Earthline Journal of Mathematical Sciences
E-ISSN: 2581-8147
Volume 14, Number 3, 2024, Pages 421-441
https://doi.org/10.34198/ejms.14324.421441

Mathematical Model and Optimal Control of Covid-19 in Nigeria

M. C. Anyanwu[1,*], S. N. Neossi-Nguetchue[2], G. M. Moremedi[3] and A. S. Eegunjobi[2]

[1] Department of Mathematics, Michael Okpara University of Agriculture, Umudike, Abia State, Nigeria
e-mail: pmanyanwu71@yahoo.com

[2] Mathematics Department, Namibia University of Science and Technology, Windhoek, Namibia

[3] Mathematics Department, University of South Africa, Johannessburg, South Africa

Abstract

The global pandemic, Covid-19, caused by corona virus disease is responsible for a significant number of deaths and huge economic losses in almost all the countries of the world, including Nigeria. In order to manage the spread of this disease in Nigeria, the Nigeria Center for Disease Control(NCDC) has proposed and implemented various control and preventive measures such as vaccination, use of alcohol-based hand sanitizers, social distancing, and others. The aim of this paper is to model the transmission dynamics of Covid-19 in Nigeria, and obtain, by using Pontryagin Maximum Principle, the combination of these control strategies for effective control of the disease in Nigeria. Numerical experiments with Nigeria Covid-19 data show the effectiveness optimal use of these preventive and control measures for Covid-19.

Received: January 12, 2024; Accepted: February 19, 2024; Published: March 5, 2024
2020 Mathematics Subject Classification: 92B05, 34D20, 34D23, 49K15.
Keywords and phrases: Covid-19, vaccination, protection, isolation, optimal control, stability.
*Corresponding author

1 Introduction

The outbreak of corona virus disease reportedly started in Wuhan, China in December, 2019, from where the disease spread to Europe, America, Asia, Africa and across the globe. It was claimed that the virus was manufactured in a laboratory in Wuhan, from where it escaped and spread around the globe. However, this claimed was refuted by Chinese government authorities. Globally, over 689,912,675 confirmed cases and over 6,887,334 deaths have been recorded since the outbreak started. In Nigeria, the first documented case of Covid-19 was on February 29th 2020, from an Italian citizen who arrived the country from Milan, Italy on January 27, 2020 through Murtala Muhammed International Airport. After this official index case, Nigeria began to experience increase in number of new cases. This led Nigerian government to respond quickly to the outbreak by enforcing total lock-down on human and economic activities in all the states of the federation in order to contain the spread of disease. This was followed by deployment of Rapid Response teams with states leading contact tracing and other response activities.

Significant number of mathematical and statistical models have been proposed in different research articles to understand the dynamics of Covid-19 in different countries that are affected by the disease. In Abioye et al. [1], the impact of the pandemic on Nigeria and potential strategies to manage and mitigate its spread are explored. They proposes three primary control measures: face masks, hand sanitizers, social distancing, treatment, active screening, and prevention against recurrence and reinfection in recovered patients. The article used the basic reproduction number (R_0) as a key metric to measure and predict the virus spread. Although the paper could benefit from more in-depth discussion on practical implementation, its use of real-time data and mathematical model contributes to the ongoing global dialogue on Covid-19 control and prevention.

The study by Iboi et al. [2], evaluates the impact of control and mitigation strategies in various jurisdictions. Their findings suggests that moderate social

distancing measures can effectively control Covid-19 in Nigeria. Combining face mask usage with social distancing measures was crucial for reducing disease transmission. Maintaining community lock-down measures for at least three to four months was essential to contain outbreaks.

Linear regression models were used by Ogundokun et al. [3] to predict the impact of traveling history and contacts on Covid-19 spread in Nigeria. Their study used data from the Nigeria Centre for Disease Control (NCDC) website and found that traveling history and contacts increased the chances of individuals being infected by 85% and 88%, respectively. The study supported the government's decision to enforce travel restrictions as an effective measure in controlling the virus's spread. The study suggested travel agencies should have better precautions and preparations in place before reopening, based on the prediction of Covid-19 cases.

Amzat et al. [4] examined the Covid-19 pandemic in Nigeria, focusing on the country's first confirmed case and government response measures. The study revealed a steady increase in cases, transitioning from imported cases to community transmission, and a 2.8% case fatality rate. The study emphasized the importance of a combined approach, incorporating social and medical responses, to effectively curb the virus's spread. They highlighted the potential ongoing threat of Covid-19 in Nigeria, particularly as the country aims to reopen its economy. It emphasized the need for continued vigilance and balanced decision-making to ensure public health gains are not compromised.

Avus et al. [5] examined the impact of self-medication and stigmatization on the co-infection dynamics of Covid-19 and malaria in Nigeria. They argued that stigmatization and misdiagnosis contributed to self-medication, increasing the prevalence of Covid-19. Their model used compartmental ordinary differential equations (ODEs) to represent transmission dynamics and incorporated the self-medicated population and the impact of stigmatization. Their findings provided valuable insights for policymakers and healthcare authorities in developing effective strategies to combat both diseases in the Global South context.

Using non-linear ordinary differential equations, Obsu and Balcha [8] developed a novel coronavirus (Covid-19) transmission model using optimal control theory. By taking into account the price of implementation, they were able to find the most effective methods of control that would have the fewest people exposed and affected. In order to prove the existence of optimal controls and characterization, they used Pontryagin's Maximum Principle. The agreement between the analytical results and the numerical simulation results was quite high. They came to the conclusion that the best way to reduce the disease epidemic at the lowest possible cost is to use a combination of preventative measures, intense medical care, and surface disinfection. In this paper, we present a mathematical model that describes the transmission dynamics of Covid-19 in Nigeria, in the presence of several control and preventive measures. Further, Pontryagin Maximum Principle is used to obtain optimal combination of the control strategies for effective control of the disease in Nigeria.

