



# On the Modified Optimal Investment Strategy for Annuity Contracts under the Constant Elasticity of Variance (CEV) Model

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## Abstract

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In this work, the optimal pension wealth investment strategy during the decumulation phase, in a defined contribution (DC) pension scheme is constructed. The pension plan member is allowed to invest in a risk free and a risky asset, under the constant elasticity of variance (CEV) model. The explicit solution of the constant relative risk aversion (CRRA) and constant absolute risk aversion (CARA) utility functions are obtained, using Legendre transform, dual theory, and change of variable methods. It is established herein that the elastic parameter,  $\beta$ , say, must not necessarily be equal to one ( $\beta \neq 1$ ). A theorem is constructed and proved on the wealth investment strategy. Observations and significant results are made and obtained, respectively in the comparison of our various utility functions and some previous results in literature.

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## 1. Introduction

There are two major designs of pension plan, namely, the defined benefit (DB) pension, and the defined contribution (DC) pension plan. As the names implies, in that of the DB, the benefits of the plan member are defined, and the sponsor bears the financial risk. Whereas, in the DC pension plan, the contributions are defined, the retirement benefits depends on the contributions and the investment returns, and the contributors (the plan members) bears the financial risk. Recently, the DC pension has taken

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dominance over the DB pension plan in the pension scheme, since DC pension plan is fully funded, which makes it easier for the plan managers (Pension Fund Administrators (PFAs')) and the Pension Fund Custodians (PFCs') to invest equitably in the market, and also makes it easier for the plan members to receive their retirement benefit as and when due.

Investment strategies of the contributions, which in turn is a strong determinant of the investment returns vis-a-vis the benefits of the contributors at retirement must be given optimum attention. Recent publications in economic journals and other reputable mathematics and science journals have brought to light, variety of methods of optimizing investment strategies and returns. For instance, some researchers have made various contributions in this direction, particularly, in DC Pension Plan. Cairns et al. [4] did a work on, "stochastic life styling: optimal dynamic asset allocation for defined contribution pension plans. In their work, various properties and characteristics of the optimal asset allocation strategy, both with and without the presence of non-hedge able salary risk were discussed. The significance of alternative optimal strategy by pension providers was established. Wang and Chen [15] investigated a defined contribution (DC) pension plan investment problem during the accumulation phase under the multi-period mean-variance criterion. Mwanakatwe et al. [14] analysed the optimal investment strategies for a DC pension fund under the Hull-White interest rate model. Under this model, the pension fund manager can invest capital in the bank account, stock index, and real estates. More so, Battocchio and Menoncin [2] studied optimal pension management in a stochastic framework, they came out with a significant result.

In order to deal with optimal investment strategy, the need for maximization of the expected utility of the terminal wealth became necessary. Example, the Constant Relative Risk Aversion (CRRA) utility function, and (or) the Constant Absolute Risk Aversion (CARA) utility function were used to maximize the terminal wealth. Cairns et al. [4], Gao [8], Boulier et al. [3], Deelstra et al. [7], and Xiao et al. [16] used CRRA to maximize terminal wealth. However, Gao [10] used the CRRA and the CARA to maximize terminal wealth, and this triggered our research. Ours is a modification of his work, by considering different categories of contributors, with some other additional assumptions made. Our task in this work is to establish, with a theorem the fact that the elastic parameter  $\beta \neq 1$ , which is lacking in his work. We used a similar approach in obtaining some of our results.

### 1.1. Preliminaries

We start with a complete and frictionless financial market that is continuously open over the fixed time interval  $[0, T]$ , for  $T > 0$ , representing the retirement time of any plan member.

We assume that the market is composed of the risk-free asset (cash), and risky asset (stock). Let  $(\Omega, F, P)$  be a complete probability space, where  $\Omega$  is a real space and  $P$  is a probability measure,  $\{w_s(t), w_t(t)\}$  are two standard unorthogonal Brownian motions,  $\{F_t(t), F_s(t)\}$  are right continuous filtrations whose information are generated by the two standard Brownian motions  $\{w_s(t), w_t(t)\}$ , whose sources of uncertainties are respectively to the stock market and time evolution.

## 2. Methodology

### 2.1. Hamilton-Jacobi-Bellman (HJB) equation

Assume we represent  $u = u_s$  as the strategy and we define the utility attained by the contributor from a given state  $y$  at time  $t$  as

$$\phi_u(t, r, y) = E_u[U(Y(t)) : r(t) = r, y(t) = y], \tag{2.1.1}$$

where  $t$  is the time,  $r$  is the short interest rate and  $y$  is the wealth. Our interest here is to find the optimal value function

$$\phi(t, r, y) = \text{Sup}_u \phi_u(t, r, y) \tag{2.1.2}$$

and the optimal strategy  $u^* = u_s^*$  such that

$$\phi_{u^*}(t, r, y) = \phi(t, r, y). \tag{2.1.3}$$

### 2.2. Legendre transformation

The Legendre transform and dual theory help to transform the nonlinear partial differential equation that is formed due to (2.1.1), to a linear partial differential equation.

**Theorem 2.2.** (Jonsson and Sircar [11]) *Let  $f : R^n \rightarrow R$  be a convex function for  $z > 0$ . Then the Legendre transform is defined as*

$$L(z) = \max_y \{f(y) = zy\}, \tag{2.2.1}$$

where  $L(z)$  is the Legendre dual of  $f(y)$ .

Since  $f(y)$  is convex, from Theorem 2.2 we defined the Legendre transform

$$\hat{\phi}(t, r, z) = \text{Sup}\{\phi(t, r, y) - zy : 0 < y < \infty\}, \quad 0 < t < T \quad (2.2.2)$$

where  $\hat{\phi}$  is the dual of  $\phi$  and  $z > 0$  is the dual variable of  $x$ .

