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Two Families of m-fold Symmetric Bi-univalent Functions Involving a Linear Combination of Bazilevic Starlike and Convex Functions

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Abstract

In the present paper, we define two new families $KM_{\Sigma_m}(\lambda,\gamma,\delta;\alpha)$ and $KM^*_{\Sigma_m}(\lambda,\gamma,\delta;\beta)$ of holomorphic and m-fold symmetric bi-univalent functions associated with the Bazilevic starlike and convex functions in the open unit disk U. We find upper bounds for the first two Taylor-Maclaurin $|a_{m+1}|$ and $|a_{2m+1}|$ for functions in these families. Further, we point out several special cases for our results.

1. Introduction

Denote by $\mathcal{A}$ the family of functions f that are holomorphic in the open unit disk $U=\{z\in\mathbb{C}:|z|<1\}$ and normalized by the conditions $f(0)=f'(0)-1=0$ and having the form:

$$f(z)=z+\sum_{k=2}^{\infty}a_kz^k\,. \tag{1.1}$$

We also denote by S the subfamily of $\mathcal{A}$ consisting of functions satisfying (1.1) which are also univalent in U.

A function $f\in\mathcal{A}$ is called starlike of order δ $(0\le\delta<1)$, if

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$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \delta, \quad (z \in U).$$

Singh [21] introduced and studied Bazilevic function that is the function f such that

$$\operatorname{Re}\left\{\frac{z^{1-\gamma} f'(z)}{(f(z))^{1-\gamma}}\right\} > 0, \quad (z \in U, \gamma \geq 0).$$

According to the Koebe one-quarter theorem (see [8]), every function $f \in S$ has an inverse f^{-1} which satisfies

$$f^{-1}(f(z)) = z, (z \in U)$$

and

$$f(f^{-1}(w)) = w, \qquad \left(|w| < r_0(f), r_0(f) \geq \frac{1}{4}\right),$$

where

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots. \quad (1.2)$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in U if both f and f^{-1} are univalent in U. We denote by Σ the family of bi-univalent functions in U given by (1.1). For a brief history and interesting examples in the family Σ see the pioneering work on this subject by Srivastava et al. [24], which actually revived the study of bi-univalent functions in recent years. In a considerably large number of sequels to the aforementioned work of Srivastava et al. [24], several different sub families of the bi-univalent function family Σ were introduced and studied analogously by the many authors (see, for example, [2,3,5,11,15,18,22,27,28,30,33]).

For each function $f \in S$, the function $h(z) = \sqrt[m]{f(z^m)}$, $(z \in U, m \in \mathbb{N})$ is univalent and maps the unit disk U into a region with m-fold symmetry. A function is said to be m-fold symmetric (see [12]) if it has the following normalized form:

$$f(z) = z + \sum_{k=1}^{\infty} a_{mk+1} z^{mk+1}, \quad (z \in U, m \in \mathbb{N}). \quad (1.3)$$

We denote by S_m the family of m-fold symmetric univalent functions in U, which are normalized by the series expansion (1.3). In fact, the functions in the family S are one-fold symmetric.

In [25] Srivastava et al. defined m-fold symmetric bi-univalent functions analogues to the concept of m-fold symmetric univalent functions. They gave some important results, such as each function $f \in \Sigma$ generates an m-fold symmetric bi-univalent function for each $m \in \mathbb{N}$. Furthermore, for the normalized form of f given by (1.3), they obtained the series expansion for f^{-1} as follows:

$$g(w) = w - a_{m+1}w^{m+1} + [(m+1)a_{m+1}^2 - a_{2m+1}]w^{2m+1}$$
$$-\left[\frac{1}{2}(m+1)(3m+2)a_{m+1}^3 - (3m+2)a_{m+1}a_{2m+1} + a_{3m+1}\right]w^{3m+1} + \cdots, \quad (1.4)$$

where $f^{-1} = g$. We denote by Σ_m the family of m-fold symmetric bi-univalent functions in U. It is easily seen that for $m = 1$, the formula (1.4) coincides with the formula (1.2) of the family Σ. Some examples of m-fold symmetric bi-univalent functions are given as follows:

$$\left(\frac{z^m}{1-z^m}\right)^{\frac{1}{m}}, \quad \left[\frac{1}{2}\log\left(\frac{1+z^m}{1-z^m}\right)\right]^{\frac{1}{m}} \quad \text{and} \quad [-\log(1-z^m)]^{\frac{1}{m}}$$

with the corresponding inverse functions

$$\left(\frac{w^m}{1+w^m}\right)^{\frac{1}{m}}, \quad \left(\frac{e^{2w^m}-1}{e^{2w^m}+1}\right)^{\frac{1}{m}} \quad \text{and} \quad \left(\frac{e^{w^m}-1}{e^{w^m}}\right)^{\frac{1}{m}},$$

respectively.

Recently, many authors investigated bounds for various subfamilies of m-fold bi-univalent functions (see [1,4,7,13,17,19,20,23,25,26,29,31,32]).

In order to prove our main results, we require the following lemma.

