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The Double XRAMA Distribution: Theory and Applications

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Abstract

In this paper, a new distribution is proposed by a mixture of two distributions; Exponential and Exponential-Rama the proposed distribution is referred to as the Double XRama distribution. It is flexible in modeling lifetime data. The properties of the XRama distribution were derived and an analysis of the behaviour was conducted. The mathematical properties which include moments, the shape of the distribution, Quantile function, hazard function, survival function, stochastic ordering, mean deviation, Bonferroni and Lorenz curve, order statistic, and Renyi entropy have been studied. From the results, the proposed model competes favorably among the members of the XRama class of distributions.

1 Introduction

One parameter distributions in the class of Lindley which is commonly derived from a component mixture of two or more heavy-tailed distributions are littered in the statistical literature. Among them is the Sujatha distribution by [1]. Modification of probability models since its inception has expanded the field of probability distribution hence creating more distributions that are useful

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in explaining complex situations that classical distributions such as Normal distribution, Ch-Square, Gamma, Weibull, and Exponential could not explain. Today, the literature is rich with models that are suited to engineering, medical and biomedical, stock and financial records, agriculture, and commerce applications, finance, and epidemiology. Since the work of [2], who first used mixing proportion of Gamma and Exponential distributions, the statistical literature has been flooded with research items with this method. Some of the articles extended the two-component mixture to three while others are four and five components. One, interesting thing about these classes of distributions is the ease of deriving them and often some of them have closed-form quantile functions lending them to many applications and aiding the generation of data. Very popular among these classes is the work by [3] who combine Exponential with scale parameter θ and Gamma distribution with shape parameter 2 and scale parameter θ. The mixing proportion $p = \frac{\theta^2}{\theta^2+1}$. This distribution has attracted many modifications such as Power Shanker by [4] which handles polynomial data. It is a two-parameter distribution with an additional parameter α which accounts for the shape of the distribution. There is also a weighted power Shanker by [5]. This model attached weight to the random variable that assumes the power Shanker distribution. The new distribution has three parameters, the third c being the weight parameter. [6] proposed the extended Lomax distribution named McDonald distribution having five parameters hence exhibiting some complexities in mathematical manipulations. [7] proposed three heavy-tailed models based on the Student's t distribution with its scale parameter randomized that model financial data. [8] introduced and studied a new family of continuous distributions called Kumaraswamy Weibull-generated family of distributions which is an extension of the Weibull-G family of probability distributions proposed by [9]. [2] was the first to explore a two-component distribution to obtain a one-parameter distribution called Lindley distribution using Exponential distribution with scale parameter θ and a Gamma distribution having shape parameter 2 and scale parameter θ with mixing proportion $p = \frac{\theta}{\theta+1}$. [10] proposed the alpha power transformed power Lindley distribution, a generalization of the power Lindley distribution that provides a better fit. An extension of the Lindley distribution which offers a more flexible model for lifetime data was introduced by [11]. [12] derived a one-parameter distribution called Pranav distribution from two distributions namely Exponential distribution with scale parameter θ and Gamma distribution having shape parameter 4 and scale parameter θ. [13] introduced a two-parameter lifetime distribution named, 'Shukla distribution' which includes several one-parameter lifetime distributions. A new one-parameter lifetime distribution named Sujatha Distribution with an increasing hazard rate for modeling lifetime data was suggested by [1]. [14] studied a one-parameter lifetime distribution named Ishita distribution based on a two-component mixture of an Exponential distribution having a shape parameter θ and a Gamma distribution having a shape parameter 3 and scale parameter θ with mixing proportion $\frac{\theta^3}{\theta^3+2}$. [15] studied a one-parameter lifetime distribution named Akash distribution based on a two-component mixture of an Exponential distribution having a shape parameter θ and a Gamma distribution having a shape parameter 2 and scale parameter θ with mixing proportion $\frac{\theta}{\theta+1}$. [16] studied a one-parameter lifetime distribution named Rani distribution based on a two-component mixture of an Exponential distribution having a shape parameter θ and a Gamma distribution having a shape parameter 5 and scale parameter θ with mixing proportion $\frac{\theta^5}{\theta^5+24}$. [17] studied a one-parameter lifetime distribution named Rama distribution based on a two-component mixture of an Exponential distribution having a shape parameter θ and a Gamma distribution having a shape parameter 4 and scale parameter θ with mixing proportion $\frac{\theta^3}{\theta^3+6}$. [18] studied a one-parameter lifetime distribution named XGamma distribution based on a two-component mixture of an Exponential distribution having a shape parameter θ and a Gamma distribution having a shape parameter 3 and scale parameter θ with mixing proportion $\frac{\theta}{\theta+1}$. [19] studied a one-parameter lifetime distribution named Aradhana distribution based on a two-component mixture of an Exponential distribution having a shape parameter θ and a Gamma distribution having a shape parameter 2 and scale parameter θ with mixing proportion $\frac{1}{\theta+1}$. [3] studied a one-parameter lifetime distribution named Shanker based on a two-component mixture of an Exponential distribution having a shape [20] parameter θ and a Gamma distribution having a shape parameter 2 and scale parameter θ with mixing proportion $\frac{\theta^2}{\theta^2+1}$. Other related works are [13], [21], [19], [22], [23], [24], [12], [25], [26], [27], [28], [29], [30], [31], [32], [33], [34], [35], [36], [37], [38], [27], [39], [40], [41] and [42].

The rest of this article is in the following sequence of arrangement. In Section 2, the proposed model is discussed, and Section 3 is dedicated to the derivation of the important characteristics. In Section 4, applications to lifetime data are done while a discussion of results is done in Section 5. The article is concluded in Section 6.