2 Mathematical Formulation of the Model

We devised a new Covid-19 transmission model using ordinary differential equations by putting into consideration the information that was already available regarding the progression of Covid-19 infection in humans as well as the recently implemented control strategies. In terms of the epidemiologically significant stages of Covid-19 transmission, people are categorized as follows: $S(t)$ susceptible human to Covid-19; $P_1(t)$ vaccinated human against Covid-19; $P_2(t)$ human who have protected themselves against the Covid-19; $P_{12}(t)$ humans who have received vaccination and are protecting themselves against the Covid-19; $E(t)$ humans who are exposed to the Covid-19; $I_1(t)$ humans that are symptomatically infectious; $I_2(t)$ humans that are asymptomatically infectious; $T(t)$ humans infected with an infectious disease who are being treated while they are isolated and $R(t)$ Humans who are infectious and who have recovered from the Covid-19 infection.

At time t, individuals enter the susceptible compartment $S(t)$ at a rate of

Λ. Susceptible individuals can transition to three different compartments: $P_1(t)$, $P_2(t)$, and $P_{12}(t)$, with transition proportions of ρ_1, ρ_3, and ρ_4, respectively. All four compartments $(S(t), P_1(t), P_2(t)$, and $P_{12}(t))$ can then move to an exposed compartment with the force of infection λ multiplied by transition proportions $\rho_2, \alpha_1, \alpha_2$, and α_{12}, respectively. After being exposed to the disease, individuals from an exposed compartment become fully infectious with an incubation rate of β and an exposure proportion of α and move to either $I_1(t)$ or $I_2(t)$. Individuals can die from naturally or due to disease while in compartments $I_1(t), I_2(t)$, and $T(t)$, respectively. Individuals in compartment $I_2(t)$ can transition to the recovery compartment with a transition proportion of γ_2 or to the treatment compartment with a transition proportion of τ_2. Individuals in compartment $I_1(t)$ transits to treatment compartment with proportion τ_1. In addition to the natural death rate and disease death rate in the treatment compartment, individuals in this compartment can also transition to the recovery compartment with a transition proportion of γ_1. Furthermore, individuals in the recovery compartment can become susceptible again with a relapse rate of δ. As a result, the relevant aspects of the transmission process are illustrated in Fig. 1. The following is an illustration of the governing equations that are used in the simulation of the emergence and spread of Covid-19:

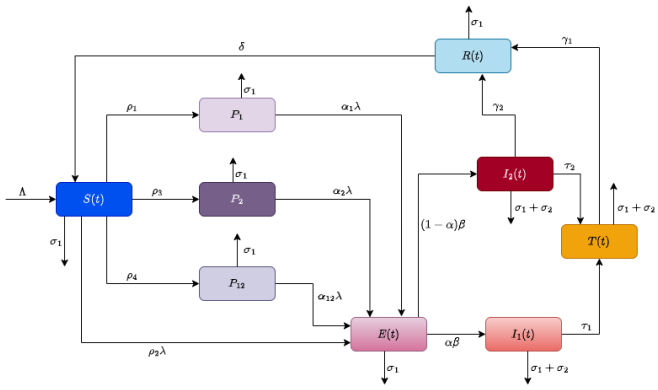

Figure 1: Schematic diagram depicting the Covid-19 transmission dynamics of model.

$$
\begin{aligned}
\frac{dS(t)}{dt} &= \Lambda - (\rho_1 + \rho_3 + \rho_4)S(t) - \rho_2\lambda(t)S(t) - \sigma_1 S(t),\\
\frac{dP_1(t)}{dt} &= \rho_1 S(t) - \lambda(t)\alpha_1 P_1(t) - \sigma_1 P_1(t),\\
\frac{dP_2(t)}{dt} &= \rho_3 S(t) - \lambda(t)\alpha_2 P_2(t) - \sigma_1 P_2(t),\\
\frac{dP_{12}(t)}{dt} &= \rho_4 S(t) - \lambda(t)\alpha_{12} P_{12}(t) - \sigma_1 P_{12}(t),\\
\frac{dE(t)}{dt} &= \lambda(t)(\rho_2 S(t) + \alpha_1 P_1(t) + \alpha_2 P_2(t) + \alpha_{12} P_{12}(t)) - (\beta + \sigma_1)E(t),\\
\frac{dI_1(t)}{dt} &= \alpha\beta E(t) - (\tau_1 + \sigma_1 + \sigma_2)I_1(t),\\
\frac{dI_2(t)}{dt} &= (1-\alpha)\beta E(t) - (\tau_2 + \gamma_2 + \sigma_1 + \sigma_2)I_2(t),\\
\frac{dT(t)}{dt} &= \tau_1 I_1(t) + \tau_2 I_2(t) - (\gamma_1 + \sigma_1 + \sigma_2)T(t),\\
\frac{dR(t)}{dt} &= \gamma_1 T(t) + \gamma_2 I_2(t) - \sigma_1 R(t),
\end{aligned}
\tag{1}
$$

where $\lambda(t) = \frac{\kappa_1 I_1(t) + \kappa_2 I_2(t) + \kappa_3 T(t)}{N(t)}$ is the force of infection.

Table 1: State variables used in this model.

variable	*Description*
S	Humans that are susceptible to Covid-19
P_1	Humans that have been vaccinated against Covid-19
P_2	Humans that are protecting themselves against Covid-19
P_{12}	Humans that have received vaccination and are protecting themselves
E	Humans that are exposed to the Covid-19
I_1	Humans that are symptomatically infectious
I_2	Humans that are asymptomatically infectious
T	Infectious humans that are isolated and are being treated
R	Infectious humans that have recovered from Covid-19 infection

Table 2: Description of the parameter used in this model.

parameter	*Description*
β	incubation rate of corona virus
ρ_1	Proportion of Humans that have been vaccinated against Covid-19
ρ_2	Proportion of Humans that do not have any form of protection against Covid-19
ρ_3	Proportion of Humans that are protecting themselves
ρ_4	Proportion of Humans that have received vaccination and are protecting themselves
α_1	Proportion of P_1 that got infected due to vaccine ineffectiveness
α_2	Proportion of P_2 that relaxed their protection and got infected
α_{12}	Proportion of P_{12} whose vaccine is ineffective and relaxed their protection and got infected
κ_1	probability of infection by symptomatically infected humans
κ_2	probability of infection by asymptomatically infected humans
κ_3	probability of infection by quarantined and treated humans
α	Proportion of exposed humans that are symptomatically infectious
τ_1	proportion of symptomatically infectious humans that are quarantined and treated
τ_2	proportion of asymptomatically infectious humans that are quarantined and treated
γ_1	proportion of quarantined and treated that have recovered
γ_2	proportion of asymptomatically infectious humans that have recovered
σ_1	Natural death rate of humans
σ_2	Death rate of humans due to Covid-19