Lemma 1.1 [3]. *If $h \in \mathcal{P}$, then $|c_k| \le 2$ for each $k \in \mathbb{N}$, where $\mathcal{P}$ is the family of all f all functions h holomorphic in U for which*

$$Re(h(z)) > 0, \ (z \in U),$$

where

$$h(z) = 1 + c_1z + c_2z^2 + \cdots, \ \ (z \in U).$$

2. Coefficient Estimates for the Function Family $\boldsymbol{KM_{\Sigma_m}(\lambda\,,\gamma\,,\delta;\alpha)}$

Definition 2.1. A function $f \in \Sigma_m$ given by (1.3) is said to be in the family $KM_{\Sigma_m}(\lambda,\gamma,\delta;\alpha)$ $(0 \le \delta \le 1, 0 \le \gamma \le 1, \lambda \ge 0; 0 < \alpha \le 1, m \in \mathbb{N},\ z,w \in U)$ if it satisfies the following conditions:

$$\left|arg\left(\lambda\gamma\left(\frac{z^{2-\delta}f''(z)}{\left(zf'(z)\right)^{1-\delta}}-2\right)+(\gamma(\lambda+1)+\lambda)\frac{z^{1-\delta}f'(z)}{\left(f(z)\right)^{1-\delta}}+(1-\lambda)(1-\gamma)\frac{f(z)}{z}\right)\right| < \frac{\alpha\pi}{2}, \quad (z \in U) \tag{2.1}$$

and

$$\left|arg\left(\lambda\gamma\left(\frac{w^{2-\delta}g''(w)}{\left(wg'(w)\right)^{1-\delta}}-2\right)+(\gamma(\lambda+1)+\lambda)\frac{w^{1-\delta}g'(w)}{\left(g(w)\right)^{1-\delta}}+(1-\lambda)(1-\gamma)\frac{g(w)}{w}\right)\right| < \frac{\alpha\pi}{2}, \quad (w \in U), \qquad (2.2).$$

where the function $g = f^{-1}$ is given by (1.4).

In particular, for one-fold symmetric bi-univalent functions, we denote the family $KM_{\Sigma_1}(\lambda,\gamma,\delta;\alpha) = KM_{\Sigma}(\lambda,\gamma,\delta;\alpha)$.

Remark 2.1. It should be remarked that the families $KM_{\boldsymbol{\Sigma_m}}(\lambda,\gamma,\delta;\alpha)$ and $KM_{\Sigma}(\lambda,\gamma,\delta;\alpha)$ are a generalization of well-known families consider earlier. These families are:

(1) For $\gamma = 0$, and $\delta = 1$, the class $KM_{\Sigma_m}(\lambda,\gamma,\delta;\alpha)$ reduce to the class $A_{\Sigma,m}^{\alpha,\lambda}$ which was introduced by Eker [9];

(2) For $\gamma = 0$, and $\lambda = \delta = 1$, the class $KM_{\Sigma_m}(\lambda,\gamma,\delta;\alpha)$ reduce to the class $\mathcal{H}_{\Sigma,m}^{\alpha}$ which was introduced by Srivastava et al. [25];

(3) For $\gamma = 0$ and $\delta = 1$, the class $KM_{\Sigma}(\lambda,\gamma,\delta;\alpha)$ reduce to the family $\mathcal{B}_{\Sigma}^{\alpha}(\alpha,\lambda)$ which was given by Frasin and Aouf [10];

(4) For $\gamma = 0$, and $\lambda = \delta = 1$ the class $KM_{\Sigma}(\delta,\gamma;\alpha)$ reduce to the class $\mathcal{H}_{\Sigma}^{\alpha}$ which was investigated by Srivastava et al. [24].

Theorem 2.1. *Let* $f \in KM_{\Sigma_m}(\lambda, \gamma, \delta; \alpha)$ $(0 \le \delta \le 1, 0 \le \gamma \le 1, \lambda \ge 0; 0 < \alpha \le 1, m \in \mathbb{N},\ z, w \in U)$ *be given by* (1.3). *Then*

$$|a_{m+1}| \le \frac{2\alpha}{\sqrt{\left| \begin{array}{c} \alpha \begin{bmatrix} (m+1)(\lambda\gamma(4m(m+1)+\delta+1)+(2m+\delta-1)(\lambda+\gamma)+1)+ \\ 2(\delta-1)\left(\lambda\gamma m(m+1)^2+\frac{1}{2}(\gamma(\lambda+1)+\lambda)(2m+\delta)\right) \end{bmatrix} \\ +(1-\alpha)[\lambda\gamma((m+1)^2+\delta)+(\delta+m-1)(\lambda+\gamma)+1]^2 \end{array} \right|}} \tag{2.3}$$

and

$$|a_{2m+1}| \le \frac{4(m+1)\alpha^2}{[\lambda\gamma((m+1)^2+\delta)+(\delta+m-1)(\lambda+\gamma)+1]^2} + \frac{2\alpha}{\lambda\gamma(4m(m+1)+\delta+1)+(2m+\delta-1)(\lambda+\gamma)+1}. \tag{2.4}$$