The value of  $y$  where this optimum is attained is denoted by  $h(t, r, z)$  so that

$$h(t, r, z) = \inf\{y : \phi(t, r, y) \geq zy + \hat{\phi}(t, r, z)\}, \quad 0 < t < T. \quad (2.2.3)$$

The functions  $h$  and  $\hat{\phi}$  are closely related and can be referred to as the dual of  $\phi$ . These functions are related as follows

$$\hat{\phi}(t, r, z) = \phi(t, r, h) - zh, \quad (2.2.4)$$

where

$$h(t, r, z) = y, \quad \phi_y = z, \quad h = -\hat{\phi}_z. \quad (2.2.5)$$

At terminal time,  $T$ , we denote

$$\hat{U}(z) = \text{Sup}\{U(y) - zy : 0 < y < T\}, \quad \text{and} \quad \phi(z) = \text{Sup}\{y : U(y) \geq zy + \hat{U}(z)\}.$$

As a result

$$\phi(z) = U^{-1}(z), \quad (2.2.6)$$

where  $\phi$  is the inverse of the marginal utility  $U$  and note that  $\phi(T, r, y) = U(y)$ .

At terminal time  $T$ , we can define

$$h(T, r, y) = \inf_{y>0}\{y : U(y) \geq zy + \hat{\phi}(t, r, z)\} \quad \text{and} \quad \hat{\phi}(t, r, z) = \text{Sup}_{y>0}\{U(y) - zy\}$$

so that  $h(T, r, z) = U^{-1}(z)$ .

### 3. The Model

This session introduces the financial market and proposes the optimization problems in the decumulation phase.

#### 3.1. The financial market

Here, we consider a financial market that consists of a risk-free asset (i.e., cash in the bank) and a risky asset (stock).

Let the risk-free asset,  $C_t$ , say, at any positive time,  $t$ , evolve thus;

$$dC_t = rC_t dt, \quad (3.1.1)$$

where  $r$  represents constant rate of interest.

Next, we denote the price of the risky asset (stock) at any positive time,  $t$ , by  $S_t$ , as in Gao [10], Akpanibah et al. [1], and Njoku et al. [12] thus;

$$dS_t = \mu S_t dt + k S_t^{\beta+1} dW_t, \quad (3.1.2)$$

where  $\mu$  ( $\mu > r$ ) represents the instantaneous rate of return on stock,  $\beta$  ( $\beta \leq 0$ ) is the elastic constant parameter,  $k$  is a constant,  $k S_t^{\beta}$  represents the instantaneous volatility.

Let  $\{W_t; t \geq 0\}$  denote a standard Brownian motion, defined on a probability space,  $(\Omega, F, P)$ , where  $F = \{F_t\}$  is an augmented filtration generated by the Brownian motion.

### 3.2. Model assumption

Consistent with the Nigerian Pension Reform Act of 2004 [13], we make the following assumptions

- (a) The pension scheme accumulates wealth.
- (b) There are different categories of contributions.
- (c) The contributions will not willingly withdraw from the scheme.
- (d) Payments are made to the retirees.
- (e) An accumulated amount is paid to the next-of-kin of the dead contributors, at the instance of death by any contributor(s).
- (f) A certain amount is retained from the payment made to the families of dead contributors, by the pension managers (i.e., management fee).

### 3.3. Model formation (i.e., the optimization program)

The fund accruing from the contributors can be invested in both bank and stock. Particularly, the fund to be invested by the fund manager is the surplus, which is the fund that is available after each period of routine disbursements. That is, let the contribution process be

$$dy = (1 + \theta_i)C_{i+1}dt \quad (3.3.1)$$

and the payment process

$$dj = b_{i+1}dt + (a_{i+1} - \eta) dW_s. \quad (3.3.2)$$

Then the surplus

$$\begin{aligned} dP &= dy - dj = (1 + \theta_i)C_{i+1}dt - [b_{i+1}dt + (a_{i+1} - \eta) dW_s] \\ &= (C_{i+1} + \theta_i C_{i+1} - b_{i+1})dt - (a_{i+1} - \eta) dW_s. \end{aligned} \quad (3.3.3)$$

Therefore our task here is to construct an optimal investment strategy for the assets for the remaining periods after retirement, to enable us maximize the expected utility at each retirement period.

Without loss of generality, the pension wealth is denoted by  $Y(t)$  at any time  $0 < t < T < T + N$ , and it evolves stochastically, thus:

$$\begin{aligned} dY(t) &= u_s Y(t) \frac{dS_t}{S_t} + (1 - u_s) Y(t) \frac{dC_t}{C_t} + (C_{i+1} + \theta_i C_{i+1} - b_{i+1}) dt \\ &\quad - (a_{i+1} - \eta) dW_s, \end{aligned} \quad (3.3.4)$$

$i = 0, 1, \dots, n - 1$  and  $\theta_i = 0, \theta_1 = 1, \theta_2 = 2, \dots, \theta_i$  (an integer) = staff loading, where;  $a_{i+1} > 0$  represents various amount that is paid to the next-of-kin of the dead contributors,  $b_{i+1} > 0$  represents various amount paid to retired contributors,  $c_{i+1} > 0$  represents various amount contributed,  $\eta$  is the service charge deducted from the  $a_{i+1}$ .

However, relevant to the provisions of the Nigerian Pension Reform Act of 2004 [13], on the eligibility condition for signing up on the pension scheme, by both government and private sectors, we have;

$$\begin{aligned} dY(t) &= u_s Y(t) \frac{dS_t}{S_t} + (1 - u_s) Y(t) \frac{dC_t}{C_t} + (C_{i+1} + \theta_i C_{i+1} - b_{i+1}) \\ &\quad - (a_{i+1} - \eta) dW_s, \end{aligned} \quad (3.3.5)$$

$i = 4, 5, \dots, n - 1$ , and  $\theta_4 = 4, \theta_5 = 5, \theta_6 = 6, \dots, \theta_n = n$  (a positive integer) = staff loading.