2.1 The control reproduction number

The disease-free equilibrium, E^0 of the model system (1) is given by $E^0 = (S^0, P_1^0, P_2^0, P_{12}^0, 0, 0, 0, 0, 0)$, where $S^0 = \frac{\Lambda}{(\rho_1+\rho_3+\rho_4+\sigma_1)}, P_1^0 = \frac{\rho_1\Lambda}{\sigma_1(\rho_1+\rho_3+\rho_4+\sigma_1)}, P_2^0 = \frac{\rho_3\Lambda}{\sigma_1(\rho_1+\rho_3+\rho_4+\sigma_1)}, P_{12}^0 = \frac{\rho_4\Lambda}{\sigma_1(\rho_1+\rho_3+\rho_4+\sigma_1)}$. This is used to obtain the control reproduction number $\mathcal{R}_c$, of Covid-19 in Nigeria, which is the average number of persons that can be infected by a single infectious person in his entire infectious life when introduced in a Covid-19-free population when some control measures are applied. The control reproduction number determines the efficacy or otherwise, of applied control measures. $\mathcal{R}_c < 1$ implies that the measures are effective, and otherwise if $\mathcal{R}_c > 1$. Here, the expression for the control reproduction number $\mathcal{R}_c$, of Covid-19 in Nigeria is derived using the next generation matrix method [9], in which $\mathcal{R}_c$ is defined as $\mathcal{R}_c = \rho(FV^{-1})$, where FV^{-1} is the next generation matrix, F and V are Jacobian matrices of $\mathcal{F}$ and

$\mathcal{V}$, respectively, evaluated at the disease-free equilibrium, E_0, $\mathcal{F}(x)$ is a column vector of rates of appearance of new infections in compartment, while $\mathcal{V}(x)$ is a column vector of rates of transfer of individuals into and out of compartments by any other means including death.

The matrices F and V are given by

$$F = \begin{pmatrix} 0 & \kappa_1\phi & \kappa_2\phi & \kappa_3\phi \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$V = \begin{pmatrix} (\beta+\sigma_1) & 0 & 0 & 0 \\ -\alpha\beta & \tau_1+\sigma_1+\sigma_2 & 0 & 0 \\ -(1-\alpha)\beta & 0 & \tau_2+\gamma_2+\sigma_1+\sigma_2 & 0 \\ 0 & -\tau_1 & -\tau_2 & \gamma_1+\sigma_1+\sigma_2 \end{pmatrix}$$

where

$$\phi = \frac{\rho_2 S^0 + \alpha_1 P_1^0 + \alpha_2 P_2^0 + \alpha_{12} P_{12}^0}{S^0}$$

Hence, the control reproduction is given by

$$\mathcal{R}_c = \rho\Big(FV^{-1}\Big) = \mathcal{R}_{I_1} + \mathcal{R}_{I_2} + \mathcal{R}_T$$

where

$$\mathcal{R}_{I_1} = \frac{\kappa_1\alpha\beta\phi}{(\beta+\sigma_1)(\tau_1+\sigma_1+\sigma_2)}$$

,

$$\mathcal{R}_{I_2} = \frac{\kappa_2(1-\alpha)\beta\phi}{(\beta+\sigma_1)(\tau_2+\gamma_2+\sigma_1+\sigma_2)}$$

,

$$\mathcal{R}_T = \frac{\kappa_3\beta\phi\left(\alpha(\tau_1+\gamma_2+\sigma_1+\sigma_2)\tau_1 + (1-\alpha)(\tau_1+\sigma_1+\sigma_2)\tau_2\right)}{(\beta+\sigma_1)(\tau_1+\sigma_1+\sigma_2)(\tau_2+\gamma_2+\sigma_1+\sigma_2)(\gamma_1+\sigma_1+\sigma_2)}$$

.

Here, $\mathcal{R}_{I_1}$ is the average number of persons that can be infected by symptomatically infectious humans, $\mathcal{R}_{I_2}$ is the average number of persons that can be infected by asymptomatically infectious humans, and $\mathcal{R}_T$ is the average number of persons that can be infected by those that are receiving treatment in the isolation facility.

3 Local Asymptotic Stability of E^0

The Jacobian matrix of (1) evaluated at the disease-free equilibrium has some of its eigenvalues as $-\sigma_1(3\ times), -(\delta+\sigma_1), -(\sigma_1+\rho_1+\rho_2+\rho_3)$. These eigenvalues correspond to the uninfected compartments in the population. The remaining eigenvalues are obtained from the sub-matrix J_0 given by

$$J_0 = \begin{pmatrix} -(\beta+\sigma_1) & \kappa_1\phi & \kappa_2\phi & \kappa_3\phi \\ \alpha\beta & -(\tau_1+\sigma_1+\sigma_2) & 0 & 0 \\ (1-\alpha)\beta & 0 & -(\tau_2+\gamma_2+\sigma_1+\sigma_2) & 0 \\ 0 & \tau_1 & \tau_2 & -(\gamma_1+\sigma_1+\sigma_2) \end{pmatrix} \tag{2}$$