Proof. It follows from conditions (2.1) and (2.2) that

$$\left(\lambda\gamma\left(\frac{z^{2-\delta}f''(z)}{\left(zf'(z)\right)^{1-\delta}} - 2\right) + (\gamma(\lambda+1)+\lambda)\frac{z^{1-\delta}f'(z)}{\left(f(z)\right)^{1-\delta}} + (1-\lambda)(1-\gamma)\frac{f(z)}{z}\right) = [p(z)]^{\alpha} \tag{2.5}$$

and

$$\left(\lambda\gamma\left(\frac{w^{2-\delta}g''(w)}{\left(wg'(w)\right)^{1-\delta}} - 2\right) + (\gamma(\lambda+1)+\lambda)\frac{w^{1-\delta}g'(w)}{\left(g(w)\right)^{1-\delta}} + (1-\lambda)(1-\gamma)\frac{g(w)}{w}\right) = [q(w)]^{\alpha}, \tag{2.6}$$

where $g = f^{-1}$ and p, q in $\mathcal{P}$ have the following series representations:

$$p(z) = 1 + p_m z^m + p_{2m} z^{2m} + p_{3m} z^{3m} + \cdots \tag{2.7}$$

and

$$q(w) = 1 + q_m w^m + q_{2m} w^{2m} + q_{3m} w^{3m} + \cdots. \tag{2.8}$$

Comparing the corresponding coefficients of (2.5) and (2.6) yields

$$[\lambda\gamma((m+1)^2+\delta)+(\delta+m-1)(\lambda+\gamma)+1]a_{m+1} = \alpha p_m. \tag{2.9}$$

$$[\lambda\gamma(4m(m+1)+\delta+1)+(2m+\delta-1)(\lambda+\gamma)+1]a_{2m+1}$$

$$+(\delta-1)\left[\lambda\gamma m(m+1)^2+\frac{1}{2}(\gamma(\lambda+1)+\lambda)(2m+\delta)\right]a_{m+1}^2$$

$$=\alpha p_{2m}+\frac{\alpha(\alpha-1)}{2}p_m^2, \tag{2.10}$$

$$-[\lambda\gamma((m+1)^2+\delta)+(\delta+m-1)(\lambda+\gamma)+1]a_{m+1}=\alpha q_m \tag{2.11}$$

and

$$[\lambda\gamma(4m(m+1)+\delta+1)+(2m+\delta-1)(\lambda+\gamma)+1]\left((m+1)a_{m+1}^2-a_{2m+1}\right)$$

$$+(\delta-1)\left[\lambda\gamma m(m+1)^2+\frac{1}{2}(\gamma(\lambda+1)+\lambda)(2m+\delta)\right]a_{m+1}^2$$

$$=\alpha q_{2m}+\frac{\alpha(\alpha-1)}{2}q_m^2. \tag{2.12}$$

Making use of (2.9) and (2.11), we obtain

$$p_m=-q_m \tag{2.13}$$

and

$$2[\lambda\gamma((m+1)^2+\delta)+(\delta+m-1)\big((\lambda+\gamma)+1\big]^2a_{m+1}^2=\alpha^2(p_m^2+q_m^2). \tag{2.14}$$

Also, from (2.10), (2.12) and (2.14), we find that

$$\Bigg[(m+1)(\lambda\gamma(4m(m+1)+\delta+1)+(2m+\delta-1)(\lambda+\gamma)+1)$$

$$+2(\delta-1)\left(\lambda\gamma m(m+1)^2+\frac{1}{2}(\gamma(\lambda+1)+\lambda)(2m+\delta)\right)\Bigg]a_{m+1}^2$$

$$=\alpha(p_{2m}+q_{2m})+\frac{\alpha(\alpha-1)}{2}(p_m^2+q_m^2)$$

$$=\alpha(p_{2m}+q_{2m})+\frac{(\alpha-1)[\lambda\gamma((m+1)^2+\delta)+(\delta+m-1)(\lambda+\gamma)+1]^2}{\alpha}a_{m+1}^2.$$

Therefore, we have

$$a_{m+1}^2=\frac{\alpha^2(p_{2m}+q_{2m})}{\alpha\begin{bmatrix}(m+1)(\lambda\gamma(4m(m+1)+\delta+1)+(2m+\delta-1)(\lambda+\gamma)+1)+\\ 2(\delta-1)\left(\lambda\gamma m(m+1)^2+\frac{1}{2}(\gamma(\lambda+1)+\lambda)(2m+\delta)\right)\end{bmatrix}\\ +(1-\alpha)[\lambda\gamma((m+1)^2+\delta)+(\delta+m-1)(\lambda+\gamma)+1]^2}. \tag{2.15}$$

Now, taking the absolute value of (2.15) and applying Lemma 1.1 for the coefficients p_{2m} and q_{2m} , we obtain

$$|a_{m+1}| \le \frac{2\alpha}{\sqrt{\left|\alpha\begin{bmatrix}(m+1)(\lambda\gamma(4m(m+1)+\delta+1)+(2m+\delta-1)(\lambda+\gamma)+1)+\\ 2(\delta-1)\left(\lambda\gamma m(m+1)^2+\frac{1}{2}(\gamma(\lambda+1)+\lambda)(2m+\delta)\right)\end{bmatrix}\\ +(1-\alpha)[\lambda\gamma((m+1)^2+\delta)+(\delta+m-1)(\lambda+\gamma)+1]^2\right|}}.$$

This gives the desired estimate for $|a_{m+1}|$ as asserted in (2.3).

