y(t) \approx Y(t) = \sum_{j=0}^{k+3} b_{n,j}(t-t_n)^j, \tag{5}$$

where $b_{n,i}$ are required coefficients. The resultant coefficients $b_{n,i}$ are derived from a system of $k+4$ equations with $k+4$ unknowns obtained from

$$\begin{aligned} Y(t_{n+i-1}) = y_{n+i-1}, \quad i = t \\ Y''_{n+i} = f'(t_{n+i}, y_{n+i}) \\ Y''_{n+i-1} = f'(t_{n+i-1}, y_{n+i-1}) \\ Y'_{n+j} = f(t_{n+j}, y_{n+j}), \quad j = 0, 1, 2, \cdots, k. \end{aligned} \tag{6}$$

In the spirit of Motsa [44], the unknown coefficients with the constituent methods are generated through the Mathematica code at the collocation points t_{n+i} to obtain

$$y(t) = y_{n+i-1} - h\sum_{j=0}^{k}\alpha_{i,j} f_{n+j} + h^2 \sum_{j=i-1}^{i}\beta_{i,j} f'_{n+j}; \quad i = 1, 2, \cdots, k. \tag{7}$$

The general code for deriving the continuous schemes in (7) is given as follows:

$$\begin{aligned}
k &= \text{"input integer value for } k \geq 2\text{" }_; \\
points &= \text{Table}[i, \{i, 0, k\}]; \\
Table[t_{n+i} &= h(n+i), \{i, 0, k\}]; \\
Y &= \textstyle\sum_{j=0}^{k+3} (t-t_n)^j b_{n,j}; \\
equation1 &= Simplify\left[Table\left[\left(\tfrac{\partial Y}{\partial t}/.\ \{t \to t_{i+n}\}\right) = f_{i+n}, \{i, 0, k\}\right]\right] \\
equation2 &= (D[Y, \{t, 2\}]/.t- > t_{n+i-1} = f'_{n+i-1} \\
equation3 &= (D[Y, \{t, 2\}]/.t- > t_{n+i} = f'_{n+i} \\
unknown &= Table\left[b_{n,i}, \{i, 0, k+3\}\right] \\
initial &= b_{n,0} = y_{n+i-1} \\
Allequations &= Join[equation1, \{equation2\}, \{equation3\}, \{initial\}] \\
solution &= Solve[Allequations, unknown] \\
work(7) &= Y/.\, solution \\
Constituentmethodsin(9) &= Simplify\left[Table\left[work/.\ \{t \to t_{n+i}\}, \{i, 1, k\}\right]\right].
\end{aligned} \tag{8}$$

The resulting scheme from (7) can easily be represented as a matrix of finite difference equation, which supplies the approximation at the first k-mesh point as the solution of the discrete problem of the form

$$A \otimes I_k Y_{u+1} = a \otimes y_0 + h\left(B \otimes I_k F_{u+1} + b f_0\right) + h^2\left(C \otimes I_k F'_{u+1} + c \otimes f'_0\right), \quad (9)$$

where I_k is the identity matrix of size k. Matrices

$$A = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 1 & 0 & \ddots & 0 & 0 \\ 0 & -1 & 1 & \ddots & \vdots & 0 \\ 0 & 0 & 0 & \ddots & 0 & 0 \\ & \vdots & \vdots & -1 & 1 & 0 \\ 0 & 0 & 0 & \ddots & -1 & 1 \end{pmatrix}_{k\times k}, \quad \bar{A} = [D \mid a] = \left(\begin{array}{ccccc|c} 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & \vdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ & \vdots & & \ddots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{array}\right)_{k\times k} \quad (10)$$

$$C = \begin{pmatrix} \beta_{1,1} & 0 & 0 & & & \cdots & 0 \\ \beta_{2,1} & \beta_{2,2} & 0 & & & \cdots & 0 \\ 0 & \beta_{3,1} & \beta_{3,2} & 0 & & \cdots & 0 \\ \ddots & \ddots & & \ddots & & & \\ & \ddots & \ddots & & \ddots & & \\ 0 & 0 & \cdots & 0 & \beta_{k-1,k-2} & \beta_{k-1,k-1} & 0 \\ 0 & 0 & 0 & \cdots & 0 & \beta_{k,k-1} & \beta_{k,k} \end{pmatrix}_{k\times k}, \quad (11)$$

$$\bar{C} = [G \mid c] = \left(\begin{array}{ccccc|c} 0 & 0 & 0 & \cdots & 0 & \beta_{1,0} \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & \vdots & \vdots \\ & \vdots & & \ddots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{array}\right)_{k\times k},$$

$$B = \begin{pmatrix} \alpha_{1,1} & \alpha_{1,2} & \cdots & \alpha_{1,k} \\ \alpha_{2,1} & \alpha_{2,2} & \cdots & \alpha_{2,k} \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ \alpha_{k-1,1} & \alpha_{k-1,2} & \cdots & \alpha_{k-1,k} \\ \alpha_{k,1} & \alpha_{k,2} & \cdots & \alpha_{k,k} \end{pmatrix}_{k\times k}, \tag{12}$$

$$\bar{B} = [E \mid b] = \left(\begin{array}{ccccc|c} 0 & 0 & 0 & \cdots & 0 & \alpha_{1,0} \\ 0 & 0 & 0 & \cdots & 0 & \alpha_{2,0} \\ 0 & 0 & 0 & \cdots & \vdots & \vdots \\ & \vdots & & \ddots & 0 & \alpha_{k-1,0} \\ 0 & 0 & 0 & \cdots & 0 & \alpha_{k,0} \end{array}\right)_{k\times k},$$

and the vector Y_{u+1}, F_{u+1} and F'_{u+1} in the form

$$Y_{u+1} = [y_1, y_2, \cdots, y_k]^T, \quad F_{u+1} = [f_1, f_2, \cdots, f_k]^T, \quad F'_{u+1} = [f'_1, f'_2, \cdots, f'_k]^T, \tag{13}$$

contain the discrete solution values, the corresponding solution values of F_{u+1} and F'_{u+1} respectively. The continuous schemes (also referred as block methods) is self-starting and only require the initial value y_0 provided by the continuous problem (2). Thus the accumulated error is insignificant on the solution obtained since the block solution are generated concurrently.

