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Initial Coefficient Bounds Analysis for Novel Subclasses of Bi-Univalent Functions linked with Horadam Polynomials

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Abstract

In this work, we investigate some subclasses of bi-univalent and regular functions associated with Horadam polynomials in the open unit disk $\mathfrak{U} = \{\varsigma \in \mathbb{C} : |\varsigma| < 1\}$. For functions that belong to these subclasses, we find bounds on their initial coefficients. The functional problem of Fekete-Szegö is also examined. Along with presenting some new results, we also talk about pertinent connections to earlier findings.

1 Introduction

Suppose that $\mathcal{A}$ denote the class of regular functions g of the form

$$g(\varsigma) = \varsigma + \sum_{j=2}^{\infty} d_j \varsigma^j, \qquad \varsigma \in \mathfrak{U}, \tag{1.1}$$

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where $\mathfrak{U}$ is an open unit disk $\{\varsigma \in \mathbb{C} : |\varsigma| < 1\}$ with $\mathbb{C}$ representing the set of complex numbers. The sets of real and natural numbers are $\mathbb{R}$ and $\mathbb{N} := \{1, 2, 3, ...\} = \mathbb{N}_0\backslash\{0\}$, respectively. A subset of $\mathcal{A}$ that consists of univalent functions in $\mathfrak{U}$ is denoted by $\mathcal{S}$. An inverse of every function g in $\mathcal{S}$ is given by (see [6])

$$\hbar(w) = g^{-1}(w) = w - d_2w^2 + (2d_2^2 - d_3)w^3 - (5d_2^3 - 5d_2d_3 + d_4)w^4 + ..., \quad (1.2)$$

satisfying $\hbar(g(\varsigma)) = \varsigma$ and $g(\hbar(w)) = w$, $|w| < r_0(g)$, $r_0(g) \geq 1/4$, ς, $w \in \mathfrak{U}$.

If $\mathfrak{U} \subset g(\mathfrak{U})$ and if g and $\hbar = g^{-1}$ are both univalent in $\mathfrak{U}$, then a function g of $\mathcal{A}$ is bi-univalent in $\mathfrak{U}$. The set of bi-univalent functions in $\mathfrak{U}$ determined by (1.1) is denoted by σ. $\frac{1}{2}log\left(\frac{1+\varsigma}{1-\varsigma}\right)$, $-log(1-\varsigma)$, and $\frac{\varsigma}{1-\varsigma}$ indicate a few of the σ family's functions. The Koebe function, however, is not a member of the σ family. Further functions in $\mathcal{S}$, like $\varsigma - \frac{\varsigma^2}{2}$ and $\frac{\varsigma}{1-\varsigma^2}$ are not elements of σ family.

Coefficient-related studies for members of the σ family started in the 1970s. After looking at the σ family, Lewin [15] asserted that $|d_2| < 1.51$ for the elements of σ. It was demonstrated in [3] that for members of σ, $|d_2| < \sqrt{2}$. Studies related to coefficients for functions $\in \sigma$ were later found by Tan [28]. In [4], the authors examined starlike and convex subclasses of σ. As evidenced by studies [5, 9, 10, 21, 29], the last ten years have seen an increase in the study of bounds related to initial coefficients for elements belonging to particular subfamilies of σ.

The current focus is on functions that fall into specific σ subfamilies that are subordinate to a known special polynomials. Numerous researchers have discovered intriguing findings regarding coefficient estimates and Fekete-Szegö functional $|d_3 - \xi d_2^2|$, $\xi \in \mathbb{R}$, for individuals belonging to specific subfamilies of σ subordinate to a known special polynomials. For more information about these one can see [1, 2, 11, 18, 23, 24, 26, 27, 30, 31]. One such polynomials that has drawn attention from researchers is the Horadam polynomials.

Horadam polynomials are studied by Hörçum and Koçer [12, 13] and are denoted by $\mathsf{H}_j(\varkappa)$ (or $\mathsf{H}_j(\varkappa, \delta, \kappa; \varrho, \vartheta)$). They are expressed in terms of the recurrence relation given by

$$\mathsf{H}_j(\varkappa) = \varrho\varkappa \mathsf{H}_{j-1}(\varkappa) + \vartheta \mathsf{H}_{j-2}(\varkappa), \qquad \mathsf{H}_1(\varkappa) = \delta, \ \mathsf{H}_2(\varkappa) = \kappa\varkappa, \tag{1.3}$$

where $j \in \mathbb{N}\backslash\{1, 2\}$ and $\varkappa, \ \varrho, \ \vartheta, \ \delta, \ \kappa \in \mathbb{R}$. $\mathsf{H}_3(\varkappa) = \varrho\kappa\varkappa^2 + \vartheta\delta$ is evident from (1.3). As per [12], the sequence $\mathsf{H}_j(\varkappa)$, where $j \in \mathbb{N}$, has the following generating function:

$$\mathsf{H}(\varkappa, \varsigma) := \sum_{j=1}^{\infty} \mathsf{H}_j(\varkappa)\varsigma^{j-1} = \frac{(\kappa - \delta\varrho)\varkappa\varsigma + \delta}{1 - \varrho\varkappa\varsigma - \vartheta\varsigma^2}, \tag{1.4}$$

where $\mathfrak{R}(\varsigma) \neq \varkappa$, $\varkappa \in \mathbb{R}$, and $\varsigma \in \mathbb{C}$.

