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A Best Proximity Point Theorem for *G*-Proximal $(\delta, 1-\delta)$ Weak Contraction in Complete Metric Space Endowed with a Graph

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Abstract

The notion of $(\delta, 1-\delta)$ weak contraction appeared in [1]. In this paper, we consider that the map satisfying the $(\delta, 1-\delta)$ weak contraction is a non-self map, and obtain a best proximity point theorem in complete metric space endowed with a graph.

1. Introduction and Preliminaries

At first we recall the following

Definition 1.1. [1] Let (X, d) be a metric space. A map $T : X \mapsto X$ is called a $(\delta, 1-\delta)$ *weak contraction* if there exists $\delta \in (0, 1)$ such that the following holds

$$d(Tx, Ty) \leq \delta d(x, y) + (1-\delta) d(y, Tx).$$

On the other hand, let W and V be two nonempty subsets of a metric space (X, d) and let $S : W \mapsto V$ be a non-self map. If $W \cap V$ is nonempty, then the equation $Sx = x$ may not have a solution. Naturally the following arises

Question 1.2. How far is the distance between x and Sx?

The problem of global optimization for determining the minimum value of the

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distance $d(x, Sx) = \min\{d(x, y) : x \in W \text{ and } y \in V\}$ is the study of best proximity point theory. Since the early paper of [2], many best proximity point theorems have been obtained, and for example see references [9-23] contained in [3].

Notation 1.3. Throughout this paper

(a) W and V denote nonempty subsets of a metric space (X, d).

(b) $d(W, V) := \inf\{d(x, y) : x \in W \text{ and } y \in V\}$.

(c) $W_0 = \{x \in W : d(x, y) = d(W, V) \text{ for some } y \in V\}$.

(d) $V_0 = \{y \in V : d(x, y) = d(W, V) \text{ for some } x \in W\}$.

The notion of proximal contraction appeared in [4], now we introduce the following

Definition 1.4. Let $S : W \mapsto V$ be a non-self mapping. We say S a *proximal* $(\delta, 1-\delta)$ *weak contraction* if there exists $\delta \in (0, 1)$ and $u_1, u_2, x, y \in W$ such that $d(u_1, Sx) = d(W, V)$ and $d(u_2, Sy) = d(W, V)$ implies

$$d(u_1, u_2) \leq \delta d(x, y) + (1-\delta) d(y, u_1).$$

The notion of G-proximal Kannan mapping appeared in [3], now we introduce the following

Definition 1.5. Let (X, d) be a metric space, and $G = (V(G), E(G))$ be a directed graph such that $V(G) = X$. A non-self mapping $S : W \mapsto V$ is called a *G-proximal* $(\delta, 1-\delta)$ *weak contraction*, if there exists $\delta \in (0, 1)$ such that $(x, y) \in E(G)$, $d(u, Sx) = d(W, V)$ and $d(v, Sy) = d(W, V)$ implies

$$d(u, v) \leq \delta d(x, y) + (1-\delta) d(y, u),$$

where $x, y, u, v \in W$.

Definition 1.6. [3] Let (X, d) be a metric space and $G = (V(G), E(G))$ be a directed graph such that $V(G) = X$. A non-self mapping $S : W \mapsto V$ is called *proximally G-edge-preserving*, if for each $x, y, u, v \in W$, $(x, y) \in E(G)$, $d(u, Sx) = d(W, V)$, and $d(v, Sy) = d(W, V)$ implies $(u, v) \in E(G)$.

2. Main Result

Our main result is as follows, which is a best proximity point theorem for a G-proximal $(\delta, 1-\delta)$ weak contraction in complete metric space endowed with a directed graph.

Theorem 2.1. *Let* (X, d) *be a complete metric space,* $G = (V(G), E(G))$ *be a directed graph such that* $V(G) = X$. *Let W and V be nonempty closed subsets of X with* W_0 *nonempty. Let* $S : W \mapsto V$ *be a non-self mapping satisfying the following properties*:

(a) *S is proximally G-edge-preserving, continuous and G-proximal* $(\delta, 1-\delta)$ *weak contraction such that* $S(W_0) \subset V_0$,

(b) *there exists* $x_0, x_1 \in W_0$ *such that*

$$d(x_1, Sx_0) = d(W, V) \text{ and } (x_0, x_1) \in E(G).$$

Then S has a best proximity point in W, that is, there exists an element $w \in W$ *such that* $d(w, Sw) = d(W, V)$. *Further the sequence* $\{x_n\}$ *defined by*

$$d(x_n, Sx_{n-1}) = d(W, V) \tag{1}$$

for all $n \in \mathbb{N}$ *converges to the element w.*

Proof. From condition (b), there exists $x_0, x_1 \in W_0$ such that

$$d(x_1, Sx_0) = d(W, V) \text{ and } (x_0, x_1) \in E(G). \tag{2}$$

Since $S(W_0) \subseteq V_0$, we have $Sx_1 \in V_0$ and hence there exists $x_2 \in W_0$ such that

$$d(x_2, Sx_1) = d(W, V). \tag{3}$$

By the proximally G-edge preserving of S and using both (2) and (3), we get $(x_1, x_2) \in E(G)$. By continuing this process, we can form the sequence $\{x_n\}$ in W_0 such that

$$d(x_n, Sx_{n-1}) = d(W, V) \text{ with } (x_{n-1}, x_n) \in E(G), \text{ for all } n \in \mathbb{N}. \tag{4}$$