Assuming,  $b_{i+1} = C_{i+1} + r_d C_{i+1}$ ;  $r < r_d$ , and  $r_d$  represents discounted interest,

then

$$dY(t) = u_s Y(t) \frac{dS_t}{S_t} + (1 - u_s) Y(t) \frac{dC_t}{C_t} + (\theta_i - r_d) C_{i+1} dt - (a_{i+1} - \eta) dW_s, \quad (3.3.6)$$

$i = 4, 5, \dots, n - 1$ , and  $\theta_4 = 4, \theta_5 = 5, \theta_6 = 6, \dots, \theta_n = n$ ,  $\theta_i > 0$  (integer) = staff loading.

Taking into (3.1.1), (3.1.2) and (3.3.6), one obtains the wealth process

$$\begin{aligned} dY(t) &= u_s Y(t) [\mu dt + k S^\beta dW_t] + (1 - u_s) Y(t) r dt + (\theta_i - r_d) C_{i+1} dt - (a_{i+1} - \eta) dW_s \\ &= u_s Y(t) \mu dt + u_s Y(t) k S^\beta dW_t + Y(t) r dt + u_s Y(t) r dt + (\theta_i - r_d) C_{i+1} dt \\ &\quad - (a_{i+1} - \eta) dW_s \\ &= (u_s Y(t) \mu + Y(t) r - u_s Y(t) r + (\theta_i - r_d) C_{i+1}) dt + u_s Y(t) k S^\beta dW_t \\ &\quad - (a_{i+1} - \eta) dW_s, \end{aligned} \quad (3.3.7)$$

$i = 4, 5, \dots, n - 1$ , and  $\theta_4 = 4, \theta_5 = 5, \theta_6 = 6, \dots, \theta_n = n$ ,  $\theta_i > 0$  (integer) = staff loading.

Based on the wealth process in (3.3.7), the pension manager seeks a strategy,  $u_t^*$ , which maximizes the utility function, such that  $u_t^* = \max E(U(Y(T)))$ ,  $\forall u(t)$ . Where  $u(\bullet)$  is an increasing concave utility function, which satisfies the Inada conditions;

$$U'(+\infty) = 0, \text{ and } U'(0) = +\infty \text{ (cf. Gao [10]).}$$

#### 4. Applying the Associated HJB Equation to Maximize Equation (3.3.7)

Applying the associated HJB to (3.3.7), one obtains

$$\begin{aligned} &\varphi_t + \mu S \varphi_s + Y(t) [u_s (\mu - r) + r] \varphi_y + (\theta_i - r_d) C_{i+1} \varphi_y + \frac{1}{2} k^2 S^{2\beta+2} \varphi_{ss} \\ &+ \frac{1}{2} U_s^2 Y^2 k^2 S^{2\beta} \varphi_{yy} - \frac{1}{2} \alpha \gamma U_s Y(t) k S^\beta a_{i+1} \varphi_{yy} + \frac{1}{2} \alpha \gamma u_s Y(t) k S^\beta \eta \varphi_{yy} \\ &+ \frac{1}{2} (a_{i+1} - \eta)^2 \varphi_{yy} + \frac{1}{2} \alpha \gamma k^2 S^{\beta+2} Y(t) u_s \varphi_{sy} - \frac{1}{2} \alpha \gamma k S^{\beta+1} a_{i+1} \varphi_{sy} \end{aligned}$$

$$+ \frac{1}{2} \alpha \gamma k S^{\beta+1} \eta \varphi_{sy} = 0. \quad (4.1)$$

To obtain the optimal value  $u_s^*$ , we differentiate (4.1) with respect to  $u_s$

$$\begin{aligned} & \varphi_t + \mu s \varphi_s + (\theta_i - r_d) C_{i+1} \varphi_y + \frac{1}{2} k^2 s^{2\beta+2} \varphi_{ss} + \frac{1}{2} (a_{i+1} - \eta)^2 \varphi_{yy} \\ & - \frac{1}{2} \alpha \gamma k s^{\beta+1} a_{i+1} \varphi_{sy} + \frac{1}{2} \alpha \gamma k s^{\beta+1} \eta \varphi_{sy} \\ & + \sup_{u_s} \left\{ \begin{aligned} & Y(t) (u_s (\mu - r) + r) \varphi_y + \frac{1}{2} u_s^2 Y(t)^2 k^2 S^{2\beta} \varphi_{yy} \\ & - \frac{1}{2} \alpha \gamma u_s Y(t) k S^\beta a_{i+1} \varphi_{yy} + \frac{1}{2} \alpha \gamma u_s k S^\beta \eta \varphi_{yy} \\ & + \frac{1}{2} \alpha \gamma k^2 s^{\beta+2} Y(t) u_s \varphi_{sy} \end{aligned} \right\} = 0, \quad (4.2) \end{aligned}$$

so that

$$\begin{aligned} \therefore u_s^* &= \frac{\alpha \gamma a_{i+1}}{2Y(t) k S^\beta} - \frac{\alpha \gamma \eta}{2Y(t) k S^\beta} - \frac{(\mu - r)}{Y(t) k^2 S^{2\beta}} \frac{\varphi_y}{\varphi_{yy}} - \frac{\alpha \gamma}{2Y(t) S^{\beta-2}} \cdot \frac{\varphi_{sy}}{\varphi_{yy}} \\ \Rightarrow u_s^* &= \frac{\alpha \gamma (a_{i+1} - \eta)}{2Y(t) k S^\beta} - \frac{(\mu - r)}{Y(t) k^2 S^{2\beta}} \frac{\varphi_y}{\varphi_{yy}} - \frac{\alpha \gamma}{2Y(t) S^{\beta-2}} \cdot \frac{\varphi_{sy}}{\varphi_{yy}}. \quad (4.3) \end{aligned}$$