2 The Proposed Method

The work of [24] extended the Rama distribution by providing a mixture of the exponential and Rama distributions. The extension is referred to as the XRama distribution.

The new proposition in this study called the Double XRama distribution is obtained similarly by combining the exponential distribution with scale parameter θ and the XRama distribution. The mixing proportion is the same as that of the XRama distribution.

Let $X \sim$ Double XRama (θ), then the pdf and cdf are respectively

$$f(x) = \frac{\theta^4}{(\theta^3+6)^3}\left[\theta^6 + 18\theta^3 + 108 + 36x^3\right] e^{-\theta x}; \quad x > 0, \quad \theta > 0 \tag{1}$$

and

$$F(x) = 1 - \left\{1 + \frac{1}{(\theta^3+6)^3} 36x\theta\left[6 + 3x\theta + \theta^2 x^2\right]\right\} e^{-\theta x}. \tag{2}$$

The survival and hazard rate functions are respectively

$$S(x) = \left\{1 + \frac{1}{(\theta^3+6)^3} 36x\theta\left[6 + 3x\theta + \theta^2 x^2\right]\right\} e^{-\theta x} \tag{3}$$

and

$$hrf(x) = \frac{\theta^4\left(108 + 36x^3 + 18\theta^3 + \theta^6\right)}{216x\theta + 108x^2\theta^2 + 36x^3\theta^3 + (6+\theta^3)^3}. \tag{4}$$

The limiting values of the Double XRama hazard function are

$$\lim_{x\to 0} hrf(x) = \frac{\theta^4\left(108+18\theta^3+\theta^6\right)}{(6+\theta^3)^3} \quad \text{and} \quad \lim_{x\to\infty} hrf(x) = \theta.$$

This limiting behavior of the hazard function at a glance shows that it is a strictly increasing function.

The plots are displayed in the Figures 1, 2, 5, and 6 below

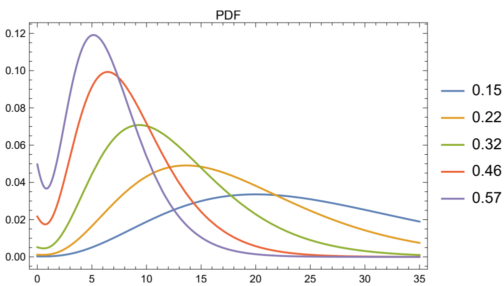

Figure 1: pdf of Double XRama distribution.

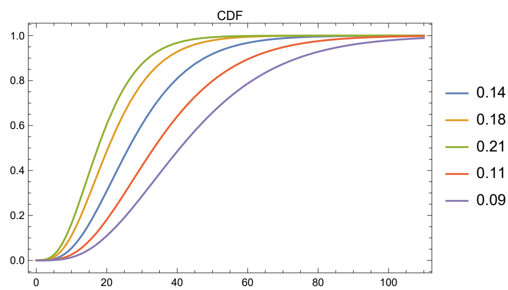

Figure 2: cdf of Double XRama distribution.

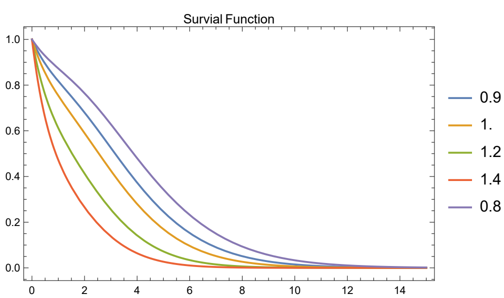

Figure 3: Survival function of Double XRama distribution.

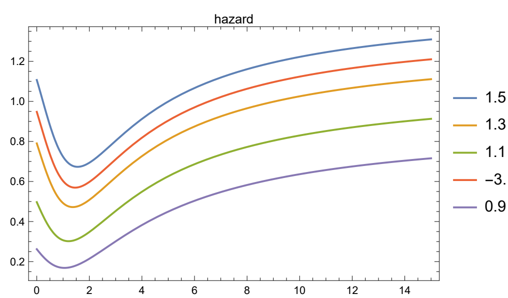

Figure 4: Hazard function of Double XRama distribution.

3 Distributional Properties of Double XRama Distribution

In this section, we derive the basic distributional properties of the proposed model.

3.1 Moments of Double XRama

Let X be a continuous random variable $\sim f(x)$, the r^{th} crude moment is generally defined as

$$\mu_r^{'} = \mathbb{E}(X) = \int_0^\infty x^r f(x)\, dx.$$

Given that

$$f(x) = \frac{\theta^4}{(\theta^3+6)^3}\left[\theta^6 + 18\theta^3 + 108 + 36x^3\right] e^{-\theta x}; \quad x > 0, \quad \theta > 0. \tag{5}$$

The r^{th} moment of the Double XRama is given as

$$\mu_r^{'} = \frac{r\theta^{-r}\left(36r\left(11 + r\left(6+r\right)\right) + \left(6+\theta^3\right)^3\right)\Gamma_r}{\left(6+\theta^3\right)^3}; \quad r = 1, 2, \cdots \tag{6}$$

The first, second, third, and fourth crude moments are obtained by replacing r with $1, 2, 3$ and 4 in the r^{th} crude moment. That is;

$$\begin{aligned} \mu &= \frac{\theta^4\left(\frac{864}{\theta^5} + \frac{108}{\theta^2} + 18\theta + \theta^4\right)}{\left(6+\theta^3\right)^3}, \quad \mu_2^{'} = \frac{\theta^4\left(36 + \frac{4320}{\theta^6} + \frac{216}{\theta^3} + 2\theta^3\right)}{\left(6+\theta^3\right)^3}, \\ \mu_3^{'} &= \frac{6\left(4320 + 108\theta^3 + 18\theta^6 + \theta^9\right)}{\theta^3\left(6+\theta^3\right)^3}, \quad \mu_4^{'} = \frac{24\left(7560 + 108\theta^3 + 18\theta^6 + \theta^9\right)}{\theta^4\left(6+\theta^3\right)^3}. \end{aligned} \tag{7}$$

3.2 The variance of Double XRama

The variance of a random variable is one important measure of dispersion in a set of observations. Its generic definition is $\sigma^2 = EX^2 - (EX)^2$.