For specific choices of δ, κ, ϱ, and ϑ, the Horadam polynomial $\mathsf{H}_j(\varkappa, \delta, \kappa\,;\, \varrho, \vartheta)$ reduces to several polynomials (for details see [23]).

For $\mathfrak{z}_1$, $\mathfrak{z}_2 \in \mathcal{A}$ regular in $\mathfrak{U}$, we say that $\mathfrak{z}_1$ is subordinate to $\mathfrak{z}_2$, if there is a function $\psi(\varsigma)$ of Schwarz, which is regular in $\mathfrak{U}$ with $\psi(0) = 0$ and $|\psi(\varsigma)| < 1$ $(\varsigma \in \mathfrak{U})$, such that $\mathfrak{z}_1(\varsigma) = \mathfrak{z}_2(\psi(\varsigma))$, $\varsigma \in \mathfrak{U}$. This principle is indicated as $\mathfrak{z}_1 \prec \mathfrak{z}_2$ or $\mathfrak{z}_1(\varsigma) \prec \mathfrak{z}_2(\varsigma)$ $(\varsigma \in \mathfrak{U})$. In particular, if $\mathfrak{z}_2 \in \mathcal{S}$, then

$$\mathfrak{z}_1(\varsigma) \prec \mathfrak{z}_2(\varsigma) \ \Leftrightarrow \ \mathfrak{z}_1(0) = \mathfrak{z}_2(0) \quad \text{and} \quad \mathfrak{z}_1(\mathfrak{U}) \subset \mathfrak{z}_2(\mathfrak{U}).$$

Motivated by the previously mentioned trends in coefficient-related investigations as well as the Fekete-Szegö issue [8] on particular subfamilies of σ, we explore two new subfamilies of σ linked with Horadam polynomials $\mathsf{H}_j(\varkappa)$ as in (1.3), namely $S\mathfrak{T}_\sigma^\tau(\beta, \nu, \varkappa)$ and $S\mathfrak{Y}_\sigma^\tau(\beta, \gamma, \mu, \varkappa)$. Unless otherwise indicated, the inverse functions $g^{-1}(w) = \hbar(w)$ as in (1.2) and $\mathsf{H}(\varkappa, \varsigma)$ as in (1.4) are used in this paper.

Definition 1.1. The class $S\mathfrak{T}_\sigma^\tau(\beta, \nu, \varkappa)$, $\tau \geq 1, \beta \in \mathbb{C} - \{0\}$, $\nu \geq 0$, and $\varkappa \in \mathbb{R}$ contains all the functions $g \in \sigma$ given by (1.1), if

$$1 + \frac{1}{\beta}\left[\nu\left(\frac{[(\varsigma g'(\varsigma))']^\tau}{g'(\varsigma)}\right) + (1-\nu)\left(\frac{\varsigma(g'(\varsigma))^\tau}{g(\varsigma)}\right) - 1\right] \prec \mathsf{H}(\varkappa, \varsigma) + 1 - \delta, \, \varsigma \in \mathfrak{U},$$

and

$$1 + \frac{1}{\beta}\left[\nu\left(\frac{[(w\hbar'(w))']^\tau}{\hbar'(w)}\right) + (1-\nu)\left(\frac{w(\hbar'(w))^\tau}{\hbar(w)}\right) - 1\right] \prec \mathsf{H}(\varkappa, w) + 1 - \delta, \, w \in \mathfrak{U}.$$

We designate the class of $\tau-$pseudo-ν-bi-starlike functions subordinate to Horadam polynomials as $\mathfrak{S}_\sigma^\tau(\nu,\varkappa) = S\mathfrak{T}_\sigma^\tau(1,\nu,\varkappa)$. $\tau-$pseudo-bi-starlike function family $\mathfrak{P}_\sigma^\tau(\varkappa) \equiv S\mathfrak{T}_\sigma^\tau(1,0,\varkappa)$ subordinate to Horadam polynomials was examined in [1] and the family $S\mathfrak{T}_\sigma^1(1,\nu,\varkappa) \equiv \mathfrak{F}_\sigma(\nu,\varkappa)$ was explored in [16].