Lemma: *The matrix $-J_0$ is a non-singular M-matrix if $\mathcal{R}_c < 1$.*

Proof: Note that the negative of (2), $-J_0$, has z-pattern [10], and its leading principal minors are $(\beta+\sigma_1), (\beta+\sigma_1)(\tau_1+\sigma_1+\sigma_2)(1-\mathcal{R}_{I_1}), (\beta+\sigma_1)(\tau_1+\sigma_1+\sigma_2)(\tau_2+\gamma_2+\sigma_1+\sigma_2)(1-\mathcal{R}_{I_1}-\mathcal{R}_{I_2})$, and $(\beta+\sigma_1)(\tau_1+\sigma_1+\sigma_2)(\tau_1+\gamma_2+\sigma_1+\sigma_2)(\gamma_1+\sigma_1+\sigma_2)(1-\mathcal{R}_c)$. Since $\mathcal{R}_c > \mathcal{R}_{I_1}+\mathcal{R}_{I_2} > \mathcal{R}_{I_1}$, we see that all the leading principal minors of $-J_0$ are positive if $\mathcal{R}_c < 1$. This proves the lemma. The above lemma proves that all the eigenvalues of $-J_0$ are positive or have positive real part if $\mathcal{R}_c < 1$. Hence, all the eigenvalues of J_0 are negative or have negative real part if $\mathcal{R}_c < 1$. We, therefore conclude that the disease-free equilibrium is locally asymptotically stable if $\mathcal{R}_c < 1$, and unstable if $\mathcal{R}_c > 1$. The implication of this result is that Covid-19 can be eradicated from the population using these control measures when $\mathcal{R}_c < 1$, if the initial sizes of the infected sub-populations are in the basin of attraction of the disease-free equilibrium, E^0.

4 A Test for Global Stability of Disease-free Equilibrium

Consider the system

$$\begin{aligned}
\frac{dS(t)}{dt} &= \Lambda - (\rho_1 + \rho_3 + \rho_4)S(t) - \sigma_1 S(t),\\
\frac{dP_1(t)}{dt} &= \rho_1 S(t) - \sigma_1 P_1(t),\\
\frac{dP_2(t)}{dt} &= \rho_3 S(t) - \sigma_1 P_2(t),\\
\frac{dP_{12}(t)}{dt} &= \rho_4 S(t) - \sigma_1 P_{12}(t),\\
\frac{dR(t)}{dt} &= -\sigma_1 R(t),
\end{aligned} \tag{3}$$

and the system

$$\begin{aligned}
\frac{dE(t)}{dt} &= \rho_2\lambda(t)S(t) + \lambda(t)\alpha_1 P_1(t) + \lambda(t)\alpha_2 P_2(t) + \lambda(t)\alpha_{12}P_{12}(t) - (\beta + \sigma_1)E(t),\\
\frac{dI_1(t)}{dt} &= \alpha\beta E(t) - (\tau_1 + \sigma_1 + \sigma_2)I_1(t),\\
\frac{dI_2(t)}{dt} &= (1-\alpha)\beta E(t) - (\tau_2 + \gamma_2 + \sigma_1 + \sigma_2)I_2(t),\\
\frac{dT(t)}{dt} &= \tau_1 I_1(t) + \tau_2 I_2(t) - (\gamma_1 + \sigma_1 + \sigma_2)T(t),
\end{aligned} \tag{4}$$

obtaned by splitting (1) into equation for uninfected compartments (3) and equations for infected infected compartments. This can be written as

$$x'(t) = f_1(x, 0) \tag{5}$$

$$y'(t) = f_2(x, y),\ \ f_2(x, 0) = 0 \tag{6}$$

where x and y represent the uninfected and infected compartments respectively. The disease-free equilibrium for (3) is $E^{01} = \left(\frac{\Lambda}{k}, \frac{\rho_1\Lambda}{\sigma_1 k}, \frac{\rho_3\Lambda}{\sigma_1 k}, \frac{\rho_4\Lambda}{\sigma_1 k}, 0\right)$. So the disease-free equilibrium of the entire system becomes $E^0 = (E^{01}, 0, 0, 0, 0)$. The

solution to (3) is easily seen to be

$$\begin{aligned} S(t) &= \frac{\Lambda}{k} - \left(\frac{\Lambda}{k} - S(0)\right) e^{-kt} \\ P_1(t) &= \frac{\rho_1\Lambda}{\sigma_1 k} - \frac{\rho_1}{\sigma_1 - k}\left(\frac{\Lambda}{k} - S(0)\right) e^{-k_t} + C_1 e^{-\sigma_1 t} \\ P_2(t) &= \frac{\rho_3\Lambda}{\sigma_1 k} - \frac{\rho_3}{\sigma_1 - k}\left(\frac{\Lambda}{k} - S(0)\right) e^{-kt} + C_2 e^{-\sigma_1 t} \\ P_{12}(t) &= \frac{\rho_4\Lambda}{\sigma_1 k} - \frac{\rho_4}{\sigma_1 - k}\left(\frac{\Lambda}{k} - S(0)\right) e^{-kt} + C_3 e^{-\sigma_1 t} \\ R(t) &= R(0)e^{-\sigma_1 t} \end{aligned} \tag{7}$$

where $k = \rho_1 + \rho_3 + \rho_4 + \sigma_1$, and $C_1 = P_1(0) - \frac{\rho_1\Lambda}{k} + \frac{\rho_1}{\sigma_1 - k}\left(\frac{\Lambda}{k} - S(0)\right)$, $C_2 = P_2(0) - \frac{\rho_3\Lambda}{k} + \frac{\rho_3}{\sigma_1 - k}\left(\frac{\Lambda}{k} - S(0)\right)$, $C_3 = P_{12}(0) - \frac{\rho_4\Lambda}{k} + \frac{\rho_4}{\sigma_1 - k}\left(\frac{\Lambda}{k} - S(0)\right)$. So that as $t \to \infty$, $x(t) \to E^{01}$. Hence, E^{01} is globally asymptotically stable. The Jacobian matrix, B of $f_2(x, y)$ at E^0 is same as the matrix, J_0. Hence,

$$By - f_2(x, y) = \begin{pmatrix} \lambda\left(\alpha_1\left(\frac{\rho_1\Lambda}{\sigma_1 k} - P_1\right) + \alpha_2\left(\frac{\rho_3\Lambda}{\sigma_1 k} - P_2\right) + \alpha_{12}\left(\frac{\rho_4\Lambda}{\sigma_1 k} - P_{12}\right) - \rho_2 S\right) \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