In order to find the bound on $|a_{2m+1}|$, by subtracting (2.12) from (2.10), we get

$$[\lambda\gamma(4m(m+1)+\delta+1)+(2m+\delta-1)(\lambda+\gamma)+1](2a_{2m+1}-(m+1)a_{m+1}^2)$$
$$=\alpha(p_{2m}-q_{2m})+\frac{\alpha(\alpha-1)}{2}(p_m^2-q_m^2). \tag{2.16}$$

It follows from (2.13), (2.14) and (2.16) that

$$a_{2m+1}=\frac{(m+1)\alpha^2(p_m^2+q_m^2)}{2[\lambda\gamma((m+1)^2+\delta)+(\delta+m-1)(\lambda+\gamma)+1]^2}$$
$$+\frac{\alpha(p_{2m}-q_{2m})}{2[\lambda\gamma(4m(m+1)+\delta+1)+(2m+\delta-1)(\lambda+\gamma)+1]}. \tag{2.17}$$

Taking the absolute value of (2.17) and applying Lemma 1.1 once again for the coefficients p_m, p_{2m}, q_m and q_{2m} , we obtain

$$|a_{2m+1}|\le\frac{4(m+1)\alpha^2}{[\lambda\gamma((m+1)^2+\delta)+(\delta+m-1)(\lambda+\gamma)+1]^2}$$
$$+\frac{2\alpha}{\lambda\gamma(4m(m+1)+\delta+1)+(2m+\delta-1)(\lambda+\gamma)+1}$$

which completes the proof of Theorem 2.1.

Remark 2.2. In Theorem 2.1, if we choose

(1) $\gamma=0$ and $\delta=1$, then we obtain the results which was obtained by Eker [10, Theorem 1];

(2) $\gamma=0$ and $\lambda=\delta=1$, then we obtain the results which were given by Srivastava et al. [25, Theorem 2].

For one-fold symmetric bi-univalent functions, Theorem 2.1 reduces to the following corollary:

Corollary 2.1. *Let* $f \in KM_{\Sigma}(\lambda, \gamma, \delta; \alpha)(0 \leq \delta \leq 1, 0 \leq \gamma \leq 1, \lambda \geq 0; 0 < \alpha \leq 1)$ *be given by* (1.1). *Then*

$$|a_2| \leq \frac{2\alpha}{\sqrt{\left|\begin{array}{c}\alpha\left[2(\lambda\gamma(9+\delta)+(1+\delta)(\lambda+\gamma)+1)+2(\delta-1)\left(4\lambda\gamma+\frac{1}{2}(\gamma(\lambda+1)+\lambda)(2+\delta)\right)\right] \\ +(1-\alpha)[\lambda\gamma(4+\delta)+\delta(\lambda+\gamma)+1]^2\end{array}\right|}}.$$

$$|a_3| \leq \frac{8\alpha^2}{[\lambda\gamma(4+\delta)+\delta(\lambda+\gamma)+1]^2} + \frac{2\alpha}{\lambda\gamma(9+\delta)+(1+\delta)(\lambda+\gamma)+1}.$$

Remark 2.3. In Corollary 2.1, if we choose

(1) $\gamma = 0$ and $\delta = 1$, then we obtain the results which was proven by Frasin and Aouf [10, Theorem 2.2];

(2) $\gamma = 0$ and $\lambda = \delta = 1$, then we have the results which was given by Srivastava et al. [25, Theorem 1].

3. Coefficient Estimates for the Function Family $KM^*_{\Sigma_m}(\lambda, \gamma, \delta; \beta)$

Definition 3.1. A function $f \in \Sigma_m$ given by (1.3) is said to be in the family $KM^*_{\Sigma_m}(\lambda, \gamma, \delta; \beta)$ $(0 \leq \delta \leq 1, 0 \leq \gamma \leq 1, \lambda \geq 0; 0 < \beta \leq 1)$ if it satisfies the following conditions:

$$Re\left\{\lambda\gamma\left(\frac{z^{2-\delta}f''(z)}{\left(zf'(z)\right)^{1-\delta}} - 2\right) + (\gamma(\lambda+1)+\lambda)\frac{z^{1-\delta}f'(z)}{\left(f(z)\right)^{1-\delta}} + (1-\lambda)(1-\gamma)\frac{f(z)}{z}\right\} > \beta, \quad (3.1)$$

and

$$Re\left\{\lambda\gamma\left(\frac{w^{2-\delta}g''(w)}{\left(wg'(w)\right)^{1-\delta}} - 2\right) + (\gamma(\lambda+1)+\lambda)\frac{w^{1-\delta}g'(w)}{\left(g(w)\right)^{1-\delta}} + (1-\lambda)(1-\gamma)\frac{g(w)}{w}\right\} > \beta, \quad (3.2)$$

where the function $g = f^{-1}$ is given by (1.4).