Definition 1.2. The class $S\mathfrak{Y}_\sigma^\tau(\beta,\gamma,\mu,\varkappa)$, $\tau \geq 1$, $0 \leq \gamma \leq 1, \beta \in \mathbb{C} - \{0\}$, $\mu \geq \gamma$, and $\varkappa \in \mathbb{R}$ contains all the functions $g \in \sigma$ given by (1.1), if

$$1 + \frac{1}{\beta}\left(\frac{\mu\varsigma^2 g''(\varsigma) + \varsigma(g'(\varsigma))^\tau}{\gamma\varsigma g'(\varsigma) + (1-\gamma)g(\varsigma)} - 1\right) \prec \mathsf{H}(\varkappa,\varsigma) + 1 - \delta,\ \varsigma \in \mathfrak{U}$$

and

$$1 + \frac{1}{\beta}\left(\frac{\mu w^2 \hbar''(w) + \omega(\hbar'(w))^\tau}{\gamma w\hbar'(w) + (1-\gamma)\hbar(w)} - 1\right) \prec \mathsf{H}(\varkappa,w) + 1 - \delta,\ w \in \mathfrak{U}.$$

For specific choices of μ and γ, the family $\mathrm{S}\mathfrak{Y}_\sigma^\tau(\beta,\gamma,\mu,\varkappa)$ contains many existing subfamilies of σ in addition to several new ones, as shown below:

1. $\mathfrak{K}_\sigma^\tau(\beta,\mu,\varkappa) \equiv S\mathfrak{Y}_\sigma^\tau(\beta,0,\mu,\varkappa), \tau \geq 1, \mu \geq 0, \beta \in \mathbb{C} - \{0\}$, and $\varkappa \in \mathbb{R}$, is the class of functions $g \in \sigma$ satisfying

$$1 + \frac{1}{\beta}\left(\frac{\mu\varsigma^2 g''(\varsigma) + \varsigma(g'(\varsigma))^\tau}{g(\varsigma)} - 1\right) \prec \mathsf{H}(\varkappa,\varsigma) + 1 - \delta,\ \varsigma \in \mathfrak{U}$$

and

$$1 + \frac{1}{\beta}\left(\frac{\mu w^2 \hbar''(w) + \omega(\hbar'(w))^\tau}{\hbar(w)} - 1\right) \prec \mathsf{H}(\varkappa,w) + 1 - \delta,\ w \in \mathfrak{U}.$$

When $\beta = 1$, the class $\mathfrak{K}_\sigma^\tau(1,\mu,\varkappa)$ was considered by Shammaky et al. [18].

2. $\mathfrak{J}_\sigma^\tau(\beta,\mu,\varkappa) \equiv S\mathfrak{Y}_\sigma^\tau(\beta,1,\mu,\varkappa), \tau \geq 1, \mu \geq 1, \beta \in \mathbb{C} - \{0\}$, and $\varkappa \in \mathbb{R}$, is the class of functions $g \in \sigma$ satisfying

$$1 + \frac{1}{\beta}\left[(g'(\varsigma))^{\tau-1}\left(1 + \mu\left(\frac{\varsigma g''(\varsigma)}{(g'(\varsigma))^\tau}\right)\right) - 1\right] \prec \mathsf{H}(\varkappa,\varsigma) + 1 - \delta, \varsigma \in \mathfrak{U}$$

and

$$1 + \frac{1}{\beta}\left[(\hbar'(w))^{\tau-1}\left(1 + \mu\left(\frac{w\hbar''(w)}{(\hbar'(w))^\tau}\right)\right) - 1\right] \prec \mathsf{H}(\varkappa,w) + 1 - \delta, w \in \mathfrak{U}.$$

3. $\mathfrak{L}_\sigma^\tau(\beta,\gamma,\varkappa) \equiv S\mathfrak{Y}_\sigma^\tau(\beta,\gamma,1,\varkappa), \tau \geq 1, 0 \leq \gamma \leq 1$, $\beta \in \mathbb{C} - \{0\}$, and $\varkappa \in \mathbb{R}$, is the family of functions $g \in \sigma$ satisfying

$$1 + \frac{1}{\beta}\left[\frac{\varsigma^2 g''(\varsigma) + \varsigma(g'(\varsigma))^\tau}{\gamma\varsigma g'(\varsigma) + (1-\gamma)g(\varsigma)} - 1\right] \prec \mathsf{H}(\varkappa,\varsigma) + 1 - \delta, \ \varsigma \in \mathfrak{U}$$

and

$$1 + \frac{1}{\beta}\left[\frac{w^2 \hbar''(w) + w(\hbar'(w))^\tau}{\gamma w \hbar'(w) + (1-\gamma)\hbar(w)} - 1\right] \prec \mathsf{H}(\varkappa,w) + 1 - \delta, \ \in \mathfrak{U}.$$