Putting back (4.3) into (4.2) gives;

$$\begin{aligned} & \varphi_t + \mu s \varphi_s + \left[ Y(t) r + (\theta_i - r_d) C_{i+1} + \frac{\alpha \gamma (\mu - r) (a_{i+1} - \eta)}{2k S^\beta} \right] \varphi_y \\ & + \frac{1}{2} k^2 S^{2\beta+2} \varphi_{ss} + c_1 \varphi_{yy} + \left[ \frac{(\alpha \gamma)^2 k S^2 (a_{i+1} - \eta)}{4} - \frac{\alpha \gamma k S^{\beta+1} (a_{i+1} - \eta)}{2} \right] \varphi_{sy} \\ & - \frac{(\mu - r)^2 \varphi_y^2}{2k^2 S^{2\beta} \varphi_{yy}} - \frac{\alpha \gamma (\mu - r) \varphi_y \varphi_{sy}}{2S^{\beta-2} \varphi_{yy}} - \frac{k^2 S^4 \varphi_{sy}^2 (\alpha \gamma)^2}{8\varphi_{yy}} = 0, \quad (4.4) \end{aligned}$$

$i = 4, 5, \dots, n-1$ ,  $r < r_d$  and  $\theta_4 = 4, \theta_5 = 5, \theta_6 = 6, \dots, \theta_n = n, \theta_{n-1} = n-1, \theta_i > 0$  (integer) = staff loading, and

$$c_1 = \frac{1}{2} (a_{i+1} - \eta)^2 + \frac{3}{4} (\alpha \gamma)^2 (a_{i+1} - \eta)^2. \quad (4.5)$$



Having seen that stochastic control problem described in the previous session has been converted to a nonlinear PDE, our next task is to solve for  $\varphi$  in (4.4) and subsequently substitute it into (4.3), to enable us obtain the optimal wealth investment strategy (i.e., the control strategy). In order to achieve this, we use employ the services of the Dual theory and Legendre transformation.

### 5. Transforming (4.4) into its Dual and Applying Legendre Transformation

Here, we transform the nonlinear second order partial differential equation (4.4) into a linear PDE, using the Dual theory and Legendre transformations in Gao [10], that is;

$$\phi_y = z \text{ and } \phi_t = \hat{\phi}_t, \phi_s = \hat{\phi}_s, \phi_{ss} = \hat{\phi}_{ss} - \frac{\hat{\phi}_{sz}^2}{\hat{\phi}_{zz}}, \phi_{yy} = \frac{-1}{\hat{\phi}_{zz}}, \phi_{ys} = \frac{-\hat{\phi}_{sz}}{\hat{\phi}_{zz}}. \quad (5.1)$$

Putting (5.1) into (4.4)

$$\begin{aligned} & \hat{\phi}_t + \mu S \hat{\phi}_s + \left[ Y(t)r + (\theta_i - r_d)C_{i+1} + \frac{\alpha\gamma(\mu - r)}{2kS^\beta}(a_{i+1} - \eta) \right] z \\ & + \frac{1}{2} k^2 S^{2\beta+2} \hat{\phi}_{ss} - \frac{c_1}{\hat{\phi}_{zz}} \\ & - \left[ \frac{(\alpha\gamma)^2 k S^2 (a_{i+1} - \eta)}{4} - \frac{\alpha\gamma k S^{\beta+1} (a_{i+1} - \eta)}{2} \right] \frac{\hat{\phi}_{sz}}{\hat{\phi}_{zz}} \\ & + \frac{(\mu - r)^2}{2k^2 S^{2\beta}} z^2 \hat{\phi}_{zz} - \frac{\alpha\gamma(\mu - r)}{2s^{\beta-2}} z \hat{\phi}_{sz} + \frac{(\alpha\gamma)^2 k^2 S^4}{8} \frac{\hat{\phi}_{sz}^2}{\hat{\phi}_{zz}} = 0, \end{aligned} \quad (5.2)$$

$i = 4, 5, \dots, n-1, r < r_d$  and  $\theta_4 = 4, \theta_5 = 5, \theta_6 = 6, \dots, \theta_n = n, \theta_{n-1} = n-1, \theta_i > 0$  (integer) = staff loading, and  $c_1 = \frac{1}{2}(a_{i+1} - \eta)^2 + \frac{3}{4}(\alpha\gamma)^2(a_{i+1} - \eta)^2$ .

Taking  $y = h = -\hat{\phi}_z$  into (5.2), and differentiating with respect to  $z$ , we obtain

$$\begin{aligned} & h_t + \mu S h_s - r z h_z - r h - (\theta_i - r_d)C_{i+1} - \frac{\alpha\gamma(\mu - r)(a_{i+1} - \eta)}{2kS^\beta} \\ & + \frac{1}{2} k^2 S^{2\beta+2} h_{ss} + \frac{c_1 h_{zz}}{h_z^2} \end{aligned}$$

$$\begin{aligned}
& + \left[ \frac{(\alpha\gamma)^2 k S^2 (a_{i+1} - \eta)}{4} - \frac{\alpha\gamma k S^{\beta+1} (a_{i+1} - \eta)}{2} \right] \frac{h_s}{h_z} - \frac{\alpha\gamma(\mu - r)}{2S^{\beta-2}} h_s \\
& - \frac{\alpha\gamma(\mu - r)}{2S^{\beta-2}} z h_{sz} + \frac{(\mu - r)^2}{k^2 S^2 \beta} z h_z + \frac{(\mu - r)^2}{2k^2 S^2 \beta} z^2 h_{zz} \\
& + \frac{(\alpha\gamma)^2 k^2 S^4}{8} \frac{h_{sz} h_s}{h_z} - \frac{(\alpha\gamma)^2 k^2 S^4 h_s^2 h_{zz}}{8h_z^2} = 0,
\end{aligned} \tag{5.3}$$

where

$$c_1 = \frac{1}{2} (a_{i+1} - \eta)^2 + \frac{3}{8} (\alpha\gamma)^2 (a_{i+1} - \eta)^2, \tag{5.4}$$

with,

$$\begin{aligned}
u_s^* &= \frac{\alpha\gamma(a_{i+1} - \eta)}{2Y(t)kS^\beta} + (\mu - r) \frac{z}{\hat{\phi}_{zz}} - \frac{\alpha\gamma}{2Y(t)S^{\beta-2}} \hat{\phi}_{sz} \\
\Rightarrow u_s^* &= \frac{\alpha\gamma(a_{i+1} - \eta)}{2Y(t)kS^\beta} - (\mu - r) \frac{z}{h_z} - \frac{\alpha\gamma}{2Y(t)S^{\beta-2}} h_s.
\end{aligned} \tag{5.5}$$

## 6. Test for Some Utility Functions

Here, we seek to obtain the explicit solutions for the CRRA and CARA utility functions, using change of variable method.