From (7), we see that $P_1(t) \leq \frac{\rho_1\Lambda}{k_1}$, $P_2(t) \leq \frac{\rho_3\Lambda}{k_1}$, $P_{12}(t) \leq \frac{\rho_4\Lambda}{k_1}, \forall t$. So, $\lambda(\alpha_1(\frac{\rho_1\Lambda}{k_1} - P_1) + \alpha_2(\frac{\rho_3\Lambda}{k_1} - P_2) + \alpha_{12}(\frac{\rho_4\Lambda}{k_1} - P_{12}) \geq 0$. Therefore, by [11], the disease-free equilibrium is globally asymptotically stable if $\lambda(\alpha_1(\frac{\rho_1\Lambda}{k_1} - P_1) + \alpha_2(\frac{\rho_3\Lambda}{k_1} - P_2) + \alpha_{12}(\frac{\rho_4\Lambda}{k_1} - P_{12}) \geq \rho_2 S$, and unstable, otherwise. Suppose the latter happens, then the global instability is caused by the proportion, ρ_2 of people who do not observe any form of control or protection against Covid-19.

5 Optimal Control of Covid-19 in Nigeria

In this section, we seek the values of the control parameters, $\rho_1, \rho_3, \rho_4, \tau_1,$ and τ_2, that would help in effective control of Covid-19 in Nigeria. To do this, we assume

that these parameters are not constants, but functions of t. Therefore, we consider an optimal control problem with objective function of the form

$$\begin{aligned} J(S,E,I_1,I_2,\rho_1,\rho_3,\rho_4,\tau_1,\tau_2) = \int_0^T & (A_1S(t) + A_2E(t) + A_3I_1(t) + A_4I_2(t) \\ & + \frac{A_5}{2}\rho_1^2(t) + \frac{A_6}{2}\rho_3^2(t) \\ & + \frac{A_7}{2}\rho_4^2(t) + \frac{A_8}{2}\tau_1^2(t) + \frac{A_9}{2}\tau_2^2(t))dt, \end{aligned} \tag{8}$$

where the constants $A_i, i = 1,2,3,\cdots,9$ are positive weights which help to balance each term in the integrand. The quadratic terms in the sum $\frac{A_5}{2}\rho_1^2(t) + \frac{A_6}{2}\rho_3^2(t) + \frac{A_7}{2}\rho_4^2(t) + \frac{A_8}{2}\tau_1^2(t) + \frac{A_9}{2}\tau_2^2(t)$ can be interpreted to be the costs associated with implementing the respective control measures. Using Pontryagin maximum principle [12], we seek the state variables $x^*(t) = (S^*(t), E^*(t), I_1^*(t), I_2^*(t))$, and the control parameters, $u^*(t) = (\rho_1^*(t), \rho_3^*(t), \rho_4^*(t), \tau_1^*(t), \tau_1^*(t))$, such that the Hamiltonian H given by

$$\begin{aligned} H = & A_1S(t) + A_2E(t) + A_3I_1(t) + A_4I_2(t) + \frac{A_5}{2}\rho_1^2(t) + \frac{A_6}{2}\rho_3^2(t) \\ & + \frac{A_7}{2}\rho_4^2(t) + \frac{A_8}{2}\tau_1^2(t) + \frac{A_9}{2}\tau_2^2(t) \\ & + \phi_1\frac{dS(t)}{dt} + \phi_2\frac{dP_1(t)}{dt} + \phi_3\frac{dP_2(t)}{dt} + \phi_4\frac{dP_{12}(t)}{dt} + \phi_5\frac{dE(t)}{dt} \\ & + \phi_6\frac{dI_1(t)}{dt} + \phi_7\frac{dI_2(t)}{dt} + \phi_8\frac{dT(t)}{dt} + \phi_9\frac{dR(t)}{dt}. \end{aligned} \tag{9}$$

satisfies

$$H(t,x,u,\phi) \geq H(t,x^*,u^*,\phi) \tag{10}$$

$$\frac{H(t,x^*,u^*,\phi)}{du} = 0 \tag{11}$$

$$\frac{d\phi(t)}{dt} = -\frac{dH(t,x^*,u^*,\phi)}{dx} \tag{12}$$

$$\frac{dx(t)}{dt} = \frac{dH(t,x^*,u^*,\phi)}{d\phi}, \tag{13}$$

where $\phi_i(t), i = 1, 2, 3, 4, \cdots, 9$ are the adjoint variables. Using (12), we have that $\phi(t)$ satisfy the following system of differential equations