It is observed that i) $\mathfrak{K}_\sigma^\tau(\beta,1,\varkappa) \equiv \mathfrak{L}_\sigma^\tau(\beta,0,\varkappa), \beta \in \mathbb{C} - \{0\}, \tau \geq 1$, and $\varkappa \in \mathbb{R}$. ii) $\mathfrak{J}_\sigma^\tau(\beta,1,\varkappa) \equiv \mathfrak{L}_\sigma^\tau(\beta,1,\varkappa), \beta \in \mathbb{C} - \{0\}, \tau \geq 1$, and $\varkappa \in \mathbb{R}$. iii) Magesh et al. [16] looked at the class $\mathfrak{K}_\sigma^1(1,\mu,\varkappa), \mu \geq 0$, and $\varkappa \in \mathbb{R}$. iv) Srivastava et al. [21] studied the family $\mathfrak{K}_\sigma^1(1,0,\varkappa) \equiv S_\sigma^*(\varkappa), \varkappa \in \mathbb{R}$ for $\mu = 0$ and $\tau = 1$.

For functions in $S\mathfrak{T}_\sigma^\tau(\beta,\nu,\varkappa)$, we find estimates for $|d_2|$, $|d_3|$, and $|d_3 - \xi d_2^2|, \xi \in \mathbb{R}$ in Section 2. For functions in $S\mathfrak{Y}_\sigma^\tau(\beta,\gamma,\mu,\varkappa)$, we derive the upper bounds for $|d_2|$, $|d_3|$, and $|d_3 - \xi d_2^2|, \xi \in \mathbb{R}$ in Section 3. Presentations of intriguing outcomes and pertinent links to the established findings are made.

2 The Function Class $S\mathfrak{T}_\sigma^\tau(\beta,\nu,\varkappa)$

For $g \in S\mathfrak{T}_\sigma^\tau(\beta,\nu,\varkappa)$, the class specified in the Section 1, we first find the coefficient related estimates.

Theorem 2.1. *Let $\xi \in \mathbb{R}$, $\varkappa \in \mathbb{R}$, $\tau \geq 1$, $\beta \in \mathbb{C} - \{0\}$, and $\nu \geq 0$. If $g \in S\mathfrak{T}_\sigma^\tau(\beta,\nu,\varkappa)$, then*

$$|d_2| \leq |\beta||\kappa\varkappa|\sqrt{\frac{|\kappa\varkappa|}{|(\tau(6\nu+1)(\tau-1)+\nu+\tau^2)\beta(\kappa\varkappa)^2 - (\nu+1)^2(2\tau-1)^2(\varrho\kappa\varkappa^2+\vartheta\delta)|}}, \tag{2.1}$$

$$|d_3| \leq |\beta|\left[\frac{|\beta|(\kappa\varkappa)^2}{(\nu+1)^2(\tau-1)^2} + \frac{|\kappa\varkappa|}{(2\nu+1)(3\tau-1)}\right] \tag{2.2}$$

and

$$|d_3 - \xi d_2^2| \leq \begin{cases} \frac{|\beta||\kappa\varkappa|}{(2\nu+1)(3\tau-1)} & ; |1-\xi| \leq \mathcal{J} \\ \frac{|\beta|^2|\kappa\varkappa|^3\,|1-\xi|}{|(\tau(6\nu+1)(\tau-1)+\nu+\tau^2)\beta(\kappa\varkappa)^2-(\nu+1)^2(2\tau-1)^2(\varrho\kappa\varkappa^2+\vartheta\delta)|} & ; |1-\xi| \geq \mathcal{J}, \end{cases} \tag{2.3}$$

where

$$\mathcal{J} = \left| \frac{(\tau(6\nu+1)(\tau-1)+\nu+\tau^2)\beta\kappa^2\varkappa^2 - (\nu+1)^2(2\tau-1)^2(\varrho\kappa\varkappa^2+\vartheta\delta)}{(2\nu+1)(3\tau-1)\beta\kappa^2\varkappa^2} \right|. \tag{2.4}$$

Proof. Let $g \in S\mathfrak{T}_\sigma^\tau(\beta,\nu,\varkappa)$. Then, we get

$$1 + \frac{1}{\beta}\left[\nu\left(\frac{[(\varsigma g'(\varsigma))']^\tau}{g'(\varsigma)}\right) + (1-\nu)\left(\frac{\varsigma(g'(\varsigma))^\tau}{g(\varsigma)}\right) - 1\right] = \mathsf{H}(\varkappa,\mathfrak{m}(\varsigma)) + 1 - \delta,\ \varsigma \in \mathfrak{U} \tag{2.5}$$

and

$$1 + \frac{1}{\beta}\left[\nu\left(\frac{[(w\hbar'(w))']^\tau}{\hbar'(w)}\right) + (1-\nu)\left(\frac{w(\hbar'(w))^\tau}{\hbar(w)}\right) - 1\right] = \mathsf{H}(\varkappa,\mathfrak{n}(w)) + 1 - \delta,\ w \in \mathfrak{U}, \tag{2.6}$$