### 6.1. Explicit solution to the CRRA utility

Following Gao [10] and Zhang and Rong [5], we use

$$h(t, s, z) = z^{\frac{1}{p-1}}, \quad p < 1, p \neq 0. \tag{6.1.1}$$

Let us conjecture a solution to (5.3), thus

$$h(t, s, z) = z^{\frac{1}{p-1}} g(t, s) + a(t); \quad a(T) = 0; \quad g(T, s) = 1. \tag{6.1.2}$$

Now, obtaining the various first and second partial derivatives with respect to  $t, s, z$ , we have

$$\begin{aligned}
 h_t &= g_t z^{\frac{1}{p-1}} + a'(t); h_s = g_s z^{\frac{1}{p-1}}; h_z = \frac{-g}{1-p} z^{\frac{1}{p-1}-1}; h_{ss} = g_{ss} z^{\frac{1}{p-1}}; \\
 h_{sz} &= \frac{-g}{1-p} z^{\frac{1}{p-1}-1}; h_{zz} = \frac{(2-p)g}{(1-p)^2} z^{\frac{1}{p-1}-2} = 0.
 \end{aligned}
 \tag{6.1.3}$$

Putting (6.1.2) and (6.1.3) into (5.3), we obtain

$$\begin{aligned}
 &g_t z^{\frac{1}{p-1}} + a'(t) + \mu s g_s z^{\frac{1}{p-1}} + \frac{r g z^{\frac{1}{p-1}}}{1-p} - r g z^{\frac{1}{p-1}} - r a(t) - (\theta_i - r_d) C_{i+1} \\
 &- \frac{\alpha \gamma (\mu - r) (a_{i+1} - \eta)}{2 k s^\beta} + \frac{1}{2} k^2 s^{2\beta+2} g_{ss} z^{\frac{1}{p-1}} + \frac{c_1 (2-p)}{g} \frac{1}{z^{\frac{1}{p-1}}} \\
 &- \left[ \frac{(\alpha \gamma)^2 k s^2 (a_{i+1} - \eta)}{4} - \frac{\alpha \gamma k s^{\beta+1} (a_{i+1} - \eta)}{2} \right] \frac{g_s (1-p) z}{g} \\
 &- \frac{\alpha \gamma (\mu - r)}{2 s^{\beta-2}} g_s z^{\frac{1}{p-1}} + \frac{\alpha \gamma (\mu - r) g_s z^{\frac{1}{p-1}}}{2 s^{\beta-2} (1-p)} - \frac{(\mu - r)^2 g}{k^2 s^{2\beta} (1-p)} z^{\frac{1}{p-1}} \\
 &+ \frac{(\mu - r)^2 (2-p) g}{2 k^2 s^{2\beta} (1-p)^2} z^{\frac{1}{p-1}} + \frac{(\alpha \gamma)^2 k^2 s^4 g_s^2 z^{\frac{1}{p-1}}}{8} \\
 &- \frac{k^2 s^4 g_s^2 z^{\frac{1}{p-1}} (2-p) (\alpha \gamma)^2}{8 g} = 0.
 \end{aligned}
 \tag{6.1.4}$$

Factoring out terms that depend on  $z^{\frac{1}{p-1}}$ , and  $z$ , and the ones that is independent of either of the two mentioned, we split (6.1.4) into three, thus

$$a'(t) - r a(t) - (\theta_i - r_d) C_{i+1} - \frac{\alpha \gamma (\mu - r) (a_{i+1} - \eta)}{2 k s^\beta} = 0, \quad r < r_d,
 \tag{6.1.5}$$

$$\left[ \frac{(\alpha\gamma)^2 ks^2(a_{i+1} - \eta)}{4} - \frac{\alpha r ks^{\beta+1}(\mu - r)(a_{i+1} - \eta)}{2} \right] \frac{g_s(1-p)}{g} = 0, \quad g \neq 0 \quad (6.1.6)$$

and

$$\begin{aligned} & g_t + \mu s g_s + \frac{rg}{1-p} - rg + \frac{1}{2} k^2 s^{2\beta+2} g_{ss} + \frac{g}{c_1(2-p)} \\ & - \frac{\alpha r(\mu - r) g_s}{2s^{\beta-2}} + \frac{\alpha\gamma(\mu - r) g_s}{2s^{2\beta-2}(1-p)} - \frac{(\mu - r)^2 g}{k^2 s^{2\beta}(1-p)} \\ & + \frac{(\mu - r)^2(2-p)g}{2k^2 s^{2\beta}(1-p)^2} - \frac{(\alpha\gamma)^2 k^2 s^4 g_s^2}{8} + \frac{k^2 s^4 g_s^2}{8g} (2-p)(\alpha\gamma)^2 = 0, \quad g \neq 0. \end{aligned} \quad (6.1.7)$$

Solving (6.1.5) at the boundary condition,  $a(T) = 0$ , we obtain the continuous annuity of duration,  $T - t$ , yields

$$a(t) = -[T - t]\ell^{-rt}[(\theta_i - r_d)C_{i+1}2k + \alpha\gamma u_s(a_{i+1} - \eta)], \quad (6.1.8)$$

where  $a(t) = -[T - t][(\theta_i - r_d)C_{i+1}2k + \alpha\gamma u_s(a_{i+1} - \eta)]$ , and  $\ell^{-rt}$  are the so called continuous annuity of duration  $T - t$ , and the continuous technical rate, respectively.