3 Analysis of the Continuous Scheme

Following the approach of Fatunla [45] and Lambert [46], the continuous scheme (4) can be expressed in form of a linear difference operator

$$\mathcal{L}_i[y(t);h] = y(t+ih) - y(t+(i-1)h) - h\sum_{j=0}^{k}\alpha_{i,j}y'(t+jh) - h^2\sum_{j=0}^{k}\beta_{i,j}y''(t+jh), \tag{14}$$

where $y(t)$ is a sufficiently differentiable function on $[a,b]$. By expanding via Taylor series the terms in (14), we obtain the expression

$$\mathcal{L}_i(y(t);h) = C_{i,0}y(t) + C_{i,1}hy'(t) + \cdots + C_{i,p}h^p y^{(p)}(t) + \cdots, \tag{15}$$

where

$$\begin{aligned}
C_{i,0} &= 0\\
C_{i,1} &= 1 - \sum_{j=0}^{k}\alpha_{i,j}\\
C_{i,2} &= \frac{1}{2!}\left(i^2 - (i-1)^2\right) - \sum_{j=0}^{k} j\alpha_{i,j} - (\beta_{i,i-1} + \beta_{i,i})\\
C_{i,3} &= \frac{1}{3!}\left(i^3 - (i-1)^3\right) - \frac{1}{2!}\sum_{j=0}^{k} j^2\alpha_{i,j} - ((i-1)\beta_{i,i-1} + i\beta_{i,i})\\
&\vdots\\
C_{i,p} &= \frac{1}{q+2!}\left(i^{q+2} - (i-1)^{q+2}\right) - \frac{1}{q+1!}\sum_{j=0}^{k} j^{q+1}\alpha_{i,j} - \frac{1}{q!}\left((i-1)^q\beta_{i,i-1} + i^q\beta_{i,i}\right)\\
&\text{for} \quad q = 0,1,2,\cdots p-2.
\end{aligned} \tag{16}$$

According to Henrici [47], we have the following definition:

Definition 3.1. *The continuous scheme* (4) *is said to be of order* p*, if*

$$\bar{C}_j = 0, \quad j = 0(1)p, \quad \bar{C}_{p+1} \neq 0, \tag{17}$$

where $\bar{C}_p = [C_{1,p}, C_{2,p}, C_{3,p}, \cdots, C_{k,p}]$. *The vector* $\bar{C}_{p+1}$ *is referred as the error constant. Thus,* (15) *can be expanded to obtained [44]*

$$\begin{aligned}\mathcal{L}_i(y(t);h) = \sum_{u=1}^{k+3}\Bigg(&\frac{i^u-(i-1)^u}{u!} - \sum_{j=0}^{k}\frac{uj^{u-1}}{u!}\alpha_{i,j}\\ -\frac{u(u-1)}{u!}&\left((i-1)^{u-2}\beta_{i,i-1}+i^{u-2}\beta_{i,i}\right)\Bigg)h^u y^u(t)+\\ &\frac{h^{k+4}}{(k+4)!}\Bigg[i^{k+4}-(i-1)^{k+4}-\sum_{j=0}^{k}(k+4)j^{k+3}\alpha_{i,j}\\ -(k+4)(k+3)&\left((i-1)^{k+2}\beta_{i,i-1}+i^{k+2}\beta_{i,i}\right)\Bigg]y^{(k+4)}(t)+O(h^{k+5}).\end{aligned} \tag{18}$$

We observe from the numerical simplification that the first five terms of (18) turns out to zero. This implies that the truncation error for the continuous scheme is of the form

$$\begin{aligned}\mathcal{L}_i(y(t);h) = &\frac{h^{k+4}}{(k+4)!}\Bigg[i^{k+4}-(i-1)^{k+4}-\sum_{j=0}^{k}(k+4)j^{k+3}\alpha_{i,j}\\ -(k+4)(k+3)&\left((i-1)^{k+2}\beta_{i,i-1}+i^{k+2}\beta_{i,i}\right)\Bigg]y^{(k+4)}(t)+O(h^{k+5}).\end{aligned} \tag{19}$$

Hence, the error constant vector from (19) is

$$\bar{C}_{p+1} = \left[C_{1,k+4}, C_{2,k+4}, C_{3,k+4}, \cdots, C_{k,k+4}\right],$$

with

$$\begin{aligned}C_{i,k+4} = &\frac{1}{(k+4)!}\Bigg[i^{k+4}-(i-1)^{k+4}-\sum_{j=0}^{k}(k+4)j^{k+3}\alpha_{i,j}\\ -(k+4)(k+3)&\left((i-1)^{k+2}\beta_{i,i-1}+i^{k+2}\beta_{i,i}\right)\Bigg], i = 01, 2, 3, \cdots, k.\end{aligned} \tag{20}$$

It is worthy to know that the constituents linear multi-step method that formed the block schemes in (9) are of the same order. Thus the block scheme (9) is consistent since each constituents linear multistep methods has order $p \geq 1$. It is confirmed from Fig. 2 that the block schemes (9) have smaller error constant when compared with the methods of Enright [5] and Cash [7]. Thus, theoretically the new scheme shows the possibility of more accurate solution to (2).

Following Fatunla [39], let

$$\bar{\rho}(R) = AR - \bar{A}, \quad \bar{\sigma}(R) = BR + \bar{B}, \quad \bar{\gamma}(R) = CR + \bar{C}, \tag{21}$$

be the first, second and third matrix polynomial respectively. Then the first stability polynomial is given as

$$\rho(R) = det(AR - \bar{A}) = R^{2k-1}(R-1), \tag{22}$$

which has a principal root $R =\mid 1 \mid$ and spurious roots $R_j = 0$, $j = 1(1)2k-1$. It is observed that the block schemes are all zero-stable since the polynomial (22) possesses only one root of unit modulus.

The stability analysis is obtained by considering the Dahlquist test equation

$$y' = \lambda y, \qquad y'' = \lambda^2 y, \quad Re(\lambda) < 0. \tag{23}$$

The application of the schemes

$$AY_{u+1} = ay_0 + h\left(BF_{u+1} + bf_0\right) + h^2\left(CF'_{u+1} + cf'_0\right), \tag{24}$$

generate the discrete solution of the form

$$Y = M(z)y_0, \quad z = \lambda h, \tag{25}$$

where a, b and c are vectors in $\mathbb{R}^k$ and $M(z) = \left(A - zB - z^2C\right)^{-1}\left(a + zb + z^2c\right)$ is the amplification matrix. The behaviour of the numerical solution Y_{u+1} will depend on the eigenvalue of $M(z)$. That is, the stability matrix $M(z)$ has eigenvalues $\{0, 0, \cdots, H(z)\}$, where $H(z)$ is the dominant eigenvalue. The region of absolute stability H for the newly derived method is