In particular, for one-fold symmetric bi-univalent functions, we denote the family $KM^*_{\Sigma_1}(\lambda, \gamma, \delta; \beta) = KM^*_{\Sigma}(\lambda, \gamma, \delta; \beta)$.

Remark 3.1. It should be remarked that the families $KM^*_{\Sigma_m}(\lambda, \gamma, \delta; \beta)$ and $KM^*_{\Sigma}(\lambda, \gamma, \delta; \beta)$ are a generalization of well-known families consider earlier. These families are:

(1) For $\gamma = 0$ and $\delta = 1$, the family $KM^{*}_{\Sigma_m}(\lambda,\gamma,\delta;\alpha)$ reduce to the class $A^{\lambda}_{\Sigma,m}(\beta)$ which was introduced by Eker [9];

(2) For $\gamma = 1$, and $\lambda = \delta = 1$, the family $KM^{*}_{\Sigma_m}(\lambda,\gamma,\delta;\beta)$ reduce to the family $\mathcal{H}_{\Sigma,m}(\beta)$ which was introduced by Srivastava et al. [25];

(3) For $\gamma = 0$ and $\delta = 1$, the family $KM^{*}_{\Sigma}(\lambda,\gamma,\delta;\beta)$ reduce to the family $\mathcal{B}^{*}_{\Sigma}(\beta,\lambda)$ which was given by Frasin and Aouf [10];

(4) For $\gamma = 0$ and $\lambda = \delta = 1$, the class $KM^{*}_{\Sigma}(\lambda,\gamma,\delta;\beta)$ reduce to the class $\mathcal{H}_{\Sigma}(\beta)$ which was investigated by Srivastava et al. [24].

Theorem 3.1. *Let* $f \in KM^{*}_{\Sigma_m}(\lambda,\gamma,\delta;\beta)$ $\big((0 \le \delta \le 1, 0 \le \gamma \le 1, \lambda \ge 0; 0 < \alpha \le 1,$ $m \in \mathbb{N},\ z,w \in U)\big)$, *be given by* (1.3). *Then*

$$|a_{m+1}| \le 2\sqrt{\frac{(1-\beta)}{\left|\begin{array}{c}(m+1)(\lambda\gamma(4m(m+1)+\delta+1)+(2m+\delta-1)(\lambda+\gamma)+1)\\ +2(\delta-1)\left(\lambda\gamma m(m+1)^2+\frac{1}{2}(\gamma(\lambda+1)+\lambda)(2m+\delta)\right)\end{array}\right|}} \tag{3.3}$$

and

$$|a_{2m+1}| \le \frac{2(1-\beta)^2(m+1)}{[\lambda\gamma((m+1)^2+\delta)+(\delta+m-1)(\lambda+\gamma)+1]^2} + \frac{2(1-\beta)}{\lambda\gamma(4m(m+1)+\delta+1)+(2m+\delta-1)(\lambda+\gamma)+1}. \tag{3.4}$$

Proof. It follows from conditions (3.1) and (3.2) that there exist $p, q \in \mathcal{P}$ such that

$$\left(\lambda\gamma\left(\frac{z^{2-\delta}f''(z)}{\left(zf'(z)\right)^{1-\delta}}-2\right)+(\gamma(\lambda+1)+\lambda)\frac{z^{1-\delta}f'(z)}{\left(f(z)\right)^{1-\delta}}+(1-\lambda)(1-\gamma)\frac{f(z)}{z}\right) = \beta+(1-\beta)p(z) \tag{3.5}$$

and

$$\left(\lambda\gamma\left(\frac{w^{2-\delta}g''(w)}{\left(wg'(w)\right)^{1-\delta}}-2\right)+(\gamma(\lambda+1)+\lambda)\frac{w^{1-\delta}g'(w)}{\left(g(w)\right)^{1-\delta}}+(1-\lambda)(1-\gamma)\frac{g(w)}{w}\right) = \beta+(1-\beta)q(w), \tag{3.6}$$

where $p(z)$ and $q(w)$ have the forms (2.7) and (2.8), respectively. Equating coefficients (3.5) and (3.6) yields

$$[\lambda\gamma((m+1)^2+\delta)+(\delta+m-1)(\lambda+\gamma)+1]a_{m+1}=(1-\beta)p_m\,, \tag{3.7}$$

$$[\lambda\gamma(4m(m+1)+\delta+1)+(2m+\delta-1)(\lambda+\gamma)+1]a_{2m+1}$$
$$+(\delta-1)\left[\lambda\gamma m(m+1)^2+\frac{1}{2}(\gamma(\lambda+1)+\lambda)(2m+\delta)\right]a_{m+1}^2=(1-\beta)p_{2m}\,, \tag{3.8}$$

$$-[\lambda\gamma((m+1)^2+\delta)+(\delta+m-1)(\lambda+\gamma)+1]a_{m+1}=(1-\beta)q_m \tag{3.9}$$

and

$$[\lambda\gamma(4m(m+1)+\delta+1)+(2m+\delta-1)(\lambda+\gamma)+1]\left((m+1)a_{m+1}^2-a_{2m+1}\right)$$
$$+(\delta-1)\left[\lambda\gamma m(m+1)^2+\frac{1}{2}(\gamma(\lambda+1)+\lambda)(2m+\delta)\right]a_{m+1}^2$$
$$=(1-\beta)q_{2m}. \tag{3.10}$$