From (6.1.6), we propose the following

**Proposition 6.1.1.** *Let, (a)  $dw_s dw_t = \frac{1}{2} \alpha\gamma dt$ ;  $\frac{\alpha\gamma}{2} \neq 1$  (b)  $\beta \neq 1$ . Then*

$$\left[ \frac{(\alpha\gamma)^2 ks^2(a_{i+1} - \eta)^2}{4} - \frac{\alpha\gamma ks^{\beta+1}(a_{i+1} - \eta)}{2} \right] \frac{g_s(1-p)}{g} = 0; \quad p < 1. \quad (6.1.9)$$

**Proof.** Suppose, for contradiction,  $\beta = 1$ ,  $\frac{\alpha\gamma}{2} = 1$ . We observe that either of the following two cases arises

Case 1:  $\frac{g_s(1-p)}{g} = 0 \Rightarrow p = 1 \Rightarrow \Leftarrow p < 1$ , by definition of  $p$  in (6.1.1).

Case 2:  $\frac{(\alpha\gamma)^2 ks^2(a_{i+1} - \eta)^2}{4} = \frac{\alpha\gamma ks^{\beta+1}(a_{i+1} - \eta)}{2} \Rightarrow \frac{\alpha\gamma s^2}{2} = s^{\beta+1}$ .

But  $s^2 = s^{\beta+1}$ , whenever  $\frac{\alpha\gamma}{2} = 1$ , satisfied by the supposition above  $\Rightarrow \beta = 1$

whenever  $\frac{\alpha\gamma}{2} = 1$ , by law of indices. But by Proposition 6.1.1,  $dw_s dw_t = \frac{1}{2} \alpha\gamma dt$

$\Rightarrow \frac{\alpha\gamma}{2} = dw_s dw_t \neq 1 \Rightarrow \Leftarrow$ , contradicting the assumption that  $\frac{\alpha\gamma}{2} = 1$ .

Hence, the proposition holds, and this completes the proof.

**Corollary 6.1.1.** *The correlation between stock and time, represented,  $dw_s dw_t$  is equal to 1, thereby making  $\frac{\alpha\gamma}{2} = 1$ , holds only if stock attains its maximum yield at every increase in time, which is not realistic.*

Lastly, solving (6.1.7), observe firstly that the equation contains some variable coefficients,  $s, s^{2\beta+2}, s^{-2\beta}, s^{-(\beta-2)}$ , and this makes obtaining solution somewhat difficult. However, in order to overcome this difficulty, we employ the services of power transformation and change of variable technique as in Cox et al. [6].

Assuming

$$g(t, s) = f(t, j), \quad j = s^{-2\beta} \tag{6.1.10}$$

such that

$$g_t = j_t; g_s = -2\beta s^{-(2\beta+1)} f_j; g_{ss} = 4\beta^2 s^{-2(2\beta+1)} f_{jj} + 2\beta(2\beta + 1) s^{-(\beta+1)} f_j, \tag{6.1.11}$$

then putting (6.1.10) and (6.1.11) into (6.1.7), gives

$$\begin{aligned} & f_t - \mu 2\beta j f_t + \frac{r}{1-p} f - rf + k^2 2\beta^2 j f_{jj} + k^2 \beta(2\beta + 1) f_j + \frac{1}{c_1(2-p)} \cdot f \\ & + \frac{(\mu - r) 4\beta j}{\alpha\gamma s^2} \cdot f_t - \frac{(\mu - r) 4\beta j}{\alpha\gamma s^2(1-p)} \cdot f_t \\ & - \frac{(\mu - r)^2}{k^2(1-p)} \cdot f + \frac{(\mu - r)^2(2-p)j}{2k^2(1-p)^2} \cdot f - k^2 \beta^2 j^2 f_t^2 + \frac{2k^2 \beta^2 j^2}{f} \cdot f_t^2 = 0, \end{aligned} \tag{6.1.12}$$

where  $s^\beta = \frac{\alpha\gamma}{2} \cdot s$ .

Following Proposition 6.1.1, we observe that some cases may arise in (6.1.12),  $\beta = 0$ ,  $\beta < 1$ , provided  $p \neq 0$ ,  $p < 1$ .

Case 1:  $\beta = 0$  (i.e., the GBM case), yields

$$f_t + \left[ \frac{r}{1-p} - r + \frac{1}{c_1(2-p)} - \frac{(\mu-r)^2}{k^2(1-p)} + \frac{2(\mu-r)^2(2-p)}{k^2(1-p)^2(\alpha\gamma)^2 s^2} \right] \cdot f = 0, \quad \beta < -1, p \neq 0. \quad (6.1.13)$$

Next, we find the solution of (6.1.13), using the structure below

$$f(t, j) = A(t) \ell^{B(t)j}; \quad A(T) = 1, B(T) = 0 \Rightarrow f_t = A_t \ell^{B(t)j} + A(t) B_t j \ell^{B(t)j}. \quad (6.1.14)$$

Putting (6.1.14) into (6.1.13) yields

$$\frac{A_t}{A(t)} + B_t j + \frac{r}{1-p} - r + \frac{1}{c_1(2-p)} - \frac{(\mu-r)^2}{k^2(1-p)} + \frac{(\mu-r)^2(2-p)j}{2k^2(1-p)^2} = 0, \quad (6.1.15)$$

where  $s^\beta = \frac{\alpha\gamma}{2} \cdot s$  is re-substituted and  $j = s^{-2\beta}$ .

Observe that we can further split (6.1.15) by separating the terms that depends on  $j$ , thus

$$B_t + \frac{(\mu-r)^2(2-p)}{2k^2(1-p)^2} = 0 \quad (6.1.16)$$

and

$$\frac{A_t}{A(t)} + \frac{r}{1-p} - r + \frac{1}{c_1(2-p)} - \frac{(\mu-r)^2}{k^2(1-p)} = 0. \quad (6.1.17)$$

Solving (6.1.16) at the initial condition  $B(T) = 0$ , yields

$$B(t) = [T - t] \left( \frac{u_s^2(2-p)}{2k^2(1-p)} \right), \quad (6.1.18)$$

where  $\mu = u_s + r$ .