where

$$\mathfrak{m}(\varsigma) = m_1\varsigma + m_2\varsigma^2 + m_3\varsigma^3 + ...,\ \text{ and }\ \mathfrak{n}(w) = n_1 w + n_2 w^2 + n_3 w^3 + ..., \tag{2.7}$$

are some regular functions that have the property that $|\mathfrak{m}(\varsigma)| < 1$ and $|\mathfrak{n}(w)| < 1$, ς, $w \in \mathfrak{U}$. Additionally, It is known that

$$|\mathfrak{m}_i| \leq 1 \text{ and } |\mathfrak{n}_i| \leq 1\, (i \in \mathbb{N}). \tag{2.8}$$

Using (1.4) and (2.5)-(2.7), it is evident that

$$1 + \frac{1}{\beta}\left[\nu\left(\frac{[(\varsigma g'(\varsigma))']^\tau}{g'(\varsigma)}\right) + (1-\nu)\left(\frac{\varsigma(g'(\varsigma))^\tau}{g(\varsigma)}\right) - 1\right] = 1-\delta+\mathsf{H}_1(\varkappa)+\mathsf{H}_2(\varkappa)\mathfrak{m}(\varsigma)+\mathsf{H}_3(\varkappa)\mathfrak{m}^2(\varsigma)+... \tag{2.9}$$

and

$$1 + \frac{1}{\beta}\left[\nu\left(\frac{[(w\hbar'(w))']^\tau}{\hbar'(w)}\right) + (1-\nu)\left(\frac{w(\hbar'(w))^\tau}{\hbar(w)}\right) - 1\right] =$$

$$1-\delta+\mathsf{H}_1(\varkappa)+\mathsf{H}_2(\varkappa)\mathfrak{n}(w)+\mathsf{H}_3(\varkappa)\mathfrak{n}^2(w)+\dots. \tag{2.10}$$

We determine from (2.9) and (2.10), in light of (1.3), that

$$1+\frac{1}{\beta}\left[\nu\left(\frac{[(\varsigma g'(\varsigma))']^\tau}{g'(\varsigma)}\right)+(1-\nu)\left(\frac{\varsigma(g'(\varsigma))^\tau}{g(\varsigma)}\right)-1\right]=$$

$$1+\mathsf{H}_2(\varkappa)\mathfrak{m}_1\varsigma+[\mathsf{H}_2(\varkappa)\mathfrak{m}_2+\mathsf{H}_3(\varkappa)\mathfrak{m}_1^2]\varsigma^2+\dots \tag{2.11}$$

and

$$1+\frac{1}{\beta}\left[\nu\left(\frac{[(w\hbar'(w))']^\tau}{\hbar'(w)}\right)+(1-\nu)\left(\frac{w(\hbar'(w))^\tau}{\hbar(w)}\right)-1\right]=$$

$$1+\mathsf{H}_2(\varkappa)\mathfrak{n}_1w+[\mathsf{H}_2(\varkappa)\mathfrak{n}_2+\mathsf{H}_3(\varkappa)\mathfrak{n}_1^2]\mathsf{w}^2+\dots. \tag{2.12}$$

Consequently, by contrasting the corresponding coefficients in (2.11) and (2.12), we get

$$(\nu+1)(2\tau-1)d_2=\beta\mathsf{H}_2(\varkappa)\mathfrak{m}_1, \tag{2.13}$$

$$(2\nu+1)(3\tau-1)d_3+(3\nu+1)(2\tau^2-4\tau+1)d_2^2=\beta[\mathsf{H}_2(\varkappa)\mathfrak{m}_2+\mathsf{H}_3(\varkappa)\mathfrak{m}_1^2], \tag{2.14}$$

$$-(\nu+1)(2\tau-1)\,d_2=\beta\mathsf{H}_2(\varkappa)\mathfrak{n}_1 \tag{2.15}$$

and

$$(2\nu+1)(3\tau-1)(2d_2^2-d_3)+(3\nu+1)(2\tau^2-4\tau+1)d_2^2=\beta[\mathsf{H}_2(\varkappa)\mathfrak{n}_2+\mathsf{H}_3(\varkappa)\mathfrak{n}_1^2]. \tag{2.16}$$

From (2.13) and (2.15), we get

$$\mathfrak{m}_1=-\mathfrak{n}_1 \tag{2.17}$$

and also

$$2(\nu+1)^2(2\tau-1)^2d_2^2=\beta^2(\mathfrak{m}_1^2+\mathfrak{n}_1^2)(\mathsf{H}_2(\varkappa))^2. \tag{2.18}$$