$$\mathbb{H} = \{z \in \mathbb{C} :\mid \mathrm{p}(H(z)) < 1\}, \tag{26}$$

and its profile is shown in Fig. 3. The methods are A-stable because the left half complex plane is contained in $\mathbb{H}$. For example the spectral radius for the block method of order p =5, 6 and 7 is as follows:

$$H(z) = \frac{7z^4 + 81z^3 + 381z^2 + 900z + 900}{7z^4 - 81z^3 + 381z^2 - 900z + 900}$$

$$H(z) = \frac{1309z^6 + 22185z^5 + 164280z^4 + 684260z^3 + 1720600z^2 + 2524800z + 1728000}{1309z^6 - 22185z^5 + 135720z^4 - 364900z^3 + 243160z^2 + 796800z - 1728000} \tag{27}$$

$$\begin{aligned} H(z) = \Big(&24347850z^8 + 541674160z^7 + 5443815132z^6 + 32530996854z^5 \\ &+127355563443z^4 + 338699439540z^3 + 599510574810z^2 + 645241282560z \\ &+322620641280\Big) \Big/ \Big(24347850z^8 - 541674160z^7 + 5443815132z^6 \\ &-32530996854z^5 + 127355563443z^4 - 338699439540z^3 \\ &+599510574810z^2 - 645241282560z + 322620641280\Big). \end{aligned} \tag{28}$$

More so, from equation (25) one can say the conventional criteria of A-stability is equivalent, to possess all $z \in \mathbb{C}^-$ ([23, 41])

$$\| y_k(z) \|_\infty \equiv \| e_k T \left(A - zB - z^2 C\right)^{-1} \left(a + zb + z^2 c\right) y_0 \|_\infty < \| y_0 \|, \tag{29}$$

where e_k is the last unit vector in $\mathbb{R}^k$ that is,

$$Re(z) < 0 \Longrightarrow g(z) \equiv \| e_k T \left(A - zB - z^2 C\right)^{-1} \left(a + zb + z^2 c\right) \|_\infty < 1. \tag{30}$$

In the spirit of Brugnano and Trigiante [41], a necessary requirement for A-stability, is to have equation in (25) to be well-defined for all such z. Thus, we have the following definition:

Definition 3.2. *[41] A block scheme is said to be pre-stable if the poles of the corresponding matrix pencil* $A - \mu B - \mu^2 C$ *is contained in* $\mathbb{C}^+$.

The defintion 3.2 holds for the block schemes (9) up to $k = 11$ (see, Fig. 4a), while from Fig. 4b the block schemes for $k = 12$ cannot be pre-stable, since it has

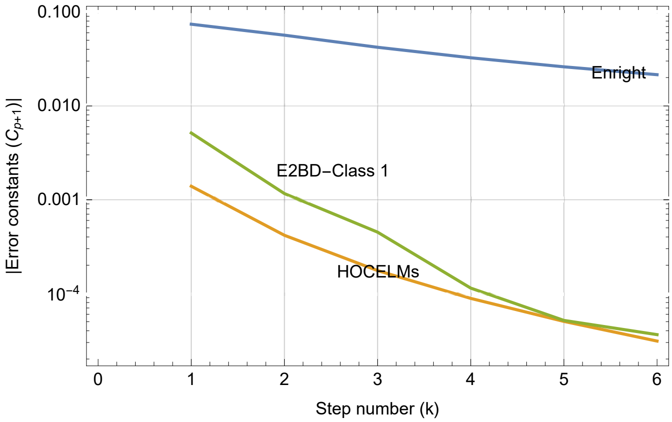


Figure 2: The plot of the error constants versus step lengths of the HOCELMs, Enright. [5], E2BD [7].

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two roots of $det(Q)$. This indicate that the proposed method is not A-stable from order $p \geq 15$.

$$Q = A - \mu B - \mu^2 C, \tag{31}$$

located in $\mathbb{C}^{-1}$. In Table 1 The maximum order of A-stability attained by the newly derived schemes is presented along block of BVMs in [23, 48]. More so, it is noticed that the new schemes have superior high order A-stable methods when compared with the block of generalized backward differentiation formulas (GBDFs), block of top order methods (TOMs), block of generalized Adams methods (GAMs) and extended trapezoidal rule of second kind (ETR_2s) in Brugnano and Trigiante [23], pp. 283 − 285 and multi-block of generalized Adams methods (MBGAMs)in [48].

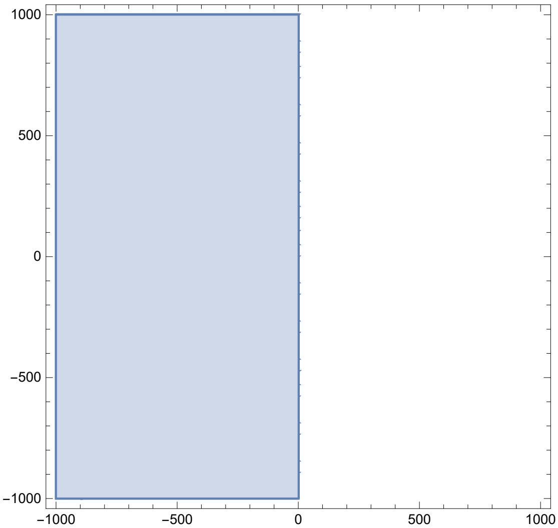

Figure 3: Region of absolute stability for the continuous schemes (9) for $k \leq 11$.
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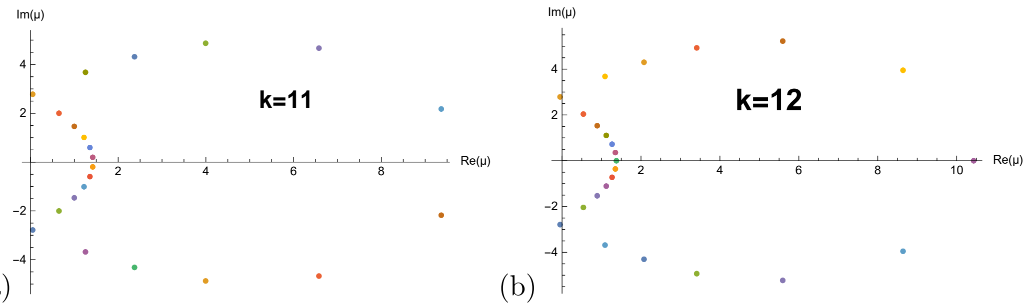

Figure 4: Eigenvalues of the pencil (31) associated with block schemes (9) of order (a) fourteenth and (b) fifteenth respectively.