But

$$E(X)=\frac{\theta^4\left(\frac{864}{\theta^5}+\frac{108}{\theta^2}+18\theta+\theta^4\right)}{\left(6+\theta^3\right)^3};\quad E(X^2)=\frac{\theta^4\left(36+\frac{4320}{\theta^6}+\frac{216}{\theta^3}+2\theta^3\right)}{\left(6+\theta^3\right)^3}. \tag{8}$$

Therefore;

$$\sigma^2=\frac{\theta^4\left(36+\frac{43206}{\theta^6}+\frac{216}{\theta^3}+2\theta^3\right)}{\left(6+\theta^3\right)^3}-\left[\frac{\theta^4\left(\frac{864}{\theta^5}+\frac{108}{\theta^2}+18\theta+\theta^4\right)}{\left(6+\theta^3\right)^3}\right]^2. \tag{9}$$

Further simplification gives

$$\sigma^2=\frac{-\theta^2\left(864+108\theta^3+18\theta^6+\theta^9\right)^2+24\left(6+\theta^3\right)^3\left(7560+108\theta^3+18\theta^6+\theta^9\right)}{\theta^4\left(6+\theta^3\right)^6}. \tag{10}$$

3.3 Skewness of Double XRama

Skewness is a measure of departure from normality. Depending on the index, a distribution is either left-skewed or right-skewed.

$$Skewness(x)=\frac{6\left(4320+108\theta^3+18\theta^6+\theta^9\right)}{\theta^3\left(6+\theta^3\right)^3\left(\frac{-\theta^2(864+108\theta^3+\theta^9)^2+24(6+\theta^3)^3(7560+108\theta^3+18\theta^6+\theta^9)}{\theta^4(6+\theta^3)^6}\right)^{\frac{3}{2}}}. \tag{11}$$

3.4 Kurtosis of Double XRama

$$Kurtosis=\frac{24\theta^4\left(6+\theta^3\right)^3\left(7560+108\theta^3+18\theta^6+\theta^9\right)}{\left(\theta^2\left(864+108\theta^3+18\theta^6+\theta^9\right)^2-24\left(6+\theta^3\right)^3\left(7560+108\theta^3+18\theta^6+\theta^9\right)\right)^2}. \tag{12}$$

3.5 Coefficient of Variation of Double XRama

$$C.V = \frac{\theta\left(6+\theta^3\right)^3\sqrt{\frac{-\theta^2(864+108\theta^3+18\theta^6+\theta^9)^2+24(6+\theta^3)^3(7560+108\theta^3+18\theta^6+\theta^9)}{\theta^4(6+\theta^3)^6}}}{(864+108\theta^3+18\theta^6+\theta^9)}. \quad (13)$$

3.6 Moment Generating Function of Double XRama

Then the Moment Generating Function (MGF)

$$E(e^{tx}) = \int_0^{\infty} e^{tx} f(x)dx \quad (14)$$

$$E(e^{tx}) = \int_0^{\infty} e^{tx}\left[\frac{\theta^4}{(\theta^3+6)^3}\left[\theta^6+18\theta^3+108+36x^3\right]e^{-\theta x}\right]dx \quad (15)$$

hence

$$E(e^{tx}) = \frac{\theta^4\left(216-(x-\theta)^3\left(108+18\theta^6+\theta^6\right)\right)}{(x-\theta)^4\left(6+\theta^3\right)^3}. \quad (16)$$

3.7 Characteristics Function of Double XRama

$$E(e^{itx}) = \int_0^{\infty} e^{itx}\left[\frac{\theta^4}{(\theta^3+6)^3}\left[\theta^6+18\theta^3+108+36x^3\right]e^{-\theta x}\right]dx \quad (17)$$

hence

$$E(e^{itx}) = \frac{\theta^4\left(216-(ix-\theta)^3\left(108+18\theta^6+\theta^6\right)\right)}{(ix-\theta)^4\left(6+\theta^3\right)^3}. \quad (18)$$

3.8 Odd Function for Double XRama

$$OddFunction = -1 + \frac{e^x\theta}{1+\frac{(36x\theta(6+3x\theta+x^2\theta^2))}{(6+\theta^3)^3}}. \quad (19)$$

3.9 Stress Strength Reliability of Double XRama

Let $X \sim DXR(\theta_1)$ be the stress variable and $Y \sim DXR(\theta_2)$ be the strength variable. Then, the reliability of the system can be expressed as

$$\begin{aligned} R(X<Y) = & \frac{1}{(6+\theta_1^3)(\theta_1+\theta_2)^6}(\theta_2(\theta_1+\theta_2)^2(\theta_1(864+108\theta_1^3+18\theta_1^6+\theta_1^9) \\ & +3\theta_1^2(12+\theta_1^3)(36+6\theta_1^3+\theta_1^6)\theta_2+3\theta_1(288+108\theta_1^3+18\theta_1^6+\theta_1^9)\theta_2^2 \\ & +(6+\theta_1^3)^3\theta_2^3)-\frac{1}{6+\theta_2^3)^3}36x\theta_2\theta_1^4(6\theta_1^2(6+\theta_1^3)^3 \\ & +3\theta_1(1728+11\theta_1^3(108+18\theta_1^3+\theta_1^6))\theta_2+2(4104 \\ & +37\theta_1^3(108+18\theta_1^3+\theta_1^6))\theta_2^2 \\ & +84\theta_1^2(108+18\theta_1^3+\theta_1^6)\theta_1^3+48\theta_1(108+18\theta_1^3+\theta_1^6)\theta_2^4 \\ & +11(108+18\theta_1^3+\theta_1^6)\theta_2^5)). \end{aligned} \tag{20}$$