From (3.7) and (3.9), we get

$$p_m=-q_m \tag{3.11}$$

and

$$2[\lambda\gamma((m+1)^2+\delta)+(\delta+m-1)(\lambda+\gamma)+1]^2a_{m+1}^2=(1-\beta)^2(p_m^2+q_m^2). \tag{3.12}$$

Adding (3.8) and (3.10), we obtain

$$\Big[(m+1)(\lambda\gamma(4m(m+1)+\delta+1)+(2m+\delta-1)(\lambda+\gamma)+1)$$
$$+2(\delta-1)\left(\lambda\gamma m(m+1)^2+\frac{1}{2}(\gamma(\lambda+1)+\lambda)(2m+\delta)\right)\Big]a_{m+1}^2$$
$$=(1-\beta)(p_{2m}+q_{2m}). \tag{3.13}$$

Therefore, we have

$$a_{m+1}^2=\frac{(1-\beta)(p_{2m}+q_{2m})}{\substack{(m+1)(\lambda\gamma(4m(m+1)+\delta+1)+(2m+\delta-1)(\lambda+\gamma)+1)\\ +2(\delta-1)\left(\lambda\gamma m(m+1)^2+\frac{1}{2}(\gamma(\lambda+1)+\lambda)(2m+\delta)\right)}}.$$

Applying Lemma 1.1 for the coefficients p_{2m} and q_{2m} , we obtain

$$|a_{m+1}| \leq 2\sqrt{\frac{(1-\beta)}{\left|\begin{array}{c}(m+1)(\lambda\gamma(4m(m+1)+\delta+1)+(2m+\delta-1)(\lambda+\gamma)+1)\\ +2(\delta-1)\left(\lambda\gamma m(m+1)^2+\frac{1}{2}(\gamma(\lambda+1)+\lambda)(2m+\delta)\right)\end{array}\right|}}.$$

This gives the desired estimate for $|a_{m+1}|$ as asserted in (3.3).

In order to find the bound on $|a_{2m+1}|$, by subtracting (3.10) from (3.8), we get

$$[\lambda\gamma(4m(m+1)+\delta+1)+(2m+\delta-1)(\lambda+\gamma)+1](2a_{2m+1}-(m+1)a_{m+1}^2)$$
$$=(1-\beta)(p_{2m}-q_{2m}),$$

or equivalently

$$a_{2m+1}=\frac{m+1}{2}a_{m+1}^2+\frac{(1-\beta)(p_{2m}-q_{2m})}{2[\lambda\gamma(4m(m+1)+\delta+1)+(2m+\delta-1)(\lambda+\gamma)+1]}.$$

Upon substituting the value of a_{m+1}^2 from (3.12), it follows that

$$a_{2m+1}=\frac{(1-\beta)^2(m+1)(p_m^2+q_m^2)}{4[\lambda\gamma((m+1)^2+\delta)+(\delta+m-1)(\lambda+\gamma)+1]^2}$$
$$+\frac{(1-\beta)(p_{2m}-q_{2m})}{2[\lambda\gamma(4m(m+1)+\delta+1)+(2m+\delta-1)(\lambda+\gamma)+1]}.$$

Applying Lemma 1.1 once again for the coefficients p_m, p_{2m}, q_m and q_{2m} , we obtain

$$|a_{2m+1}| \leq \frac{2(1-\beta)^2(m+1)}{[\lambda\gamma((m+1)^2+\delta)+(\delta+m-1)(\lambda+\gamma)+1]^2}$$
$$+\frac{2(1-\beta)}{\lambda\gamma(4m(m+1)+\delta+1)+(2m+\delta-1)(\lambda+\gamma)+1}$$

which completes the proof of Theorem 3.1.

Remark 3.2. In Theorem 3.1, if we choose

(1) $\gamma=0$ and $\delta=1$, then we obtain the results which was obtained by Eker [9, Theorem 2];

(2) $\gamma=0$ and $\lambda=\delta=1$, then we obtain the results which were given by Srivastava et al. [25, Theorem 3].