Table 1: Maximum order of A-stable block methods

Block methods	maximum order attainable
HOCELMs ($p = k + 3$)	$p \leq 14$
MBGAMs ($p = 2(k + 1)$	$p \leq 10$
GAMs ($p = k + 1$	$p \leq 9$
ETR$_2$s ($p = k + 1$)	$p \leq 9$
GBDFs ($p = k$)	$p \leq 4$
TOMs ($k + 1$)	$p \leq 6$

Table 2: Coefficients, error constant C_{p_i+1}, and the order p of the continuous scheme (7)

k	i	$\alpha_{i,0}$	$\alpha_{i,1}$	$\alpha_{i,2}$	$\alpha_{i,3}$	$\alpha_{i,4}$	$\alpha_{i,5}$	$\alpha_{i,6}$	$\alpha_{i,7}$	$\beta_{i,0}$
	1	$\frac{11}{24}$	$\frac{8}{15}$	$\frac{1}{120}$	0	0	0	0	0	$\frac{1}{15}$
2	2	$\frac{1}{120}$	$\frac{8}{15}$	$\frac{11}{24}$	0	0	0	0	0	0
	1	$\frac{313}{720}$	$\frac{131}{240}$	$\frac{1}{48}$	$-\frac{1}{720}$	0	0	0	0	$\frac{7}{120}$
3	2	$\frac{1}{240}$	$\frac{119}{240}$	$\frac{119}{240}$	$\frac{1}{240}$	0	0	0	0	0
	3	$-\frac{1}{720}$	$\frac{1}{48}$	$\frac{131}{240}$	$\frac{313}{720}$	0	0	0	0	0
	1	$\frac{50623}{120960}$	$\frac{4153}{7560}$	$\frac{41}{1120}$	$-\frac{37}{7560}$	$\frac{53}{120960}$	0	0	0	$\frac{107}{2016}$
4	2	$\frac{53}{20160}$	$\frac{1789}{3780}$	$\frac{18}{35}$	$\frac{13}{1260}$	$-\frac{31}{60480}$	0	0	0	0
	3	$-\frac{31}{60480}$	$\frac{13}{1260}$	$\frac{18}{35}$	$\frac{1789}{3780}$	$\frac{53}{20160}$	0	0	0	0
	5	$\frac{53}{120960}$	$-\frac{37}{7560}$	$\frac{41}{1120}$	$\frac{4153}{7560}$	$\frac{50623}{120960}$	0	0	0	0
	1	$\frac{98291}{241920}$	$\frac{132521}{241920}$	$\frac{53}{960}$	$-\frac{671}{60480}$	$\frac{481}{241920}$	$-\frac{1}{5376}$	0	0	$\frac{199}{4032}$
	2	$\frac{5}{2688}$	$\frac{15803}{34560}$	$\frac{1133}{2160}$	$\frac{121}{6720}$	$-\frac{31}{17280}$	$\frac{31}{241920}$	0	0	0
5	3	$-\frac{31}{120960}$	$\frac{29}{4480}$	$\frac{3733}{7560}$	$\frac{3733}{7560}$	$\frac{29}{4480}$	$-\frac{31}{120960}$	0	0	0
	4	$\frac{31}{241920}$	$-\frac{31}{17280}$	$\frac{121}{6720}$	$\frac{1133}{2160}$	$\frac{15803}{34560}$	$\frac{5}{2688}$	0	0	0
	5	$-\frac{1}{5376}$	$\frac{481}{241920}$	$-\frac{671}{60480}$	$\frac{53}{960}$	$\frac{132521}{241920}$	$\frac{98291}{241920}$	0	0	0
	1	$\frac{2398441}{6048000}$	$\frac{9852103}{18144000}$	$\frac{2309}{30240}$	$-\frac{2231}{108864}$	$\frac{4001}{725760}$	$-\frac{6241}{6048000}$	$\frac{1279}{13608000}$	0	$\frac{6031}{129600}$
	2	$\frac{1279}{907200}$	$\frac{8072717}{18144000}$	$\frac{384773}{725760}$	$\frac{4901}{181440}$	$-\frac{367}{90720}$	$\frac{2099}{3628800}$	$-\frac{817}{18144000}$	0	0
6	3	$-\frac{817}{5443200}$	$\frac{2759}{604800}$	$\frac{173693}{362880}$	$\frac{1436}{2835}$	$\frac{1361}{120960}$	$-\frac{1621}{1814400}$	$\frac{289}{5443200}$	0	0
	4	$\frac{289}{5443200}$	$-\frac{1621}{1814400}$	$\frac{1361}{120960}$	$\frac{1436}{2835}$	$\frac{173693}{362880}$	$\frac{2759}{604800}$	$-\frac{817}{5443200}$	0	0
	5	$-\frac{817}{18144000}$	$\frac{2099}{3628800}$	$-\frac{367}{90720}$	$\frac{4901}{181440}$	$\frac{384773}{725760}$	$\frac{8072717}{18144000}$	$\frac{1279}{907200}$	0	0
	6	$\frac{1279}{13608000}$	$-\frac{6241}{6048000}$	$\frac{4001}{725760}$	$-\frac{2231}{108864}$	$\frac{2309}{30240}$	$\frac{9852103}{18144000}$	$\frac{2398441}{6048000}$	0	0
	1	$\frac{7049453}{18144000}$	$\frac{9724213}{18144000}$	$\frac{671}{6720}$	$-\frac{913}{27216}$	$\frac{26213}{2177280}$	$-\frac{6817}{2016000}$	$\frac{131}{212625}$	$-\frac{29}{544320}$	$\frac{5741}{129600}$
	2	$\frac{29}{25920}$	$\frac{7891613}{18144000}$	$\frac{9667373}{18144000}$	$\frac{6749}{181440}$	$-\frac{5}{672}$	$\frac{1159}{725760}$	$-\frac{4513}{18144000}$	$\frac{11}{567000}$	0
7	3	$-\frac{11}{113400}$	$\frac{3127}{907200}$	$\frac{8468189}{18144000}$	$\frac{74737}{145152}$	$\frac{3053}{181440}$	$-\frac{911}{453600}$	$\frac{289}{1209600}$	$-\frac{289}{18144000}$	0
	4	$\frac{289}{10886400}$	$-\frac{71}{136080}$	$\frac{797}{100800}$	$\frac{119167}{241920}$	$\frac{119167}{241920}$	$\frac{797}{100800}$	$-\frac{71}{136080}$	$\frac{289}{10886400}$	0
	5	$-\frac{289}{18144000}$	$\frac{289}{1209600}$	$-\frac{911}{453600}$	$\frac{3053}{181440}$	$\frac{74737}{145152}$	$\frac{8468189}{18144000}$	$\frac{3127}{907200}$	$-\frac{11}{113400}$	0
	6	$\frac{11}{567000}$	$-\frac{4513}{18144000}$	$\frac{1159}{725760}$	$-\frac{5}{672}$	$\frac{6749}{181440}$	$\frac{9667373}{18144000}$	$\frac{7891613}{18144000}$	$-\frac{2687}{43200}$	0
	7	$-\frac{29}{544320}$	$\frac{131}{212625}$	$-\frac{6817}{2016000}$	$\frac{26213}{2177280}$	$-\frac{913}{27216}$	$\frac{671}{6720}$	$\frac{9724213}{18144000}$	$\frac{7049453}{18144000}$	0