We add (2.14) and (2.16) to obtain the bound on $|d_2|$:

$$2(\tau(6\nu+1)(\tau-1)+\nu+\tau^2)d_2^2=\beta\mathsf{H}_2(\varkappa)(\mathfrak{m}_2+\mathfrak{n}_2)+\beta\mathsf{H}_3(\varkappa)(\mathfrak{m}_1^2+\mathfrak{n}_1^2). \tag{2.19}$$

Putting the value of $\mathfrak{m}_1^2+\mathfrak{n}_1^2$ from (2.18) in (2.19), we get

$$d_2^2=\frac{\beta^2\mathsf{H}_2^3(\varkappa)(\mathfrak{m}_2+\mathfrak{n}_2)}{2\left[(\tau(6\nu+1)(\tau-1)+\nu+\tau^2)\beta\mathsf{H}_2^2(\varkappa)-(\nu+1^2)(2\tau-1)^2\mathsf{H}_3(\varkappa)\right]}. \tag{2.20}$$

We obtain (2.1) by applying (2.8) for $\mathfrak{m}_2$ and $\mathfrak{n}_2$.

We subtract (2.16) from (2.14) to obtain the bound on $|d_3|$.

$$d_3 = d_2^2 + \frac{\beta \mathsf{H}_2(\varkappa)(\mathfrak{m}_2 - \mathfrak{n}_2)}{2(2\nu+1)(3\tau-1)}. \tag{2.21}$$

From (2.17), (2.18) and (2.21) it follows that

$$d_3 = \frac{\beta^2 \mathsf{H}_2^2(\varkappa)(\mathfrak{m}_1^2 + \mathfrak{n}_1^2)}{2(\nu+1)^2(2\tau-1)^2} + \frac{\beta \mathsf{H}_2(\varkappa)(\mathfrak{m}_2 - \mathfrak{n}_2)}{2(2\nu+1)(3\tau-1)}.$$

We obtain (2.2) by applying (2.8) for the coefficients $\mathfrak{m}_1$, $\mathfrak{m}_2$, $\mathfrak{n}_1$ and $\mathfrak{n}_2$.

Lastly, we use the values of d_2^2 and d_3 from (2.20) and (2.21), respectively, to compute the bound on $|d_3 - \xi d_2^2|$. Thus, we have

$$|d_3 - \xi d_2^2| = \frac{|\beta||\mathsf{H}_2(\varkappa)|}{2} \left| \left(\frac{1}{(2\nu+1)(3\tau-1)} + \mathcal{B}(\xi,\varkappa) \right) \mathfrak{m}_2 - \left(\frac{1}{(2\nu+1)(3\tau-1)} - \mathcal{B}(\xi,\varkappa) \right) \mathfrak{n}_2 \right|,$$

where

$$\mathcal{B}(\xi,\varkappa) = \frac{(1-\xi)\beta \mathsf{H}_2^2(\varkappa)}{\left[(\tau(6\nu+1)(\tau-1) + \nu + \tau^2)\beta \mathsf{H}_2^2(\varkappa) - (\nu+1^2)(2\tau-1)^2 \mathsf{H}_3(\varkappa) \right]}.$$

Clearly

$$|d_3 - \xi d_2^2| \leq \begin{cases} \frac{|\beta||\mathsf{H}_2(\varkappa)|}{(2\nu+1)(3\tau-1)} & ;\ 0 \leq |\mathcal{B}(\xi,\varkappa)| \leq \frac{1}{(2\nu+1)(3\tau-1)} \\ |\beta||\mathsf{H}_2(\varkappa)||\mathcal{B}(\xi,\varkappa)| & ;\ |\mathcal{B}(\xi,\varkappa)| \geq \frac{1}{(2\nu+1)(3\tau-1)}, \end{cases}$$

which leads us to (2.3) where $\mathcal{J}$ is as in (2.4). □

Remark 2.1. i) Using $\beta = 1$ and $\nu = 0$ in the above theorem, we obtain Theorem 2.1 in [1]. Additionally, we can obtain the results in [20, Corollary 1 and Corollary 3] by allowing $\tau = 1$. ii) A result in [16, Theorem 2.2] is obtained by taking $\beta = \tau = 1$ in Theorem 2.1. Additionally, by allowing $\nu = 1$, we obtain Corollary 2.3 in [16], which is also expressed as Corollary 1 in [17].

3 The Function Class $S\mathfrak{Y}_\sigma^\tau(\beta,\gamma,\mu,\varkappa)$

The coefficient bounds for the functions $g \in S\mathfrak{Y}_\sigma^\tau(\beta,\gamma,\mu,\varkappa)$ discussed in Section 1 are provided in this section. Since this proof is fairly similar to Theorem 2.1, it is excluded.