3.10 Mean Residual Function for Double XRama

$$\frac{-864-\theta\left(648x+216x^2\theta+36\left(3+x\right)\theta^2+18\theta^5+\theta^8\right)}{\theta\left(-216x\theta-108x^2\theta^2-36x^3\theta^3-\left(6-\theta^3\right)^3\right)}. \tag{21}$$

3.11 Bonferroni and Lorenz curve

Bonferroni and Lorenz's curve has applications not only in Economics but also in reliability analysis, demography, medicine, and insurance. Bonferroni curve is defined as:

$$B(p) = \frac{1}{p\mu}\int_0^q xf(x)dx = \frac{1}{p\left(\frac{864}{\theta^5} + \frac{108}{\theta^2} + 18\theta + \theta^4\right)} \times \left(\theta^4\gamma(2,q) + 18\theta\gamma(2,q) + \frac{108\gamma(2,q)}{\theta^2} + \frac{36\gamma(2,q)}{\theta^6}\right),$$
$$L(p) = \frac{1}{\mu}\int_0^q xf(x)dx = \frac{1}{\left(\frac{864}{\theta^5} + \frac{108}{\theta^2} + 18\theta + \theta^4\right)} \times \left(\theta^4\gamma(2,q) + 18\theta\gamma(2,q) + \frac{108\gamma(2,q)}{\theta^2} + \frac{36\gamma(2,q)}{\theta^6}\right). \tag{22}$$

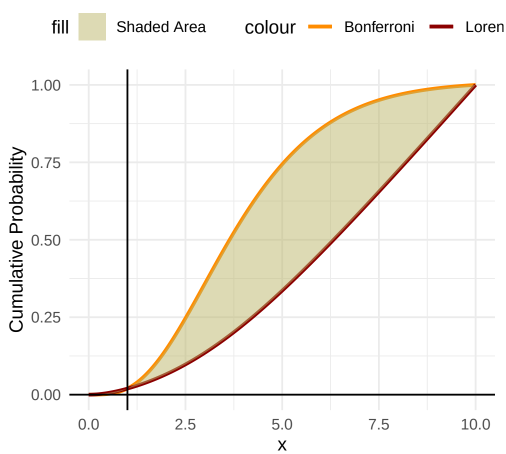

Figure 5: Bonferroni and Lorenz Curve of Double XRama distribution.

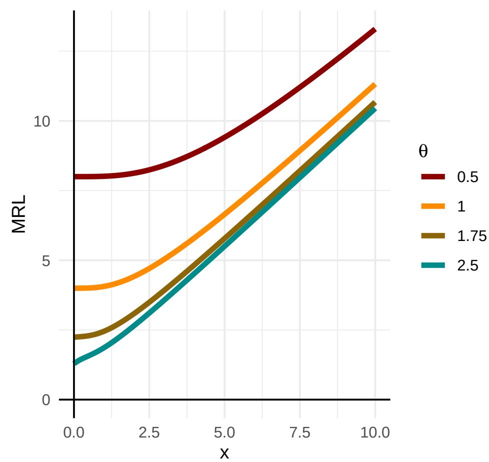

Figure 6: Mean Residual Function of Double XRama distribution.

3.12 Stochastic ordering of Double XRama distribution

A random variable X is said to be smaller than another random variable Y in the stochastic order $(X \leq_{st} Y)$ if $F_Y(x) \geq F_Y(x)\ \forall_x$; Hazard order $(X \leq_{hr} Y)$ if $h_X(x) \geq h_y(x)\ \forall_x$; Mean residual life order $(X \leq_{mrl} Y)$ if $m_X(x) \geq m_Y(x)\ \forall_x$; Likelihood ratio order $(X \leq_{lr} Y)$ if $\frac{f_X(x)}{f_Y(y)}$ decreases in x.

This implies that $X \leq_{lr} Y \Rightarrow X \leq_{hr} Y \Rightarrow X \leq_{st} Y \Rightarrow X \leq_{mrl} Y$.

Theorem 3.1. *Let $X \sim DoubleXRama(\theta_1)$ and $Y \sim DoubleXRama(\theta_2)$. If $\theta_1 \geq \theta_2$, then $X \leq_{lr} Y$ hence $X \leq_{lr} Y$ hence $X \leq_{hr} Y$, $X \leq_{mrl} Y$ and $X \leq_{st} Y$.*

$$\begin{aligned} \frac{f_x(x)}{f_y(x)} &= \frac{\frac{\theta_1^4}{(\theta_1^3+6)^3}\left[\theta_1^6 + 18\theta_1^3 + 108 + 36x^3\right] e^{-\theta_1 x}}{\frac{\theta_2^4}{(\theta_2^3+6)^3}\left[\theta_2^6 + 18\theta_2^3 + 108 + 36x^3\right] e^{-\theta_2 x}} \\ &= \frac{\theta_1^4(\theta_2^3+6)^3(\theta_1^6 + 18\theta_1^3 + 36\theta_1^3 + 108)}{\theta_2^4(\theta_1^3+6)^3(\theta_2^6 + 18\theta_2^3 + 36\theta_2^3 + 108)} e^{(\theta_2-\theta_1)x}. \end{aligned} \tag{23}$$