Table 3: Table 2 continued.

k	i	$\beta_{i,1}$	$\beta_{i,2}$	$\beta_{i,3}$	$\beta_{i,4}$	$\beta_{i,5}$	$\beta_{i,6}$	$\beta_{i,7}$	p_i	C_{p_i+1}
	1	$-\frac{7}{60}$	0	0	0	0	0	0	5	$\frac{-1}{2400}$
2	2	$-\frac{7}{60}$	$-\frac{1}{15}$	0	0	0	0	0	5	$\frac{1}{2400}$
	1	$-\frac{17}{120}$	0	0	0	0	0	0	6	$\frac{-31}{302400}$
3	2	$\frac{11}{120}$	$-\frac{11}{120}$	0	0	0	0	0	6	$\frac{53}{302400}$
	3	0	$\frac{17}{120}$	$-\frac{7}{120}$	0	0	0	0	6	$\frac{53}{302400}$
	1	$-\frac{41}{252}$	0	0	0	0	0	0	7	$-\frac{5}{56448}$
4	2	$\frac{5}{63}$	$-\frac{37}{336}$	0	0	0	0	0	7	$\frac{31}{846720}$
	3	0	$\frac{37}{336}$	$-\frac{5}{63}$	0	0	0	0	7	$\frac{-31}{846720}$
	4	0	0	$\frac{41}{252}$	$-\frac{107}{2016}$	0	0	0	7	$\frac{5}{56448}$
	1	$-\frac{731}{4032}$	0	0	0	0	0	0	8	$\frac{1279}{25401600}$
	2	$\frac{289}{4032}$	$-\frac{253}{2016}$	0	0	0	0	0	8	$-\frac{817}{50803200}$
5	3	0	$\frac{191}{2016}$	$-\frac{191}{2016}$	0	0	0	0	8	$\frac{289}{25401600}$
	4	0	0	$\frac{253}{2016}$	$-\frac{289}{4032}$	0	0	0	8	$-\frac{817}{50803200}$
	5	0	0	0	$\frac{731}{4032}$	$-\frac{199}{4032}$	0	0	8	$\frac{1279}{25401600}$
	1	$-\frac{8563}{43200}$	0	0	0	0	0	0	9	$-\frac{29}{933120}$
	2	$\frac{2863}{43200}$	$-\frac{1201}{8640}$	0	0	0	0	0	9	$\frac{11}{1360800}$
6	3	0	$\frac{23}{270}$	$-\frac{1393}{12960}$	0	0	0	0	9	$-\frac{289}{65318400}$
	4	0	0	$\frac{1393}{12960}$	$-\frac{23}{270}$	0	0	0	9	$\frac{289}{65318400}$
	5	0	0	0	$\frac{1201}{8640}$	$-\frac{2863}{43200}$	0	0	9	$-\frac{11}{1360800}$
	6	0	0	0	0	$\frac{8563}{43200}$	$-\frac{6031}{129600}$	0	9	$\frac{29}{933120}$
	1	$-\frac{27719}{129600}$	0	0	0	0	0	0	10	$\frac{146513}{7185024000}$
	2	$\frac{2687}{43200}$	$-\frac{6533}{43200}$	0	0	0	0	0	10	$-\frac{10709}{2395008000}$
7	3	0	$\frac{3391}{43200}$	$-\frac{205}{1728}$	0	0	0	0	10	$\frac{59}{29568000}$
	4	0	0	$\frac{2497}{25920}$	$-\frac{2497}{25920}$	0	0	0	10	$-\frac{317}{205286400}$
	5	0	0	0	$\frac{205}{1728}$	$-\frac{3391}{43200}$	0	0	10	$\frac{59}{29568000}$
	6	0	0	0	0	$\frac{6533}{43200}$	$-\frac{2687}{43200}$	0	10	$-\frac{10709}{2395008000}$
	7	0	0	0	0	0	$\frac{27719}{129600}$	$-\frac{5741}{129600}$	10	$\frac{146513}{7185024000}$

4 Implementation Procedure

By adopting the approach in [40,49] we present, the implementation procedure for the HOCELMs (9) for $k = 2,3,4,5$ with order $p = 5,6,7,8.9,10,11,12,13,14$ respectively. The HOCELMs of fifth, sixth, seventh, eighth ninth, tenth, eleventh, twelfth, thirteenth and fourteenth order are denoted by HOCELMs2, HOCELMs3, HOCELMs4, HOCELMs5, HOCELMs6, HOCELMs7, HOCELMs8, HOCELMs9, HOCELMs10 and HOCELMs11 respectively. The block schemes (9) is conveniently written in the form

$$AY_{u+1} = \bar{A}Y_u + h\left(BF_{u+1} + \bar{B}F_u\right) + h^2\left(CF'_{u+1} + \bar{C}F'_u\right), \tag{32}$$

where the coefficients A, $\bar{A}$, B, $\bar{B}$, C, $\bar{C}$ are given in (10), (11) and (12), (see also (13)) and

$$\begin{aligned} Y_u = \left[y_{-k+1}, y_{-k+2}, \cdots, y_0\right]^T, \quad F_u = \left[f_{-k+1}, f_{-k+2}, \cdots, f_0\right]^T, \\ F'_{u+1} = \left[f'_{-k+1}, f'_{-k+2}, \cdots, f'_0\right]^T. \end{aligned} \tag{33}$$

The schemes in (32) is used to approximate the IVPs (2) without requiring starting values. The procedure for block scheme (32) for the case of $k = 5$ is given according to the following sequences.

Given the partition
$\varpi_n : a = t_0 < t_1 < \cdots < t_n < t_{n+1} < \cdots < t_N$, $h = t_{n+1} - t_n$, $n = 0, 1, \cdots, N-1$.