For one-fold symmetric bi-univalent functions, Theorem 3.1 reduces to the following corollary:

Corollary 3.1. *Let* $f \in KM_{\Sigma}^{*}(\lambda, \gamma, \delta; \beta)(0 \leq \beta < 1, 0 \leq \delta \leq 1, 0 \leq \gamma \leq 1)$, *be given by* (1.1). *Then*

$$|a_{m+1}| \leq 2\sqrt{\frac{(1-\beta)}{\left|\begin{array}{l}2(\lambda\gamma(9+\delta)+(1+\delta)(\lambda+\gamma)+1)\\ +2(\delta-1)\left(4\lambda\gamma+\frac{1}{2}(\gamma(\lambda+1)+\lambda)(2+\delta)\right)\end{array}\right|}}$$

and

$$|a_3| \leq \frac{4(1-\beta)^2}{[\lambda\gamma(4+\delta)+\delta(\lambda+\gamma)+1]^2} + \frac{2(1-\beta)}{\lambda\gamma(9+\delta)+(1+\delta)(\lambda+\gamma)+1}.$$

Remark 3.3. In Corollary 3.1, if we choose

(1) $\gamma = 0$ and $\delta = 1$, then we obtain the results which was proven by Frasin and Aouf [10, Theorem 3.2];

(2) $\gamma = 0$ and $\lambda = \delta = 1$, then we have the results which was given by Srivastava et al. [24, Theorem 2].

4. Conclusion

This work has introduced a new families $KM_{\Sigma_m}(\lambda, \gamma, \delta; \alpha)$ and $KM_{\Sigma_m}^{*}(\lambda, \gamma, \delta; \beta)$ of Σ_m for normalized holomorphic and m-fold symmetric bi-univalent functions associated with the Bazilevic starlike and convex functions and investigated the initial coefficient bounds $|a_{m+1}|$ and $|a_{2m+1}|$ for functions in each of these new families.

References

[1] Abd, B. A., & Wanas, A. K. (2024). Coefficient bounds for a new families of m-fold symmetric bi-univalent functions defined by Bazilevic convex functions. *Earthline Journal of Mathematical Sciences*, *14*(1), 105-117. https://doi.org/10.34198/ejms.14124.105117

[2] Aldawish, I., Swamy, S. R., & Frasin, B. A. (2022). A special family of m-fold symmetric bi-univalent functions satisfying subordination condition. *Fractal Fractional*, *6*, 271. https://doi.org/10.3390/fractalfract6050271

[3] Al-Shbeil, I., Wanas, A. K., Saliu, A., & Catas, A. (2022). Applications of beta negative binomial distribution and Laguerre polynomials on Ozaki bi-close-to-convex functions. *Axioms*, *11*, Art. ID 451, 1-7. https://doi.org/10.3390/axioms11090451

[4] Altinkaya, S., & Yalçin, S. (2018). On some subclasses of m-fold symmetric bi-univalent functions. *Communications in Faculty of Sciences University of Ankara, Series A1*, *67*(1), 29-36. https://doi.org/10.1501/Commua1_0000000827

[5] Amourah, A., Alamoush, A., & Al-Kaseasbeh, M. (2021). Gegenbauer polynomials and bi-univalent functions. *Palestine Journal of Mathematics*, *10*(2), 625-632. https://doi.org/10.3390/math10142462

[6] Brannan, D. A., & Taha, T. S. (1986). On some classes of bi-univalent functions. *Studia Universitatis Babes-Bolyai Mathematica*, 31(2), 70-77.

[7] Bulut, S. (2016). Coefficient estimates for general subclasses of *m*-fold symmetric analytic bi-univalent functions. *Turkish Journal of Mathematics*, *40*, 1386-1397. https://doi.org/10.3906/mat-1511-41

[8] Duren, P. L. (1983). *Univalent functions*. Grundlehren der Mathematischen Wissenschaften, Band 259. Springer Verlag.

[9] Eker, S. S. (2016). Coefficient bounds for subclasses of m-fold symmetric bi-univalent functions. *Turkish Journal of Mathematics*, *40*, 641-646. https://doi.org/10.3906/mat-1503-58

[10] Frasin, B. A., & Aouf, M. K. (2011). New subclasses of bi-univalent functions. *Applied Mathematics Letters*, *24*, 1569-1573. https://doi.org/10.1016/j.aml.2011.03.048

[11] Khan, B., Srivastava, H. M., Tahir, M., Darus, M., Ahmad, Q. Z., & Khan, N. (2021). Applications of a certain q-integral operator to the subclasses of analytic and bi-univalent functions. *AIMS Mathematics*, *6*, 1024-1039. https://doi.org/10.3934/math.2021061

[12] Koepf, W. (1989). Coefficients of symmetric functions of bounded boundary rotations. *Proceedings of the American Mathematical Society*, *105*, 324-329. https://doi.org/10.1090/S0002-9939-1989-0930244-7

[13] Kumar, T. R. K., Karthikeyan, S., Vijayakumar, S., & Ganapathy, G. (2021). Initial coefficient estimates for certain subclasses of *m*-fold symmetric bi-univalent functions. *Advances in Dynamical Systems and Applications*, *16*(2), 789-800.

[14] Li, X. F., & Wang, A. P. (2012). Two new subclasses of bi-univalent functions. *International Mathematical Forum*, *7*(2), 1495-1504.