$$\begin{aligned}
\frac{d\phi_1(t)}{dt} &= \rho_1(\phi_1 - \phi_2) + \rho_3(\phi_1 - \phi_3) + \rho_4(\phi_1 - \phi_4) + \rho_2\lambda\left(1 - \frac{S}{N}\right)(\phi_1 - \phi_5) \\
&\quad + \frac{\alpha_1 P_1\lambda}{N}(\phi_5 - \phi_2) + \frac{\alpha_2 P_2\lambda}{N}(\phi_5 - \phi_3) + \frac{\alpha_{12} P_{12}\lambda}{N}(\phi_5 - \phi_4) + \sigma_1\phi_1 - A_1 \\
\frac{d\phi_2(t)}{dt} &= \frac{\rho_2 S\lambda}{N}(\phi_5 - \phi_1) + \alpha_1\lambda\left(1 - \frac{P_1}{N}\right)(\phi_2 - \phi_5) + \frac{\alpha_2 P_2\lambda}{N}(\phi_5 - \phi_3) \\
&\quad + \frac{\alpha_{12} P_{12}\lambda}{N}(\phi_5 - \phi_4) + \sigma_1\phi_2 \\
\frac{d\phi_3(t)}{dt} &= \frac{\rho_2 S\lambda}{N}(\phi_5 - \phi_1) + \frac{\alpha_1 P_1\lambda}{N}(\phi_5 - \phi_2) + \frac{\alpha_{12} P_{12}\lambda}{N}(\phi_5 - \phi_4) \\
&\quad + \alpha_2\lambda\left(1 - \frac{P_1}{N}\right)(\phi_3 - \phi_5) + \sigma_1\phi_3 \\
\frac{d\phi_4(t)}{dt} &= \frac{\rho_2 S\lambda}{N}(\phi_5 - \phi_1) + \frac{\alpha_1 P_1\lambda}{N}(\phi_5 - \phi_2) + \frac{\alpha_2 P_2\lambda}{N}(\phi_5 - \phi_3) \\
&\quad + \alpha_{12}\lambda\left(1 - \frac{P_{12}}{N}\right)(\phi_4 - \phi_5) + \sigma_1\phi_4 \\
\frac{d\phi_5(t)}{dt} &= \frac{\rho_2 S\lambda}{N}(\phi_5 - \phi_1) + \frac{\alpha_1 P_1\lambda}{N}(\phi_5 - \phi_2) + \frac{\alpha_2 P_2\lambda}{N}(\phi_5 - \phi_3) \\
&\quad + \frac{\alpha_{12} P_{12}\lambda}{N}(\phi_5 - \phi_4) + \alpha\beta(\phi_7 - \phi_6) - \beta\sigma_7 + (\beta + \sigma_1)\phi_5 - A_2 \\
\frac{d\phi_6(t)}{dt} &= \frac{\kappa_1 - \lambda}{N}\left(\rho_2 S(\phi_1 - \phi_5) + \alpha_1 P_1(\phi_2 - \phi_5) + \alpha_2 P_2(\phi_3 - \phi_5)\right. \\
&\quad \left. + \alpha_{12} P_{12}(\phi_4 - \phi_5) - \tau_1\phi_8 + (\tau_1 + \sigma_1 + \sigma_2)\phi_6 - A_3\right. \\
\frac{d\phi_7(t)}{dt} &= \frac{\kappa_2 - \lambda}{N}\left(\rho_2 S(\phi_1 - \phi_5) + \alpha_1 P_1(\phi_2 - \phi_5) + \alpha_2 P_2(\phi_3 - \phi_5)\right. \\
&\quad \left. + \alpha_{12} P_{12}(\phi_4 - \phi_5) - \tau_2\phi_8 - \gamma_2\phi_9 + (\tau_2 + \gamma_2 + \sigma_1 + \sigma_2)\phi_7 - A_4\right. \\
\frac{d\phi_8(t)}{dt} &= \frac{\kappa_3 - \lambda}{N}\left(\rho_2 S(\phi_1 - \phi_5) + \alpha_1 P_1(\phi_2 - \phi_5) + \alpha_2 P_2(\phi_3 - \phi_5)\right. \\
&\quad \left. + \alpha_{12} P_{12}(\phi_4 - \phi_5) - \tau_2\phi_8 - \gamma_1\phi_9 + (\gamma_1 + \sigma_1 + \sigma_2)\phi_8\right. \\
\frac{d\phi_9(t)}{dt} &= \frac{\rho_2 S\lambda}{N}(\phi_5 - \phi_1) + \frac{\alpha_1 P_1\lambda}{N}(\phi_5 - \phi_2) + \frac{\alpha_2 P_2\lambda}{N}(\phi_5 - \phi_3) \\
&\quad + \frac{\alpha_{12} P_{12}\lambda}{N}(\phi_5 - \phi_4) + \sigma_1\phi_9
\end{aligned} \tag{14}$$

with the terminal condition $\phi_n(T) = 0,\ n = 1, 2, 3, 4, \cdots, 9$. From the optimality condition (11), we obtain the optimal controls as
$\rho_1^*(t) = \frac{(\phi_1(t)-\phi_2(t))S^*(t)}{A_5},\ \rho_3^*(t) = \frac{(\phi_1(t)-\phi_3(t))S^*(t)}{A_6}, \rho_4^*(t) = \frac{(\phi_1(t)-\phi_4(t))S^*(t)}{A_7},\ \tau_1^*(t) = \frac{(\phi_6(t)-\phi_8(t))I_1^*(t)}{A_8}, \tau_2^*(t) = \frac{(\phi_7(t)-\phi_8(t))I_2^*(t)}{A_9}$.
Therefore, the optimal controls can be characterized as

$$\begin{aligned}
\rho_1^*(t) &= max\left(a_1, min\left(b_1, \frac{(\phi_1(t)-\phi_2(t))S^*(t)}{A_5}\right)\right)\\
\rho_3^*(t) &= max\left(a_2, min\left(b_2, \frac{(\phi_1(t)-\phi_3(t))S^*(t)}{A_6}\right)\right)\\
\rho_4^*(t) &= max\left(a_3, min\left(b_3, \frac{(\phi_1(t)-\phi_4(t))S^*(t)}{A_7}\right)\right)\\
\tau_1^*(t) &= max\left(a_4, min\left(b_4, \frac{(\phi_6(t)-\phi_8(t))I_1^*(t)}{A_8}\right)\right)\\
\tau_2^*(t) &= max\left(a_5, min\left(b_5, \frac{(\phi_7(t)-\phi_8(t))I_2^*(t)}{A_9}\right)\right)
\end{aligned} \tag{15}$$

Hence, (15) gives the values of the control parameters that would guarantee optimal control of Covid-19 in Nigeria.

6 Numerical Solution and Discussion

To solve the optimal control problem, the forward-backward sweep numerical algorithm is used. This algorithm uses R-K order 4 numerical method to solve first, the model system forward in time, and then the adjoint system backward in time. The optimal controls are updated at the end of each time step. The algorithm is implemented until the prescribed stopping criterion is reached. The parameter values (Table 3) used in the numerical solution were estimated from the daily Covid-19 data obtained from the website(www.ncdc.org) of Nigeria Center for Disease Control(NCDC).