Stage 1: Fixed N for $k = 5$, $h = \frac{(b-a)}{N}$ the number of block $\Gamma = \frac{N}{5}$. Using (32), $n = 0$, $u = 0$, the solution value of $(y_1, y_2, y_3, y_4, y_5)^T$ are obtained concurrently over the sub-interval $[t_0, t_5]$ since y_0 is given by the continuous problem (2).

Stage 2: $n = 5$, $u = 1$ $(y_6, y_7, y_8, y_9, y_{10})^T$ are similarly generated over the sub-interval $[t_5, t_{10}]$ since y_5 is given in the previous block.

Stage 3: the iteration process is continued for $n = 10, \cdots, N-5$ and $u = 2, \cdots, \Gamma$ to get approximate solution to (2) on sub-intervals $[t_{10}, t_{15}] \cdots [t_{N-5}, t_N]$.

In fact, the accumulated error is insignificant on the solution obtained since the block solution are generated concurrently from non-overlapping sub-interval. The implementation of (32) is achieved by using a modified Newton Raphson method for nonlinear problem while for linear problems, one require Gaussian elimination using partial pivoting technique.

5 Numerical Experiment

In this section, we considered some known linear and non-linear stiff problems to examine the accuracy of the HOCELMs. All computations were carried out using our written code in MATLAB 2010a. *First test problem is the well known integration system which was solved by Cash [7]*

$$\begin{aligned} y_1' &= -y_1 - 30y_2 + 30e^{-t}, \quad y_1(0) = 1, \\ y_2' &= 30y_1 - y_2 - 30e^{-t}, \quad y_2(0) = 1, \end{aligned}$$

and the required theoretical solution is

$$y_1(t) = e^{-t}, \quad y_2(t) = e^{-t}.$$

The Example 5 is solved using HOCELMs5 and compare with those of [7] of the same order $p = 8$. It is shown from Table 4, that our methods is superior in accuracy than the E2BD methods given in [7]. More so, a further comparison of order $p = 5$ and 6 with those of [49] is reported in Table 5 for stesize $h = 0.01$. From Table 5, it is clear that our method perform better in accuracy than the method of Akinfenwa and Jator [49]. This confirm that the proposed method is suitable for integrating problems with eigenvalues lying close to the imaginary axis. *Consider the stiff test recommended by Akinfenwa and Jator* [49]

$$\begin{aligned} y_1'(t) &= -2000y_1 + 1000y_2 + 1 \qquad y_1(0) = 0 \\ y_2'(t) &= y_1 - y_2 \qquad\qquad y_2(0)) = 0, \end{aligned} \tag{34}$$

the problem in Example 2 is stiff with the stiffness ratio $S = 4001$ *and the theoretical results are*

$$\begin{aligned} y_1(t) &= 4.97 \times 10^{-4} e^{-2000.5t} - 5.034 \times 10^{-4} e^{-0.5t} + 0.001 \\ y_2(t) &= -2.5 \times 10^{-7} e^{-2000.5t} - 1.007 \times 10^{-3} e^{-0.5t} + 0.001. \end{aligned} \tag{35}$$

In this example, the HOCELMs3 of order 6 is compared with methods of order 6 in Ismail and Ibrahim [50], and ECB_2DF in Akinfenwa and Jator [49]. It is noticed from Table 6 that the HOCELMs3 scheme performs better in accuracy than the methods of Ismail and Ibrahim [50] and Akinfenwa and Jator [49] at the point T=5 and 10. Table 7 further shows the results at higher stepsize $h = 0.1$ for Akinfenwa and Jator [49] and the proposed methods for order 5 and 6. *Given the IVPs in [51, 52]*

$$y\prime = -10(y-1)^2, \quad y(0) = 2, \quad t \in [0, 0.1]\,. \tag{36}$$

The theoretical solution is $y = 1 + \frac{1}{1+10t}$. The HOCELMs2 is applied to Example 5 and the error $(|\, y - y(t)\, |)$ in the various interval $0 < t \leq 0.1$ are reported in Table 8. It is clear from the numerical result and comparison in Table 8 that the HOCELMs2 is superior in terms of accuracy than the methods (p=5) in [51] and methods (p=6) of [52].

Consider a non-linear stiff system,

$$\begin{aligned} y_1' &= -10002 y_1 + 1000 y_2^2, \quad y_1(0) = 1 \\ y_2' &= y_1 - y_2(1 + y_2), \quad y_2(0) = 1 \end{aligned} \; , y(t) = \begin{pmatrix} e^{-2t} \\ e^{-t} \end{pmatrix}, \tag{37}$$

in [53].

The numerical results for Example 5 for various stepsizes h are presented in Table 9. The results obtained in Table 9, indicate r that the HOCELMs for $k = 2, 3$ and 5 improves in accuracy than the GSDLMMEs3 [26], BVMs in [54], ECB_2DFs in [49] and $BBDF_8$ in [55].

Consider the linear problem in [23]

$$y' = \begin{pmatrix} -21 & 19 & -20 \\ 19 & -21 & 20 \\ 40 & -40 & -40 \end{pmatrix} y, \quad y(0) = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}; \tag{38}$$

$$y(t) = \frac{1}{2} \begin{pmatrix} e^{-2t} + e^{-40t}\left(cos(40t) + sin(40t)\right) \\ e^{-2t} - e^{-40t}\left(cos(40t) + sin(40t)\right) \\ 2e^{-40t}\left(cos(40t) - sin(40t)\right) \end{pmatrix}.$$

Table 10 and 11 contains the maximum relative error $\max_{1<i<3} \mid y_i(t) - y_{i,h} \mid /(1+ \mid y_{i,h} \mid)$ in the interval $0 < t \leq 1$ using HOCELMs for order 5 to 14. The performance compares with generalized Adams methods GAMs of order 5,7 and 9 in [23]. It is seen from Table 10 and 11 that the scheme perform better and conformed to the numerical order of convergence (rate). The rate is computed from

$$rate = log_2 \left(\frac{\max_{1<i<3} \mid y_i(t) - y_{i,h} \mid /(1+ \mid y_{i,h} \mid)}{\max_{1<i<3} \mid y_i(t) - y_{i,\frac{h}{2}} \mid /(1+ \mid y_{i,\frac{h}{2}} \mid)} \right), \tag{39}$$
$$i = 1(1)m, \quad m = 3, \quad 0 < t \leq 1.$$

This rate in Table 10 and 11 is obtained from applying the SDBVMs with two different step sizes h and $\frac{h}{2}$. From which the rate is computed from the log of the absolute value of the ratio of two errors at the output point t. Here $y_i(t)$ is the exact solution at t since it is available for the ordinary differential equations in Example 5. *Consider the a scalar stiff problem recommended by Abdi and Conte [56].*

$$y' = \frac{y - cos(2\pi t)}{\sigma} - 2\pi sin(2\pi t), \quad y(0) = 1, 0 \leq t \leq 10.$$

The theoretical result is given as

$$y(t) = cos(2\pi t).$$

Using HOCELMs of order $p = 6$ on Example 5, we compare the results with the OSASM in [57] and BBDM in [58]. From Table 13, the proposed method have the lowest number of step (NS) and also improved maximum absolute error (MAE).