Taking a natural log of the ratio will yield

$$ln\frac{f_x(x)}{f_y(x)} = ln\frac{\theta_1^4(\theta_2^3+6)^3}{\theta_2^4(\theta_1^3+6)^3} + ln\frac{(\theta_1^6 + 18\theta_1^3 + 36\theta_1^3 + 108)}{(\theta_2^6 + 18\theta_2^3 + 36\theta_2^3 + 108)} + (\theta_2 - \theta_1)x. \tag{24}$$

Differentiating the natural log of the ratio w.r.t x will result

$$= -\theta_1 + \theta_2 - \frac{108x^2\left(18\theta_1^3 + \theta_1^6 - \theta_2^3\left(18 + \theta_2^3\right)\right)}{\left(36\left(3+x^3\right) + 18\theta_1^3 + \theta_1^6\right)\left(36\left(3+x^3\right) + 18\theta_2^3 + \theta_2^6\right)}. \tag{25}$$

If $\theta_2 \geq \theta_1$, $\frac{d}{dx} ln\frac{f_x(x)}{f_y(x)} \leq 0$, and $\frac{f_x(x,\theta_1)}{f_y(x,\theta_2)}$ is (CHECK!!!) in x.

3.13 Maximum Likelihood Function

Let $(X_1, X_2, .., X_n)$ be random variables of double Double XRama. Then the Maximum Likelihood Function is given as

$$\begin{aligned} \ell(f_{(x;\theta)}) &= \prod_{i=1}^{n} \frac{\theta^4}{(\theta^3+6)^3}\left[\theta^6 + 18\theta^3 + 108 + 36x^3\right] e^{-\theta x} \\ &= \frac{\theta^{4n}}{(\theta^3+6)^{3n}} \prod_{i=1}^{n}\left[\theta^6 + 18\theta^3 + 108 + 36x^3\right] e^{-\theta\sum_{i=0}^{n} x_i}. \end{aligned} \tag{26}$$

Taking the log of the above function

$$\psi = 4nln\theta - 3nln(\theta^3+6) - \theta\sum_{i=0}^{n} x_i + \sum_{i=0}^{n}\left(\theta^6 + 18\theta^3 + 108 + 36x^3\right).$$

Differentiating ψ w.r.t θ and equating to zero

$$\frac{d\psi}{d\theta} = \frac{4n}{\theta} - \frac{9\theta^2 n}{(\theta^3+6)} - \sum_{i=0}^{n} x_i + \frac{6\theta^5 + 53\theta^3}{\theta^6 + 18\theta^3 + 108 + 36x^3}. \tag{27}$$

3.14 Renyi Entropy

Entropy is the quantity of uncertainty or randomness in a system. It is an information measure for non-negative $\omega \neq 1$. The Reny Entropy for Double XRama distributed random variable X is

$$\begin{aligned} R_w(x) &= \frac{1}{1-\omega} log \int_0^\infty f_(w)^\omega dx \\ &= \frac{1}{1-\omega} log \int_0^\infty \left\{ \frac{\theta^4}{(\theta^3+6)^3} \left[\theta^6 + 18\theta^3 + 108 + 36x^3\right] e^{-\theta x} \right\}^\omega dx \quad (28) \\ &= \frac{1}{1-\omega} log \frac{\theta^{4\omega}}{(\theta^3+6)^{3\omega}} \int_0^\infty \left[\theta^6 + 18\theta^3 + 108 + 36x^3\right]^\omega e^{-\theta\omega x} dx. \end{aligned}$$

Let $(\theta^6 + 18\theta^3 + 108) = a$, $36x^3 = b$ and $\omega = n$. Then using binomial expansion, we have;

$$\begin{aligned} R_w(x) &= \frac{1}{1-\omega} log \frac{\theta^{4\omega}}{(\theta^3+6)^{3\omega}} \int_0^\infty \sum_{j=0}^{\omega} \binom{\omega}{j} (\theta^6 + 18\theta^3 + 108)^j (36x^3)^{\omega-j} e^{-\theta\omega x} dx \\ &= \frac{1}{1-\omega} log \frac{\theta^{4\omega}}{(\theta^3+6)^{3\omega}} \sum_{j=0}^{\omega} \binom{\omega}{j} (\theta^6 + 18\theta^3 + 108)^j 36^{\omega-j} \int_0^\infty x^{3\omega-3j} e^{-\theta\omega x} \\ &= \frac{1}{1-\omega} log \frac{\theta^{4\omega}}{(\theta^3+6)^{3\omega}} \sum_{j=0}^{\omega} \binom{\omega}{j} (\theta^6 + 18\theta^3 + 108)^j 36^{\omega-j} \frac{\Gamma_{3\omega-3j+1}}{(\theta\omega)^{3\omega-3j+1}} \\ &= \frac{1}{1-\omega} log \frac{\theta^{\omega_3 j-1}}{(\theta^3+6)^{3\omega}} \sum_{j=0}^{\omega} \binom{\omega}{j} (\theta^6 + 18\theta^3 + 108)^j 36^{\omega-j} \frac{(3\omega-3j)!}{\omega^{3\omega-3j+1}}. \end{aligned}$$

3.15 Order Statistics

Suppose $X_1, X_2, ..., X_n$ is a random sample of X_r; $r = (1, 2, ..., n)$ are the r^{th} order statistics obtained by arranging X_r in ascending order of magnitude $\ni X_1 \leq X_2 \leq$