Next we show that S has a best proximity point in W. Suppose there exists $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1}$. By using (4), we obtain that $d(x_{n_0}, Sx_{n_0}) = d(x_{n_0+1}, Sx_{n_0}) =$

$d(W, V)$, and so x_{n_0} is a best proximity point of S. Now we suppose that $x_{n-1} \neq x_n$ for all $n \in \mathbb{N}$. We show that $\{x_n\}$ is a Cauchy sequence in W. As S is G-proximal $(\delta, 1-\delta)$ weak contraction, and for each $n \in \mathbb{N}$, $(x_{n-1}, x_n) \in E(G)$, $d(x_n, Sx_{n-1}) = d(W, V)$, and $d(x_{n+1}, Sx_n) = d(W, V)$, then we have

$$d(x_n, x_{n+1}) \leq \delta d(x_{n-1}, x_n) + (1-\delta) d(x_n, x_n) = \delta d(x_{n-1}, x_n).$$

By the above inequality, we have

$$d(x_1, x_2) \leq \delta d(x_0, x_1)$$

and hence

$$d(x_2, x_3) \leq \delta^2 d(x_1, x_2)$$

By induction, we deduce the following

$$d(x_n, x_{n+1}) \leq \delta^n d(x_0, x_1) \tag{5}$$

for all $n \in \mathbb{N}$. From (5), for each $m, n \in \mathbb{N}$ with $m > n$, we deduce the following

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ &\leq \delta^n d(x_0, x_1) + \delta^{n+1} d(x_0, x_1) + \cdots + \delta^{m-1} d(x_0, x_1) \\ &= d(x_0, x_1) \sum_{i=n}^{m-1} \delta^i \\ &\leq \frac{\delta^n}{1-\delta} d(x_0, x_1). \end{aligned}$$

Since $\delta \in (0, 1)$, it follows that $\{x_n\}$ is a Cauchy sequence in W. Since W is closed, there exists $w \in W$ such that $x_n \to w$. By continuity of S, we have $Sx_n \to Sw$ as $n \to \infty$. As the metric function is continuous, we obtain

$$d(x_{n+1}, Sx_n) \to d(w, Sw) \text{ as } n \to \infty.$$

Similarly, by (4), we have

$$d(w, Sw) = d(W, V).$$

It follows that $w \in W$ is a best proximity point of *S*. Moreover, the sequence $\{x_n\}$ defined by

$$d(x_{n+1}, Sx_n) = d(W, V), \quad n \in \mathbb{N}$$

converges to an element *w*, and the proof is completed. □

3. Open Problem

First we recall the following

Definition 3.1. [5] Let (X, d) be a metric space. A map $T : X \mapsto X$ is called a $(\delta, 1 - 3\delta)$ *weak Reich contraction* if there exists $\delta \in \left(0, \frac{1}{3}\right)$ such that the following holds for all $x, y \in X$

$$d(Tx, Ty) \leq \delta[d(x, y) + d(x, Tx) + d(y, Ty)] + (1 - 3\delta) d(y, Tx).$$

Now we introduce the following two new concepts.

Definition 3.2. Let $S : W \mapsto V$ be a non-self mapping. We say *S* a *proximal* $(\delta, 1 - 3\delta)$ *weak Reich contraction* if there exists $\delta \in \left(0, \frac{1}{3}\right)$ and $u_1, u_2, x, y \in W$ such that $d(u_1, Sx) = d(W, V)$ and $d(u_2, Sy) = d(W, V)$ implies

$$d(u_1, u_2) \leq \delta[d(x, y) + d(x, u_1) + d(y, u_2)] + (1 - 3\delta) d(y, u_1)$$

Definition 3.3. Let (X, d) be a metric space, and $G = (V(G), E(G))$ be a directed graph such that $V(G) = X$. A non-self mapping $S : W \mapsto V$ is called a *G-proximal* $(\delta, 1 - 3\delta)$ *weak Reich contraction*, if there exists $\delta \in \left(0, \frac{1}{3}\right)$ such that $(x, y) \in E(G)$, $d(u, Sx) = d(W, V)$, and $d(v, Sy) = d(W, V)$ implies

$$d(u, v) \leq \delta[d(x, y) + d(x, u) + d(y, v)] + (1 - 3\delta) d(y, u),$$

where $x, y, u, v \in W$.

The open problem is to prove or disprove the following. The conjecture can be regarded as a best proximity point theorem for a *G*-proximal $(\delta, 1 - 3\delta)$ weak Reich contraction in complete metric space endowed with a directed graph.

Theorem 3.4. *Let* (X, d) *be a complete metric space,* $G = (V(G), E(G))$ *be a directed graph such that* $V(G) = X$. *Let W and V be nonempty closed subsets of X with* W_0 *nonempty. Let* $S : W \mapsto V$ *be a non-self mapping satisfying the following properties*:

(a) *S is proximally G-edge-preserving, continuous and G-proximal* $(\delta, 1 - 3\delta)$ *weak Reich contraction such that* $S(W_0) \subset V_0$.

(b) *there exists* $x_0, x_1 \in W_0$ *such that*

$$d(x_1, Sx_0) = d(W, V) \text{ and } (x_0, x_1) \in E(G).$$

Then S has a best proximity point in W, that is, there exists an element $w \in W$ *such that* $d(w, Sw) = d(W, V)$. *Further the sequence* $\{x_n\}$ *defined by*

$$d(x_n, Sx_{n-1}) = d(W, V)$$

for all $n \in \mathbb{N}$ *converges to the element w.*

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