Theorem 3.1. *Let $\xi \in \mathbb{R}, \varkappa \in \mathbb{R}, \tau \geq 1$, $\mu \geq \gamma, \beta \in \mathbb{C} - \{0\}$, and $0 \leq \gamma \leq 1$. If $g \in S\mathfrak{Y}_\sigma^\tau(\beta,\gamma,\mu,\varkappa)$, then*

$$|d_2| \leq |\beta||\kappa\varkappa|\sqrt{\frac{|\kappa\varkappa|}{|(\gamma^2-2\gamma(\mu+\tau)+4\mu+\tau(2\tau-1))\beta(\kappa\varkappa)^2-(2(\mu+\tau)-\gamma-1)^2(\varrho\kappa\varkappa^2+\vartheta\delta)|}},$$

$$|d_3| \leq |\beta|\left[\frac{|\kappa\varkappa|}{3(2\mu+\tau)-2\gamma-1}+\frac{|\beta|(\kappa\varkappa)^2}{(2(\tau+\mu)-\gamma-1)^2}\right]$$

and

$$|d_3-\xi d_2^2| \leq \begin{cases} \frac{|\beta||\kappa\varkappa|}{3(2\mu+\tau)-2\gamma-1} & ;\ |1-\xi| \leq \mathcal{Q} \\ \frac{|\beta|^2|\kappa\varkappa|^3\,|1-\xi|}{|(\gamma^2-2\gamma(\mu+\tau)+4\mu+\tau(2\tau-1))\beta(\kappa\varkappa)^2-(2(\tau+\mu)-2\gamma-1)^2(\varrho\kappa\varkappa^2+\vartheta\delta)|} & ;\ |1-\xi| \geq \mathcal{Q}, \end{cases}$$

where

$$\mathcal{Q} = \left|\frac{(\gamma^2-2\gamma(\mu+\tau)+4\mu+\tau(2\tau-1))\beta b^2\varkappa^2-(2(\tau+\mu)-\gamma-1)^2(\varrho b\varkappa^2+\vartheta\delta)}{(3(2\mu+\tau)-2\gamma-1)\beta b^2\varkappa^2}\right|.$$

Remark 3.1. i) When $\beta = 1$, Theorem 3.1 agrees with Corollary 2 in [18]. ii) When $\beta = \tau = 1$, Theorem 3.1 coincides with a result [16, Theorem 2.1].

Theorem 3.1 would produce the following when $\gamma = 0$:

Corollary 3.1. *Let $\xi \in \mathbb{R}, \varkappa \in \mathbb{R}, \tau \geq 1$, $\mu \geq 0$, and $\beta \in \mathbb{C} - \{0\}$. If $g \in S\mathfrak{K}_\sigma^\tau(\beta,\mu,\varkappa)$, then*

$$|d_2| \leq |\beta||\kappa\varkappa|\sqrt{\frac{|\kappa\varkappa|}{|(4\mu+\tau(2\tau-1))\beta(\kappa\varkappa)^2-(2(\mu+\tau)-1)^2(\varrho\kappa\varkappa^2+\vartheta\delta)|}},$$

$$|d_3| \leq |\beta| \left[\frac{|\kappa\varkappa|}{3(2\mu+\tau)-1} + \frac{|\beta|\kappa^2\varkappa^2}{(2(\mu+\tau)-1)^2} \right]$$

and

$$|d_3 - \xi d_2^2| \leq \begin{cases} \frac{|\beta||\kappa\varkappa|}{3(2\mu+\tau)-1} & ; |1-\xi| \leq \mathcal{Q}_1 \\ \frac{|\beta|^2|\kappa\varkappa|^3\,|1-\xi|}{|(4\mu+\tau(2\tau-1))\beta(\kappa\varkappa)^2-(2(\mu+\tau)-1)^2(\varrho\kappa\varkappa^2+\vartheta\delta)|} & ; |1-\xi| \geq \mathcal{Q}_1, \end{cases}$$

where

$$\mathcal{Q}_1 = \left| \frac{(4\mu+\tau(2\tau-1))\beta b^2\varkappa^2 - (2(\mu+\tau)-1)^2(\varrho b\varkappa^2+\vartheta\delta)}{(3(2\mu+\tau)-1)\beta b^2\varkappa^2} \right|.$$

Remark 3.2. When $\beta = 1, \mu = \gamma = 0$, Theorem 3.1 agrees with a finding in [1, Theorem 2.1]. Furthermore, we arrive at the outcomes in [20, Corollaries 1 and 3], by allowing $\tau = 1$.