... $\leq X_r$ where X_1 is the smallest of all variable and X_r is the largest of all variable, then the pdf of the r^{th} order statistics is given by

$$f_{r:n}(x) = \frac{n!}{(r-1)!(n-r)!} f(x)(F(x))^{r-1}(1-F(x))^{n-r}$$

the pdf of r^{th} order statistics of Double XRama is given as:

$$\begin{aligned} f_{r:n}(X) &= \frac{n!}{(r-1)!(n-r)!}\frac{\theta^4}{(\theta^3+6)^3}(\theta^6+18\theta^3+108+36x^3)e^{-\theta x} \\ &\times \left(1-\left\{1+\frac{1}{(\theta^3+6)^3}36x\theta(6+3x\theta+\theta^2x^2)\right\}e^{-\theta x}\right)^{r-1} \\ &\times \left(\left\{1+\frac{1}{(\theta^3+6)^3}36x\theta(6+3x\theta+\theta^2x^2)\right\}e^{-\theta x}\right)^{n-r} \end{aligned} \tag{29}$$

pdf of minimum order is obtained by setting $r = 1$

$$f_{1:n}(x) = \frac{n\theta^4}{(\theta^3+6)^3}(\theta^6+18\theta^3+108+36x^3)e^{-\theta x} \tag{30}$$

$$\times \left(\left\{1+\frac{1}{(\theta^3+6)^3}36x\theta(6+3x\theta+\theta^2x^2)\right\}e^{-\theta x}\right)^{n-1} \tag{31}$$

pdf of maximum order is obtained by setting $r = n$

$$f_{n:n}(x) = \frac{n\theta^4}{(\theta^3+6)^3}(\theta^6+18\theta^3+108+36x^3)e^{-\theta x} \tag{32}$$

$$\times \left(1-\left\{1+\frac{1}{(\theta^3+6)^3}36x\theta(6+3x\theta+\theta^2x^2)\right\}e^{-\theta x}\right)^{n-1}. \tag{33}$$

4 Applications

In this section, we present the applications of the proposed model to two-lifetime datasets.

The first application is on rainfall data reported at the Los Angeles Civic Center from 1943 to 2018 in March studied by [29].

Table 1: Rainfall data reported at the Los Angeles Civic Center from 1943 to 2018 in March.

4.55	2.47	3.43	3.66	0.79	3.07	1.40	0.87	0.44	6.14	0.48	2.99	0.56	1.02	5.30	0.31	0.57
1.10	2.78	1.79	2.49	0.53	2.5	3.34	1.49	2.36	0.53	2.70	3.78	4.83	1.81	1.89	8.02	5.85
4.79	4.10	3.54	8.37	0.28	1.29	5.27	0.95	0.26	0.81	0.17	5.92	7.12	2.74	1.86	6.98	2.16
4.06	1.24	2.82	1.17	0.32	4.32	1.47	2.14	2.87	0.05	0.01	0.35	0.48	3.96	1.75	0.54	1.18
0.87	1.60	0.09	2.69													

Table 2: Metrics of Model performance and fitness for the rainfall data.

Dist.	NLL	AIC	CAIC	BIC	HQIC	W^*	A^*	K-S	P-value	$\hat{\theta}$	std.err
DXR	135.42	272.841	272.898	275.117	273.747	0.291	0.208	0.054	0.9842	1.0984	0.0532
DOJE	153.92	309.830	309.887	312.107	310.737	0.345	1.971	0.184	0.0150	2.3715	0.0823
Lindley	135.82	273.630	273.687	275.907	274.537	0.026	0.170	0.080	0.7469	0.6547	0.5683
Akash	136.42	274.838	274.895	277.114	275.744	0.031	0.210	0.089	0.6172	0.9650	0.0633
Pranav	137.63	277.267	277.324	279.543	278.173	0.068	0.433	0.083	0.7073	1.2761	0.0592
CJ	136.3	274.596	274.653	276.872	275.502	0.030	0.204	0.090	0.5999	0.9463	0.0718
Rama	138.63	279.261	279.319	281.538	280.168	0.077	0.483	0.104	0.4206	1.3029	0.0689
XRama	136	273.998	274.055	276.275	274.905	0.047	0.309	0.062	0.9426	1.1725	0.0590

The second application is on vinyl chloride data (g/L) from ground-water monitoring wells that are located in clean-up-gradient areas studied by [43] and [28].

Table 3: vinyl chloride data (g/L) from ground-water monitoring wells.

5.1	1.2	1.3	0.6	0.5	2.4	0.5	1.1	8.0	0.8	0.4	0.6	0.9	0.4	2.0	0.5	5.3
3.2	2.7	2.9	2.5	2.3	1.0	0.2	0.1	0.1	1.8	0.9	2.0	4.0	6.8	1.2	0.4	0.2

Dist.	NLL	AIC	CAIC	BIC	HQIC	W^*	A^*	K-S	P-value	$\hat{\theta}$	std.err
DXR	56.49	114.978	115.103	116.505	115.499	0.084	0.547	0.104	0.8551	1.2419	0.8805
DOJE	67.888	137.752	137.877	139.278	138.272	0.384	2.176	0.225	0.0642	2.6644	0.1268
Lindley	56.3	114.607	114.732	116.134	115.128	0.063	0.405	0.133	0.5878	0.8238	0.1054
Akash	57.57	117.149	117.274	118.676	117.670	0.099	0.630	0.156	0.3758	1.1656	0.1126
Pranav	58.34	118.672	118.797	120.198	119.192	0.136	0.847	0.146	0.4606	1.4664	0.0980
CJ	57.93	117.854	117.979	119.38	118.374	0.103	0.655	0.178	0.2303	1.1645	0.1321
Rama	59.34	120.683	120.808	122.210	121.204	0.154	0.952	0.177	0.2383	1.5310	0.1176
XRama	57.28	116.556	116.681	118.082	117.077	0.111	0.703	0.127	0.6465	1.3490	0.0984