Consider a non-linear stiff first-order system of ODEs [56].

$$\begin{array}{ll} y_1' = -10004y_1 + 1000y_2^4, & y_1(0) = 1 \\ y_2' = y_1 - y_2\left(1 + y_2^3\right), & y_2(0) = 1 \end{array}, y(t) = \begin{pmatrix} e^{-4t} \\ e^{-t} \end{pmatrix}. \tag{40}$$

Using HOCELMs of order $p = 6$ on Example 5, we compare the results with the SDGLM in [56], OSASM in [57] and ode15s in [56]. However, it is clear From Table 14 that HOCELMs3 performs better than the compared methods. Here NFEVAL denote the number of function evaluations.

Table 4: Results for Example 5 $h = 0.09$.

	E2BD-Class2 k=5, p=8		E2BD-Class1 k=5, p=8		HOCELMs5 k=5, p=8	
t	$\lvert$ Error in y_1 $\rvert$	$\lvert$ Error in y_2 $\rvert$	$\lvert$ Error in y_1 $\rvert$	$\lvert$ Error in y_2 $\rvert$	$\lvert$ Error in y_1 $\rvert$	$\lvert$ Error in y_2 $\rvert$
4.9	0.1×10^{-10}	0.1×10^{-10}	0.1×10^{-10}	0.1×10^{-10}	0.1×10^{-16}	0.1×10^{-16}
9	0.1×10^{-12}	0.1×10^{-12}	0.1×10^{-12}	0.1×10^{-12}	0.4×10^{-19}	0.1×10^{-18}
13.5	0.1×10^{-15}	0.1×10^{-15}	0.8×10^{-11}	0.6×10^{-11}	0.5×10^{-22}	0.5×10^{-21}
18	0.1×10^{-17}	0.1×10^{-17}	0.1×10^{-11}	0.1×10^{-11}	0.4×10^{-23}	0.1×10^{-23}

Table 5: Results for Example 5 $h = 0.01$.

T	y_i	ECBBDFs4 [49] order 5	HOCELMs2 order 5	ECBBDFs5 [49] order 6	HOCELMs3 order 6
	y_1	0.128×10^{-16}	0.124×10^{-17}	0.407×10^{-17}	0.111×10^{-17}
1	y_2	0.117×10^{-15}	0.351×10^{-17}	0.222×10^{-17}	0.165×10^{-17}
	y_1	0.108×10^{-20}	0.203×10^{-21}	0.108×10^{-20}	0.474×10^{-21}
10	y_2	0.162×10^{-19}	0.142×10^{-20}	0.407×10^{-21}	0.115×10^{-20}
	y_1	0.724×10^{-24}	0.381×10^{-25}	0.207×10^{-25}	0.289×10^{-25}
20	y_2	0.529×10^{-23}	0.381×10^{-25}	0.290×10^{-25}	0.000

Table 6: Absolute errors for Example 5, using *error* $y_i = | y_i - y(t_i) |$, $i = 1, 2$, $h = 0.0001$.

T	y_i	Ismail etal [50]	ECBBDFs5 [49]	HOCELMs3
5	y_1	3.64920×10^{-7}	2.32895×10^{-7}	2.32895×10^{-7}
	y_2	7.670023×10^{-7}	5.027468×10^{-7}	5.027468×10^{-7}
10	y_1	2.45035×10^{-7}	1.70086×10^{-8}	1.70086×10^{-8}
	y_2	4.94295×10^{-7}	3.70518×10^{-8}	3.70518×10^{-8}

Table 7: Absolute errors for Example 5, using *error* $y_i = | y_i - y(t_i) |$, $i = 1, 2$, $h = 0.1$.

T	y_i	ECBBDFs4 [49] order 5	HOCELMs2 order 5	ECBBDFs5 [49] order 6	HOCELMs3 order 6
5	y_1	4.924191×10^{-5}	9.1734×10^{-4}	3.163426×10^{-4}	2.58944×10^{-7}
	y_2	5.274907×10^{-7}	5.16607×10^{-7}	6.6107427×10^{-7}	5.85455×10^{-7}
10	y_1	1.763163×10^{-4}	1.51971×10^{-6}	2.005234×10^{-4}	2.37629×10^{-7}
	y_2	1.252704×10^{-7}	3.78203×10^{-8}	1.373470×10^{-7}	5.02685×10^{-8}

Table 8: Numerical results for Example 5, $error\ y = \mid y - y(t) \mid$, $h = 0.01$.

T	HOCELMs2 *error y*	[51] *error y*	[52] *error y*
0.01	1.671×10^{-7}	2.829×10^{-7}	1.558×10^{-6}
0.02	1.721×10^{-8}	4.045×10^{-7}	2.399×10^{-6}
0.03	6.580×10^{-8}	4.472×10^{-7}	2.830×10^{-6}
0.04	1.724×10^{-8}	4.509×10^{-7}	3.020×10^{-6}
0.05	3.360×10^{-8}	4.356×10^{-7}	3.069×10^{-6}
0.06	1.469×10^{-8}	4.117×10^{-7}	3.034×10^{-6}
0.07	2.069×10^{-8}	3.846×10^{-7}	2.951×10^{-6}
0.08	1.216×10^{-8}	3.572×10^{-7}	2.840×10^{-6}
0.09	1.441×10^{-8}	3.307×10^{-7}	2.717×10^{-6}
0.10	1.007×10^{-8}	3.058×10^{-7}	2.588×10^{-6}