Theorem 3.1 would produce the following when $\gamma = 1$:

Corollary 3.2. *Let* $\xi \in \mathbb{R}, \varkappa \in \mathbb{R}, \mu \geq 1, \beta \in \mathbb{C} - \{0\}$, *and* $\tau \geq 1$. *If* $g \in \mathfrak{K}_\Sigma^\tau(\beta,\mu,\varkappa) \equiv S\mathfrak{Y}_\Sigma^\tau(1,\mu,\varkappa)$, *then*

$$|d_2| \leq |\beta||\kappa x| \sqrt{\frac{|\kappa x|}{|(2\mu+2\tau^2-3\tau+1)\beta(\kappa\varkappa)^2 - 4(\mu+\tau-1)^2(\varrho\kappa\varkappa^2+\vartheta\delta)|}},$$

$$|d_3| \leq |\beta| \left[\frac{|\kappa\varkappa|}{3(2\mu+\tau-1)} + \frac{|\beta|(\kappa\varkappa)^2}{4(\mu+\tau-1)^2} \right]$$

and

$$|d_3 - \xi d_2^2| \leq \begin{cases} \frac{|\beta|\kappa\varkappa|}{3(2\mu+\tau-1)} & ; |1-\xi| \leq \mathcal{Q}_2 \\ \frac{|\beta|^2|\kappa\varkappa|^3\,|1-\xi|}{|(2\mu+2\tau^2-3\tau+1)\beta(\kappa\varkappa)^2-4(\mu+\tau-1)^2(\varrho\kappa\varkappa^2+\vartheta\delta)|} & ; |1-\xi| \geq \mathcal{Q}_2, \end{cases}$$

where $\mathcal{Q}_2 = \frac{1}{3(\tau+2\mu-1)} \left| 2\mu+2\tau^2-3\tau+1-4(\tau+\mu-1)^2 \left(\frac{\varrho\kappa\varkappa^2+\vartheta\delta}{\beta\kappa^2\varkappa^2} \right) \right|.$

Remark 3.3. In Corollary 3.2, if we allow $\beta = \tau = 1$, we get the outcome in [25, Corollary 2.2].

Theorem 3.1 would produce the following when $\mu = 1$:

Corollary 3.3. *Let* $\xi \in \mathbb{R}, \varkappa \in \mathbb{R}, 0 \leq \gamma \leq 1, \beta \in \mathbb{C} - \{0\}$, *and* $\tau \geq 1$. *If* $g \in \mathfrak{L}_{\Sigma}^{\tau}(\beta, \varkappa, \gamma) \equiv S\mathfrak{Y}_{\Sigma}^{\tau}(\varkappa, \beta, \gamma, 1)$, *then*

$$|d_2| \leq |\beta||\kappa\varkappa|\sqrt{\frac{|\kappa\varkappa|}{|((1-\gamma)^2 + 3 - \tau + 2\tau(\tau-\gamma))\beta(\kappa\varkappa)^2 - (2\tau+1-\gamma)^2(\varrho\kappa\varkappa^2 + \vartheta\delta)|}},$$

$$|d_3| \leq |\beta| \left[\frac{|\kappa\varkappa|}{3\tau + 5 - 2\gamma} + \frac{|\beta|(\kappa\varkappa)^2}{(2\tau+1-\gamma)^2}\right]$$

and

$$|d_3 - \xi d_2^2| \leq \begin{cases} \frac{|\beta||\kappa\varkappa|}{3\tau+5-2\gamma} & ; |1-\xi| \leq \mathcal{Q}_3 \\ \frac{|\beta|^2|\kappa\varkappa|^3\,|1-\xi|}{|((1-\gamma)^2+3-\tau+2\tau(\tau-\gamma))\beta(\kappa\varkappa)^2-(2\tau+1-\gamma)^2(\varrho\kappa\varkappa^2+\vartheta\delta)|} & ; |1-\xi| \geq \mathcal{Q}_3, \end{cases}$$

where

$$\mathcal{Q}_3 = \frac{1}{3\tau+5-2\gamma}\left|((1-\gamma)^2 + 3 - \tau + 2\tau(\tau-\gamma)) - (2\tau+1-\gamma)^2\left(\frac{\varrho\kappa\varkappa^2 + \vartheta\delta}{\beta\kappa^2\varkappa^2}\right)\right|.$$

Remark 3.4. By letting $\gamma = \tau = \beta = 1$ in the above corollary, we arrive at a result in [16, Corollary 2.3]. Orhan et al. [17] also express this result as Corollary 1.

4 Conclusion

The upper bounds of $|d_2|$ and $|d_3|$ for functions belonging to the introduced subfamilies of σ linked with Horadam polynomials are obtained, in the present paper. Additionally, for functions in these subfamilies, we have determined the Fekete-Szegö problem $|d_3 - \xi d_2^2|, \xi \in \mathbb{R}$. By adjusting the parameters in Theorems 2.1 and 3.1, we have highlighted a number of implications. Additionally, pertinent links to the current findings are found. The open problem is to estimate the bound of $|d_j|, (j \in \mathbb{R}/\{1, 2, 3\})$ for the subfamilies studied in this article. Many researchers may be motivated to concentrate on a variety of recent publications, such as the i) operators on fractional q-calculus [19], ii) q-derivative and q-integral operators [7, 14, 22], based on the subfamilies this investigation examines.

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