[15] Magesh, N., & Yamini, J. (2018). Fekete-Szegö problem and second Hankel determinant for a class of bi-univalent functions. *Tbilisi Mathematical Journal*, *11*(1), 141-157. https://doi.org/10.32513/tbilisi/1524276036

[16] Murugusundaramoorthy, G., Magesh, N., & Prameela, V. (2013). Coefficient bounds for

certain subclasses of bi-univalent function. *Abstract and Applied Analysis*, Art. ID 573017, 1-3. https://doi.org/10.1155/2013/573017

[17] Sakar, F. M., & Aydogan, S. M. (2018). Coefficients bounds for certain subclasses of m-fold symmetric bi-univalent functions defined by convolution. *Acta Universitatis Apulensis*, *55*, 11-21. https://doi.org/10.17114/j.aua.2018.55.02

[18] Sakar, F. M., & Aydogan, S. M. (2019). Bounds on initial coefficients for a certain new subclass of bi-univalent functions by means of Faber polynomial expansions. *Mathematics in Computer Science*, *13*, 441-447. https://doi.org/10.1007/s11786-019-00406-7

[19] Sakar, F. M., & Canbulat, A. (2019). Inequalities on coefficients for certain classes of m-fold symmetric and bi-univalent functions equipped with Faber polynomial. *Turkish Journal of Mathematics*, *43*, 293-300. https://doi.org/10.3906/mat-1808-82

[20] Sakar, F. M., & Tasar, N. (2019). Coefficients bounds for certain subclasses of m-fold symmetric bi-univalent functions. *New Trends in Mathematical Sciences*, *7*(1), 62-70. https://doi.org/10.20852/ntmsci.2019.342

[21] Singh, R. (1973). On Bazilevic functions. *Proceedings of the American Mathematical Society*, *38*(2), 261-271. https://doi.org/10.1090/S0002-9939-1973-0311887-9

[22] Srivastava, H. M., Eker, S. S., & Ali, R. M. (2015). Coefficient bounds for a certain class of analytic and bi-univalent functions. *Filomat*, *29*, 1839-1845. https://doi.org/10.2298/FIL1508839S

[23] Srivastava, H. M., Gaboury, S., & Ghanim, F. (2016). Initial coefficient estimates for some subclasses of m-fold symmetric bi-univalent functions. *Acta Mathematica Scientia* *36*, 863-871. https://doi.org/10.1016/S0252-9602(16)30045-5

[24] Srivastava, H. M., Mishra, A. K., & Gochhayat, P. (2010). Certain subclasses of analytic and bi-univalent functions. *Applied Mathematics Letters*, *23*, 1188-1192. https://doi.org/10.1016/j.aml.2010.05.009

[25] Srivastava, H. M., Sivasubramanian, S., & Sivakumar, R. (2014). Initial coefficient bounds for a subclass of m-fold symmetric bi-univalent functions. *Tbilisi Mathematical Journal*, *7*(2), 1-10. https://doi.org/10.2478/tmj-2014-0011

[26] Srivastava, H. M., & Wanas, A. K. (2019). Initial Maclaurin coefficient bounds for new subclasses of analytic and m-fold symmetric bi-univalent functions defined by a linear combination, *Kyungpook Math. J.*, *59*, 493-503.

[27] Srivastava, H. M., Wanas, A. K., & Murugusundaramoorthy, G. (2021). A certain family of bi-univalent functions associated with the Pascal distribution series based upon the Horadam polynomials. *Surveys in Mathematics and its Applications*, *16*, 193-205.

[28] Swamy, S. R., & Cotîrlă, L-I. (2022). On τ-pseudo-*ν*-convex κ-fold symmetric bi-univalent function family. *Symmetry*, *14*(10), 1972. https://doi.org/10.3390/sym14101972

[29] Swamy, S. R., & Cotîrlă, L-I. (2023). A new pseudo-type κ-fold symmetric bi-univalent function class. *Axioms*, *12*(10), 953. https://doi.org/10.3390/axioms12100953

[30] Swamy, S. R., Frasin, B. A., & Aldawish, I. (2022). Fekete-Szegö functional problem for a special family of *m*-fold symmetric bi-univalent functions. *Mathematics*, *10*, 1165. https://doi.org/10.3390/math10071165

[31] Wanas, A. K., & Raadhi, H. K. (2016). Maclaurin coefficient estimates for a new subclasses of *m*-fold symmetric bi-univalent functions. *Earthline Journal of Mathematical Sciences*, *11*(2), 199-210. https://doi.org/10.34198/ejms.11223.199210

[32] Wanas, A. K., & Tang, H. (2020). Initial coefficient estimates for a classes of *m*-fold symmetric bi-univalent functions involving Mittag-Leffler function. *Mathematica Moravica*, *24*(2), 51-61. https://doi.org/10.5937/MatMor2002051K

[33] Yalçin, S., Muthunagai, K., & Saravanan, G. (2020). A subclass with bi-univalence involving (p,q)-Lucas polynomials and its coefficient bounds. *Bol. Soc. Mat. Mex.*, *26*, 1015-1022. https://doi.org/10.1007/s40590-020-00294-z