Next, we solve (6.1.17) at initial condition  $A(T) = 1$ , yields

$$A(t) = \ell^{\left[ \frac{r}{1-p} - r + \frac{1}{c_1(2-p)} - \frac{u_s^2}{k^2(1-p)} \right] [T-t]}, \quad p \neq 0, p < -1, \quad (6.1.19)$$

where  $\mu = u_s + r$ .

Taking into (6.1.18), (6.1.19) and (6.1.14), we have

$$f(t, j) = \ell^{\left[ \frac{r}{1-p} - r + \frac{1}{c_1(2-p)} - \frac{u_s^2}{k^2(1-p)} + \frac{u_s^2(2-p)j}{2k^2(1-p)} \right] [T-t]}. \quad (6.1.20)$$

In this sequel, we state;

**Theorem 6.1.** *Let equations (6.1.2), (6.1.8), (6.1.10) and (6.1.20) hold. Then the optimal wealth investment made in stock is given as*

$$u_s^* = \frac{\alpha\gamma(a_{i+1} - \eta)}{2Y(t)k} - u_s(p-1)z^{\frac{2p-3}{p-1}} \ell^{\left[ \frac{rp}{1-p} - r + \frac{1}{c_1(2-p)} - \frac{u_s^2}{k^2(1-p)} + \frac{u_s^2(2-p)j}{2k^2(1-p)} \right] [T-t]}.$$

**Proof.** Taking into (6.1.8), (6.1.10), (6.1.20) and (6.1.2), we have

$$h(t, s, z) = z^{\frac{1}{p-1}} \ell^{\left[ \frac{r}{1-p} - r + \frac{1}{c_1(2-p)} - \frac{u_s^2}{k^2(1-p)} + \frac{u_s^2(2-p)j}{2k^2(1-p)} \right] [T-t]} - \ell^{-rt} [2k(\theta_i - r_d)C_{i+1} + \alpha\gamma u_s(a_{i+1} - \eta)] [T-t]. \quad (6.1.21)$$

Differentiating (6.1.21) w.r.t.  $s$  and  $z$ , yields

$$h_s = 0 \quad (6.1.22)$$

and

$$h_z = \frac{1}{p-1} z^{\frac{1}{p-1}-1} \ell^{\left[ \frac{r}{1-p} - r + \frac{1}{c_1(2-p)} - \frac{u_s^2}{k^2(1-p)} + \frac{u_s^2(2-p)j}{2k^2(1-p)} \right] [T-t]}. \quad (6.1.23)$$

Therefore, taking into (6.22), (6.23) and (5.5), thus

$$u_s^* = \frac{\alpha\gamma(a_{i+1} - \eta)}{2Y(t)k}$$

$$-u_s(p-1)z^{\frac{2p-3}{p-1}} \ell \left[ \frac{rp}{1-p} - r + \frac{1}{c_1(2-p)} - \frac{u_s^2}{k^2(1-p)} + \frac{u_s^2(2-p)j}{2k^2(1-p)} \right] [T-t], \quad (6.1.24)$$

where  $c_1 = \frac{1}{2}(a_{i+1} - \eta)^2 + \frac{3}{8}(\alpha\gamma)^2(a_{i+1} - \eta)^2$ , as required.

## 6.2. Explicit solution for the CARA utility

Suppose, the plan number takes an exponential utility function

$$U(y) = -\frac{1}{q}e^{-qy}, \quad q > 0. \quad (6.2.1)$$

The decision maker's absolute risk aversion, presented in (6.2.1) is constant, and is called CARA utility.

Combining  $h(T, s, z) = (U')^{-1}(z)$  and (6.2.1) above, we have

$$h(T, s, z) = -\frac{1}{q}hz. \quad (6.2.2)$$

Let us now conjecture a solution to (5.3), thus

$$h(T, s, z) = -\frac{1}{q}[h(t)(hz + m(t, s))] + a(t) \quad (6.2.3)$$

such that  $h(T) = 1$ ;  $a(T) = 0$ ;  $m(T, s) = 0$ .

Obtaining the first, second and mixt derivative of (6.2.3) with respect to  $t, s, z$

$$\begin{aligned} h_t &= -\frac{1}{q}[b'(t)(\ln z + m(t, s)) + b(t)m_t] + a'(t); \quad h_s = -\frac{1}{q}h(t)m_s; \quad h_z = \frac{-b(t)}{q^2}; \\ h_{zz} &= \frac{b(t)}{qz^2}; \quad h_{ss} = -\frac{1}{q}b(t)m_{ss}; \quad h_{sz} = 0. \end{aligned} \quad (6.2.4)$$

Putting (6.2.4) into (5.3) yields

$$\begin{aligned} &b'(t)\ln z + b'(t)m(t, s) + b(t)m_t + a'(t)q - \mu sb(t)m_s + \frac{rz b(t)}{q} + rb(t)\ln z \\ &+ rb(t)m(t, s) + ra(t)q - (\theta_i - r)C_{i+1}q - \frac{\alpha\gamma(\mu - r)(a_{i+1} - \eta)}{s^\beta}q \end{aligned}$$



$$\begin{aligned}
 & -\frac{k^2 s^{2\beta+2} b(t)}{2} m_{ss} + \frac{c_1 q^4}{z^2 b(t)} + \frac{(\alpha\gamma)^2 k s^2 (a_{i+1} - \eta)}{4} m_s q^2 \\
 & -\frac{\alpha\gamma k s^{\beta+1} (a_{i+1} - \eta)}{2} m_s q^2 + \frac{\alpha\gamma(\mu - \gamma) b(t) m_s}{2s^{\beta-2}} - \frac{(\mu - \gamma)^2 z b(t)}{k^2 s^{2\beta} q} + \frac{(\mu - \gamma)^2 b(t)}{2k^2 s^{2\beta}} \\
 & + \frac{k^2 s^4 m_s^2 b(t) q^2}{8z^2} (\alpha\gamma)^2 = 0.
 \end{aligned} \tag{6.2.5}$$