In all the figures (i.e. Figures 2–6), the label "with control" refers to the control values given in Table 6 and the label "with optimal control" corresponds

Table 3: Parameter values used in this model.

parameter	*value*	*source*	*parameter*	*value*	*source*
β	1/6	[2]	κ_1	0.045100437278262	fitted
ρ_1	0.000280181157232	fitted	κ_2	0.061036824980133	fitted
ρ_2	0.085133094732974	fitted	κ_3	0.020000015483272	fitted
ρ_3	0.000289546300842	fitted	α	0.012575411288940	fitted
ρ_4	0.000290717830511	fitted	τ_1	0.035099986980849	fitted
α_1	0.025410008732941	fitted	τ_2	0.016556854673704	fitted
α_2	0.0321	fitted	γ_1	0.025100026262081	fitted
α_{12}	0.078100023039961	fitted	γ_2	0.011635794641596	fitted
σ_1	0.0035		σ_2	0.00003	

to control parameters taking the optimal controls values as obtained at the end of the previous section. In our model, we consider five control measures, namely vaccination, self-treatment, the combination of both vaccination and self-treatment, quarantine and treatment of symptomatically infectious humans and finally quarantine and treatment of asymptomatically infectious humans.

Figure 2 compares the simulation of the dynamics of the Nigerian population that is susceptible to Covid-19 for the case where all the five controls are implemented on the one hand and where all five optimal controls are implemented. In the latter case, it is observed that the number of susceptible individuals decreases drastically from the 27th day and reaches zero from the 30th day indicating the eradication of the Covid-19 virus, which is completely different from the former case where the size of the susceptible population remains constant all along the simulation time.

Figures 3 and 4 compare the simulation of the dynamics of the Nigerian population that is symptomatically-infected and asymptomatically-infected by Covid-19 respectively. Both figures exhibit the same trends but with different

proportions. For the first case, i.e. "with control", we observe in both figures an increase in the number of infections which reaches at peak on the twelve day and thereafter starts decreasing till the last day of the simulation (here the 50th) where it reaches a maximum value. For the second case, i.e. "with optimal control", we observe a different situation. The optimal control measures are so effective that from the initial time, we observe a fast decrease in the infected population and the disease is eradicated after the 25th day.

Figure 5 compares the simulation of the dynamics of the Nigerian population treated from Covid-19 when control measures are implemented and when optimal control measures are implemented. From the figure, we observe a fast increase in the number of treated individuals during the first twelve days under optimal control measures compared to the case of simple control where there is initially a slight decrease followed by a slow increase that suddenly reaches a plateau. After the twelfth-day period, the first curve (with optimal control) starts decreasing linearly, this behaviour is the consequence of the fact that the spread of the disease is being eradicated.

Figure 6 shows the simulations of the recovered individuals under the same conditions as those of the previous figures. We observe that both curves (with control and with optimal control) are increasing but with the difference that the optimal control curve increases faster indicating the efficiency of the related measures.

From all the simulations, it is clear that under optimal control measures, the spread of Covid-19 is well controlled and eradicated at a fast rate.

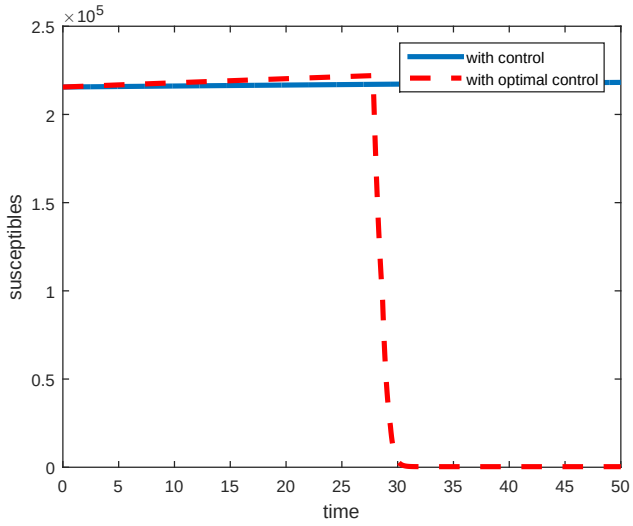

Figure 2: Graph of individuals who are susceptible to Covid-19.

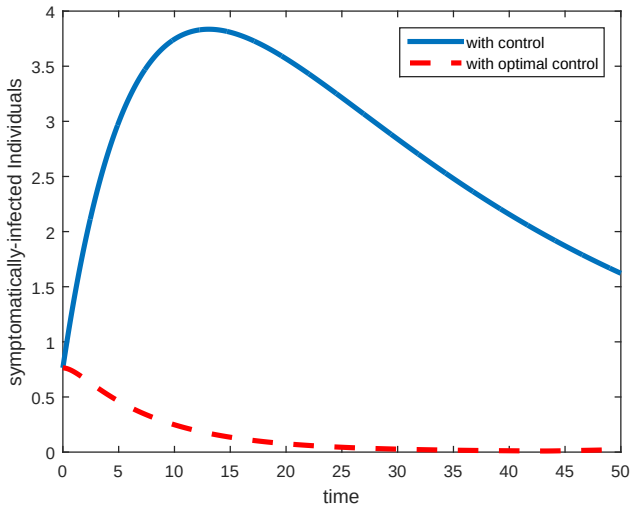

Figure 3: Graph of individuals that infectious with symptoms.

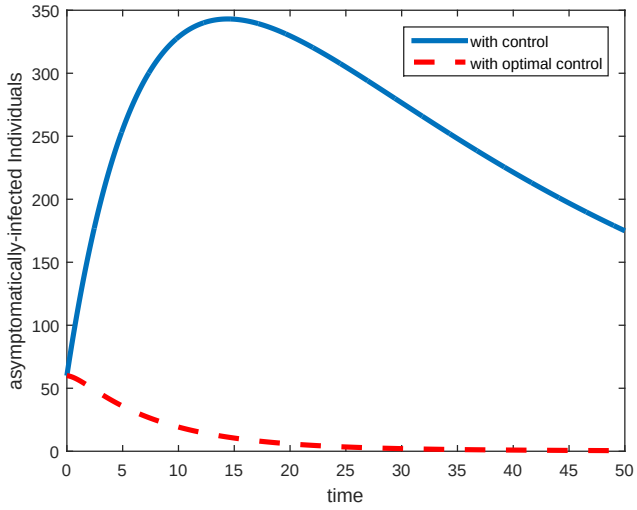

Figure 4: Graph of individuals that infectious without symptoms.