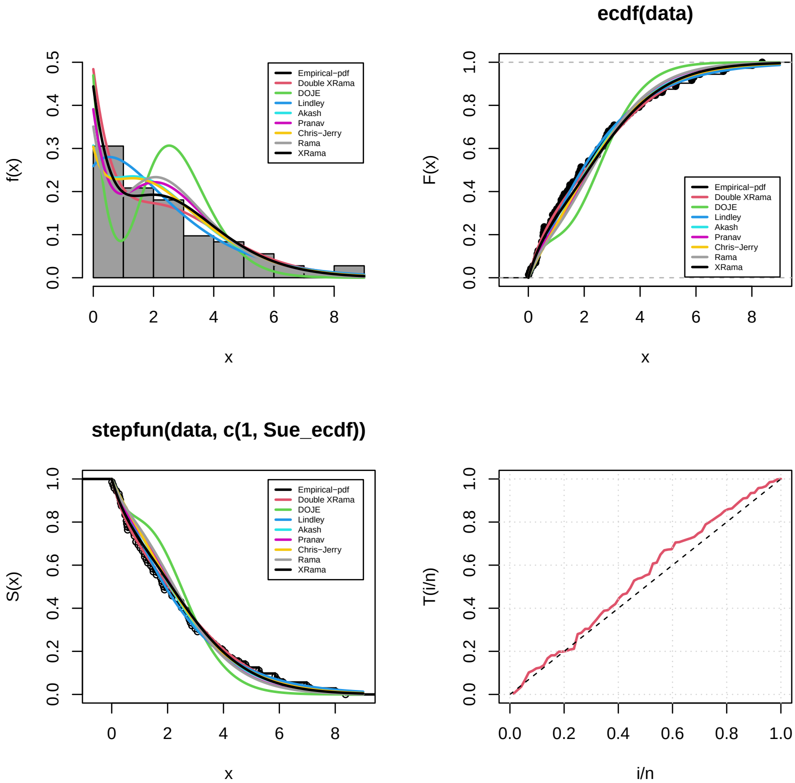

Figure 7: Density, cdf, survival and TTT plots of rainfall data.

5 Discussion of Results

The mixture of two distributions namely exponential and XRama birthed a member in the Lindley class of distribution, now known as Double XRama distribution. The distribution statistical properties namely moments variance, skewness, kurtosis, coefficient of variation, Moment generating function, characteristics function, odd function, and Stress-strength reliability function were derived and discussed. Analysis of lifetime data using data namely rainfall at Los Angeles Civic Center from 1943 to 2018 in March was conducted. We compared the

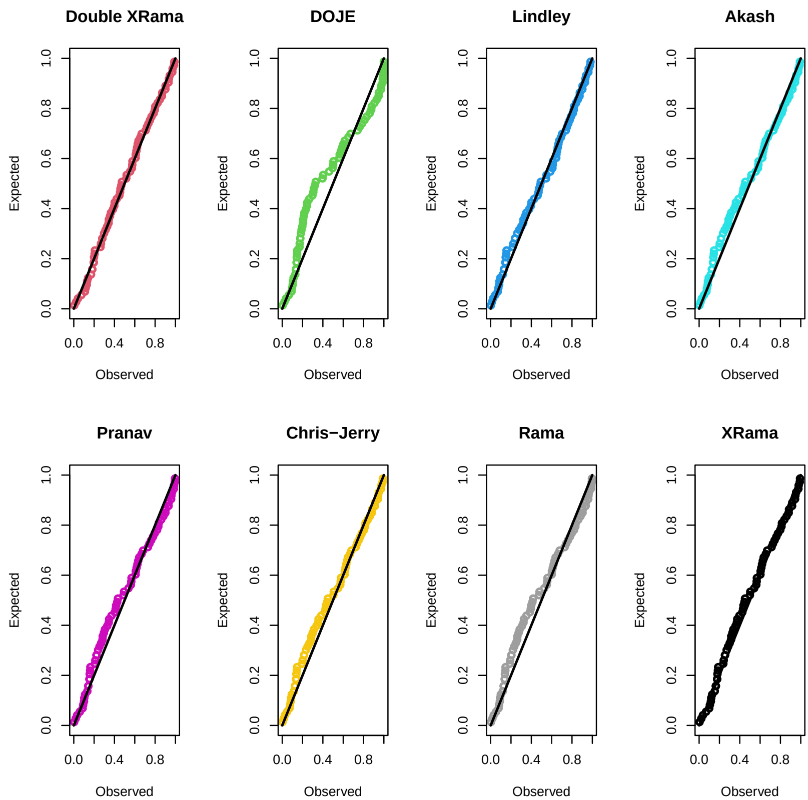

Figure 8: pp plots of the rainfall data.

proposed double XRama distribution with other Lindley class of distribution which included; DOJE, Lindley, Akash, PRANAV, Chris-Jerry, Rama, and XRama. Having compared them we found the DXR to be of best fit in most cases and of course, followed by the XRama then the Rama distribution the DXR had the lowest NLL. This shows that this model assumes lesser loss with respect to its parameters as opposed to the other distributions. Of course, XRama came close as followed by Rama due to their common similarities as to being birthed from a direct parent. Comparing the outcome of the p-value. DXR came up with the highest p-value of 0.9842 compared to other distributions. In essence, this