Table 9: Absolute errors for Example 5

Method	p	N	h	Error y_1	Error y_2
HOCELMs2	5	50	0.02	1.02×10^{-14}	8.55×10^{-15}
BVM [54]	5	50	0.02	3.20×10^{-12}	3.02×10^{-12}
GSDLMMEs [?]	5	50	0.02	3.07×10^{-13}	1.17×10^{-13}
HOCELMs2	5	1000	0.01	2.17×10^{-21}	5.85×10^{-17}
ECBBDFs4 [49]	5	1000	0.01	3.13×10^{-17}	3.45×10^{-13}
HOCELMs3	6	500	0.02	1.86×10^{-22}	2.05×10^{-18}
ECBBDFs5 [49]	6	500	0.02	1.33×10^{-20}	1.35×10^{-16}
Wu-Xia [53]	6	500	0.002	2.56×10^{-14}	8.02×10^{-8}
HOCELMs5	8	20	0.02	4.71×10^{-16}	2.77×10^{-16}
BBDF$_8$ [55]	8	20	0.02	4.56×10^{-13}	6.26×10^{-13}
Wu-Xia [53]	8	500	0.002	2.56×10^{-7}	8.01×10^{-8}
HOCELMs5	8	1000	0.01	2.89×10^{-23}	3.18×10^{-19}
BBDF$_8$ [55]	8	1000	0.01	6.64×10^{-20}	2.39×10^{-13}
Wu-Xia [53]	8	10000	0.001	5.54×10^{-16}	6.09×10^{-12}

Table 10: A comparison of method for Example 5.

h	GAMs $k=4$, $p=5$	rate	GAMs $k=6$, $p=7$	rate	GAMs $k=8$, $p=9$	rate
0.05	2.249×10^{-1}	-	1.266×10^{-1}	-	8.308×10^{-2}	-
0.025	4.413×10^{-2}	3.31	1.449×10^{-2}	2.40	8.392×10^{-3}	2.92
0.0125	6.490×10^{-3}	3.21	1.508×10^{-3}	5.71	9.097×10^{-4}	7.75
0.00625	8.859×10^{-4}	5.05	1.114×10^{-4}	7.27	2.749×10^{-5}	7.37
0.003125	9.881×10^{-5}	5.59	4.877×10^{-6}	7.46	5.694×10^{-7}	9.16
h	HOCELMs2 $k=2$, $p=5$	rate	HOCELMs3 $k=3$, $p=6$	rate	HOCELMs4 $k=4$, $p=7$	rate
0.05	3.102×10^{-2}	-	2.460×10^{-2}	-	1.051×10^{-2}	-
0.025	3.614×10^{-3}	3.10	1.800×10^{-3}	3.8	5.833×10^{-4}	4.17
0.0125	1.487×10^{-4}	4.60	4.537×10^{-5}	5.31	1.032×10^{-5}	5.82
0.00625	4.614×10^{-6}	5.01	7.391×10^{-7}	5.94	7.470×10^{-8}	7.11
0.003125	1.412×10^{-7}	5.03	1.146×10^{-8}	6.01	4.773×10^{-10}	7.29

Table 11: A comparison of method for Example 5.

h	HOCELMs5 $k=5$, $p=8$	rate	HOCELMs6 $k=6$, $p=9$	rate	HOCELMs7 $k=7$, $p=10$	rate
0.05	5.781×10^{-3}	-	3.620×10^{-2}	-	6.704×10^{-3}	-
0.025	1.508×10^{-4}	5.26	7.200×10^{-4}	5.65	4.402×10^{-5}	7.25
0.0125	1.725×10^{-6}	6.45	3.142×10^{-6}	7.84	2.253×10^{-7}	7.61
0.00625	5.906×10^{-9}	8.19	5.847×10^{-9}	9.07	2.458×10^{-10}	9.84
0.003125	1.712×10^{-11}	8.43	9.873×10^{-12}	9.21	2.164×10^{-13}	10.15
h	HOCELMs5 $k=8$, $p=11$	rate	HOCELMs6 $k=9$, $p=12$	rate	HOCELMs7 $k=10$, $p=13$	rate
0.05	1.600×10^{-3}	-	1.254×10^{-3}	-	1.004×10^{-3}	-
0.025	6.577×10^{-6}	7.91	8.238×10^{-6}	7.26	2.832×10^{-6}	8.47
0.0125	1.920×10^{-8}	8.42	6.013×10^{-9}	10.42	1.195×10^{-9}	11.21
0.00625	1.246×10^{-11}	10.59	1.136×10^{-12}	12.37	1.735×10^{-13}	12.75
0.003125	6.754×10^{-15}	10.85	2.587×10^{-16}	12.10	2.104×10^{-17}	13.01

Table 12: Continuation of Table 11.

h	HOCELMs5 $k=11$, $p=14$	rate
0.05	1.000×10^{-3}	-
0.025	3.284×10^{-6}	8.25
0.0125	9.022×10^{-10}	11.83
0.00625	4.760×10^{-14}	14.21
0.003125	3.431×10^{-18}	13.76

Table 13: A comparison of methods for Example 5 with MAXAE=max $\mid y(t) - y_h \mid$, $h = 10^{-2}$, $\sigma = 10^{-3}$, $0 \leq t \leq 10$.

Method	p	ATOL	NS	MAXAE	ATOL	NS	MAXAE
HOCELMs3	6	10^{-4}	48	2.1296×10^{-8}	10^{-6}	144	7.2218×10^{-10}
OSASM	6	10^{-4}	64	3.2142×10^{-7}	10^{-6}	159	6.1606×10^{-9}
BBDM	6	10^{-4}	116	1.5439×10^{-6}	10^{-6}	398	3.1184×10^{-8}

Table 14: A comparison of methods for Example 5, MAXRE=$\max_{1<i<2} \mid (y_i(t) - y_{i,h})/y_i(t) \mid$, $h = 10^{-3}$.

ATOL	Method	p	NS	NFEVAL	MAXRE
10^{-6}	HOCELMs3	6	20	120	2.1296×10^{-12}
10^{-6}	OSASM	6	25	125	3.8579×10^{-10}
10^{-6}	BBDM	6	23	136	5.0000×10^{-5}
10^{-6}	ode15s		88	122	6.6300×10^{-6}
10^{-8}	HOCELMs3	6	37	222	2.1296×10^{-13}
10^{-8}	OSASM	6	45	225	4.6893×10^{-11}
10^{-8}	BBDM	6	37	230	3.8400×10^{-6}
10^{-8}	ode15s		159	224	2.0000×10^{-8}

6 Concluding Remarks

A new schemes of HOCELMs (7) for integrating stiff system of ODEs has been considered. The continuous schemes is self-starting and possesses good accuracy. This continuous methods is A-stable up to order fourteen (see Table 1). The well-known IVPs considered indicate that the new continuous schemes performs better in term of accuracy than some known schemes in the literature. (see Tables 13 to 9 , Fig 2).

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