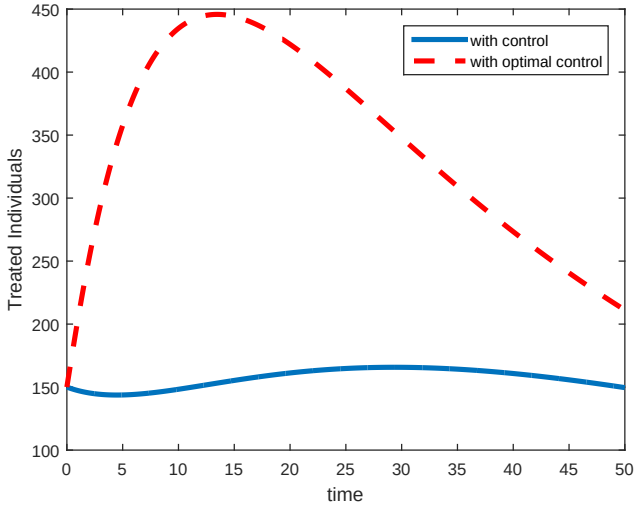

Figure 5: Graph of individuals who are being treated of Covid-19.

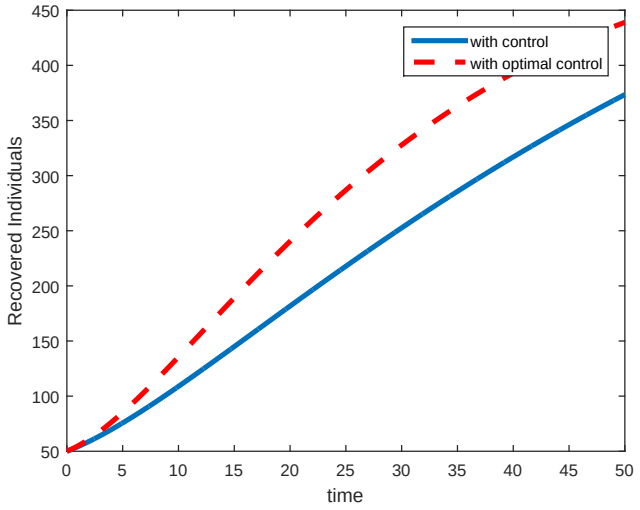

Figure 6: Graph of individuals who have recovered from Covid-19 infection.

7 Conclusion

In this paper, a mathematical model with control has been proposed for the dynamics of Covid-19 in Nigeria. From the proposed model, an optimal control model is formulated by seeking the maximum of an objective function that measures susceptible individuals, total infected individuals and total cost associated with implementing the control measures. The optimal controls are obtained using Pontryagin Maximum Principle. The numerical simulations show the effectiveness of the optimal control measures in the control and eradication of Covid-19 in Nigeria.

References

[1] Abioye, A. I., Peter, O. J., Ogunseye, H. A., Oguntolu, F. A., Oshinubi, K., Ibrahim, A. A., & Khan, I. (2021). Mathematical model of COVID-19 in Nigeria with optimal

control. *Results in Physics*, *28*, 104598. https://doi.org/10.1016/j.rinp.2021.104598

[2] Iboi, E. A., Sharomi, O., Ngonghala, C. N., & Gumel, A. B. (2020). Mathematical modeling and analysis of COVID-19 pandemic in Nigeria. *Mathematical Biosciences and Engineering*, *17*(6), 7192-7220. https://doi.org/10.3934/mbe.2020369

[3] Ogundokun, R. O., Lukman, A. F., Kibria, G. B. M., Awotunde, J. B., & Aladeitan, B. B. (2020). Predictive modelling of COVID-19 confirmed cases in Nigeria. *Infectious Disease Modelling*, *5*, 543-548. https://doi.org/10.1016/j.idm.2020.08.003

[4] Amzat, J., Aminu, K., Kolo, V. I., Akinyele, A. A., Ogundairo, J. A., & Danjibo, M. C. (2020). Coronavirus outbreak in Nigeria: Burden and socio-medical response during the first 100 days. *International Journal of Infectious Diseases*, *98*, 218-224. https://doi.org/10.1016/j.ijid.2020.06.067

[5] Avusuglo, W. S., Han, Q., Woldegerima, W. A., Asgary, A., Wu, J., Orbinski, J., Bragazzi, N. L., Ahmadi, A., & Kong, J. D. COVID-19 and Malaria Co-Infection: Do Stigmatization and Self-Medication Matter? A Case for Nigeria. Available at SSRN: https://ssrn.com/abstract=4220775

[6] Herbert, W. H. (2000). The mathematics of infectious diseases.*SIAM Review*, *42*(4), 599-653. https://doi.org/10.1137/S0036144500371907

[7] Seidu, B. (2020). Optimal strategies for control of COVID-19: A mathematical perspective. *Scientifica*, 2020, Article ID 4676274, 12 pages. https://doi.org/10.1155/2020/4676274

[8] Obsu, L. L., & Balcha, S. F. (2020). Optimal control strategies for the transmission risk of COVID-19. *Journal of Biological Dynamics*, *14*(1), 590-607. https://doi.org/10.1080/17513758.2020.1788182

[9] Van Den Driessche, P., & Watmough, J. (2002). Reproduction numbers and the sub-threshold endemic equilibria for compartmental models of infectious disease transmission. *Mathematical Biosciences*, *180*, 29-48. https://doi.org/10.1016/S0025-5564(02)00108-6

[10] Katri, P. (2010). Modeling the Transmission Dynamics of the Dengue Virus. *Open Access Dissertations*, Paper 147.

[11] Castillo-Chavez, C., Feng, Z., & Huang, W. (2012). On the computation of R_0 and its role on global stability. *Mathematical Approaches for Emerging and Reemerging Infectious Diseases: An Introduction*, Springer, Berlin, 229. https://doi.org/10.1007/978-1-4757-3667-0_13

[12] Pontryagin, L. S., Boltayanskii, V. G., Gamkrelidze, R. V., & Mishchenko, E. F. (1962). *The mathematical theory of optimal processes*. Wiley.