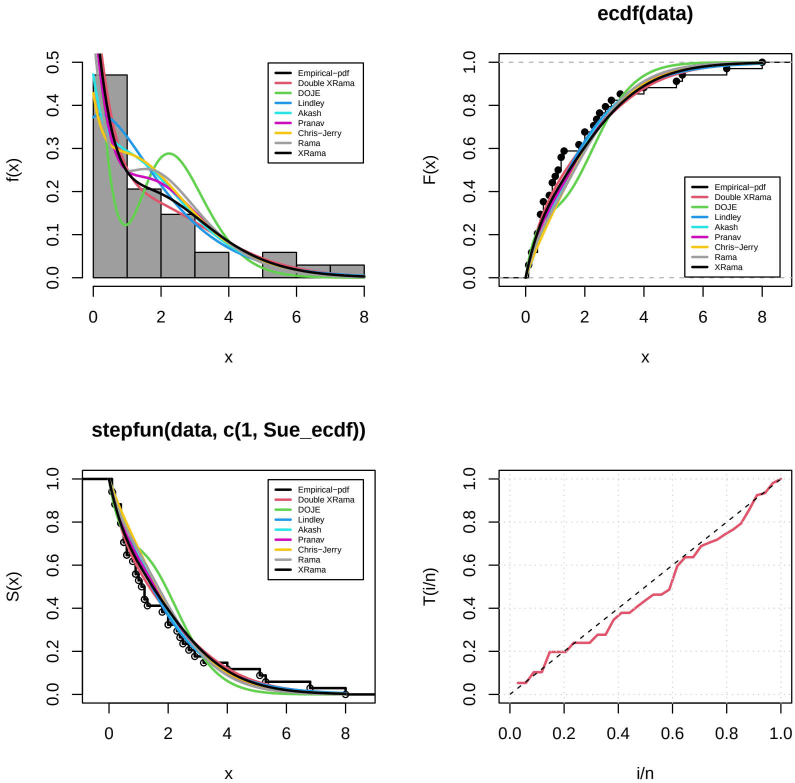

Figure 9: Density, cdf, surv and TTT plots of vinyl chloride data.

proves that DXR has a higher probability of supporting the null hypothesis in the given data observe how the DXR performed better across the measures of the fitness model with respect to the next data application on Vinyl chloride(g/L) from groundwater monitoring wells that are located in clean-up gradient areas studied by [43] and [28]. See Table 3. It is observed that DXR shows the same level of competence in performance, being the best fit to all measures of fitness model in the respective data. Taking a close look at the pp-plots of vinyl Chloride in Figure 6, it is observed that DXR has the finest line of best fit accompanied by the mothering distribution XRama.

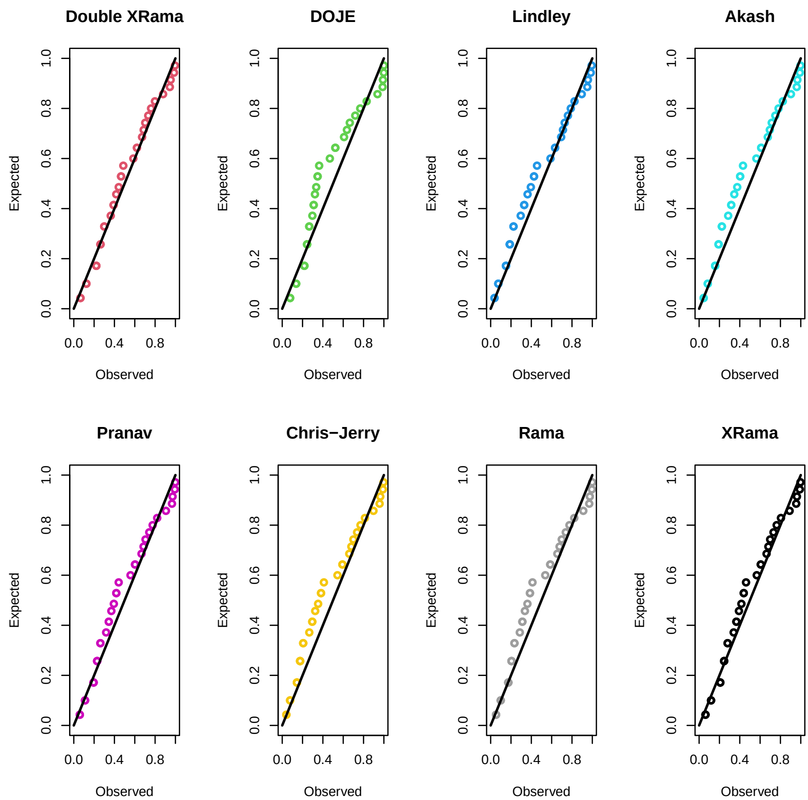

Figure 10: pp plots of vinyl chloride data.

6 Conclusion

The proposed Double XRama distribution is birthed from parent distributions; Exponential and XRama. Its mathematical statistical properties for this distribution were derived. It is no contingency that DXR came out with the prime competence on all measures of the fitness model even when compared with the real-life data set, it has been presented to show the application and goodness of fit of the two-parameter double XRama distribution with other Lindley class of distribution which included; DOJE, Lindley, Akash, Pranav, Chris-Jerry, Rama, XRama. The results suggest that the Double Xrama presents a prominent substitute when modeling some varieties of lifetime data. Over time, various families of probability distributions have been examined for analyzing different real-life data. Developing further extension of existing models by introducing additional parameter(s) or other forms of transformation has proven to advance the flexibility and applicability of the attendant models. On this note, it is recommended that the Double XRama distribution be modified either by developing a two-parameter Double XRama, power-transformed Double XRama, Exponentiated Double XRama, or Invers Double XRama. This will inadvertently improve the Double XRama and further enrich the statistical literature.

Conflict of Interest

The authors declare that there is no conflict of interest.

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