Splitting (6.2.5) into three in order to remove dependency on  $\ln z$ ,  $q$  and  $h(t)$  gives

$$b'(t) + rb(t) = 0, \tag{6.2.6}$$

$$a'(t) + ra(t) - (\theta_i - r_d) C_{i+1} - \frac{\alpha\gamma(\mu - r) (a_{i+1} - \eta)}{2k s^\beta} = 0 \tag{6.2.7}$$

and

$$\begin{aligned}
 & \frac{b'(t) m(t, s)}{b(t)} + m_t - \mu s m_s + \frac{rz}{q} + rm(t, s) - \frac{k^2 s^{2\beta+2}}{2} m_{ss} \\
 & + \frac{c_1 q^4}{z^2 b^2} + \frac{(\alpha\gamma)^2 k s^2 (a_{i+1} - \eta)}{4} m_s q - \frac{\alpha\gamma k s^{\beta+1} (a_{i+1} - \eta)}{2} m_s q + \frac{\alpha\gamma(\mu - r) m_s}{2s^{\beta-2}} \\
 & - \frac{(\mu - r)^2 z}{k^2 s^{2\beta} q} + \frac{(\mu - r)^2}{2k^2 s^{2\beta}} + \frac{k^2 s^4 m_s^2 q^2}{8z^2} (\alpha\gamma)^2 = 0.
 \end{aligned} \tag{6.2.8}$$

Solving (6.2.6) at the initial condition,  $b(T) = 1$ , we have

$$b(t) = \ell^{r(T-t)}. \tag{6.2.9}$$

Next, solving (6.2.7) at the initial condition,  $a(T) = 0$ , we have

$$\begin{aligned}
 a(t) = e^{-\gamma t} \left[ \int e^{rt} \frac{[2k(\theta_i - r_d) C_{i+1} + \alpha\gamma u_L (a_{i+1} - \eta)]}{2k} dt \right. \\
 \left. - \int e^{rT} \frac{[2k(\theta_i - r_d) C_{i+1} + \alpha\gamma u_L (a_{i+1} - \eta)]}{2k} dT \right].
 \end{aligned} \tag{6.2.10}$$

More so, to solve (6.2.8), observe that (6.2.8) contains the coefficients,

$s, s^{-(\beta-2)}, s^{-2\beta}, s^{\beta+1}, s^2$ . Therefore, we assume that

$$m(t, s) = f(t, v) \ni v = s^{-2\beta}. \quad (6.2.11)$$

Obtaining the first and second derivative of (6.2.11) with respect to  $t, s$ , yields

$$\begin{aligned} m_t &= f_t; m_s = -2\beta s^{-2\beta-1} f_v; m_s^2 = 4\beta^2 s^{-4\beta-2} f_v^2 \\ m_{ss} &= 2\beta(2\beta+1) s^{-2\beta-2} f_v + 4\beta^2 s^{-4\beta-2} f_{vv}. \end{aligned} \quad (6.2.12)$$

Putting (6.2.11), and (6.2.12) into (6.2.8), and assuming that  $\mu = u_L + r$ , since the stock appreciates instantaneously

$$\begin{aligned} &\frac{b'(t)}{b(t)} f + f_t + 2(u_L + r)\beta v f_v + \frac{rz}{q} + rf - k^2\beta(2\beta+1) f_v - 2k^2\beta^2 v f_{vv} + \frac{c_1 q^4}{z^2 b(t)^2} \\ &- \frac{(\alpha\gamma)^2 ks(a_{i+1} - \eta)}{2} q v f_v + \alpha\gamma ks^{-\beta}(a_{i+1} - \eta) q f_v - \frac{\alpha\gamma u_L \beta v f_v}{s^{\beta-1}} - \frac{u_L^2 z v}{kq} + \frac{u_L^2}{2k^2} v \\ &+ \frac{k^2 s^4 \beta^2 q^2 v^2 f_v^2}{2z^2} = 0. \end{aligned} \quad (6.2.13)$$

We also conjecture a solution of (6.2.13), thus

$$f(t, v) = \tilde{A}(t) + \tilde{B}(t)v; \tilde{A}(T) = 0, \tilde{B}(T) = 0, \quad (6.2.14)$$

$$\Rightarrow f_t = \tilde{A}(t) + \tilde{B}(t)v; f_v = \tilde{B}(t); f_{vv} = 0; f_v^2 = \tilde{B}^2(t). \quad (6.2.15)$$

Putting (6.2.15) and (6.2.14) into (6.2.13)

$$\begin{aligned} &\frac{b'(t)}{b(t)} (\tilde{A}(t) + \tilde{B}(t)v) + \tilde{A}_t + \tilde{B}_t v + 2(u_L + r)\beta v \tilde{B}(t) + \frac{rz}{q} \\ &+ r(\tilde{A}(t) + \tilde{B}(t)v) - k^2\beta(2\beta+1) \tilde{B}(t) + \frac{c_1 q^4}{z^2 b(t)^2} - \frac{(\alpha\gamma)^2 ks(a_{i+1} - \eta)}{2} q \tilde{B}(t) \\ &+ \alpha\gamma ks^{-\beta}(a_{i+1} - \eta) q \tilde{B}(t) - \frac{\alpha\gamma u_L \beta v}{s^{\beta-1}} \tilde{B}(t) - \frac{u_L^2 z v}{kq} + \frac{u_L^2}{2k^2} v \\ &+ \frac{k^2 s^4 \beta^2 q^2 v^2}{2z^2} \tilde{B}^2(t) = 0. \end{aligned} \quad (6.2.16)$$

But from (6.2.6)

$$\frac{b'(t)}{b(t)} = -r. \quad (6.2.17)$$

Putting (6.2.17) into (6.2.16), gives

$$\begin{aligned} & \tilde{A}_t + \tilde{B}_t v + 2(u_L + r)\beta y \tilde{B}(t) + \frac{rz}{q} - k^2 \beta (2\beta + 1) \tilde{B}(t) + \frac{c_i q^4}{z^2 b^2(t)} \\ & - \frac{(\alpha\gamma)^2 ks(a_{i+1} - \eta)}{2} qv \tilde{B}(t) + \alpha\gamma ks^{-\beta}(a_{i+1} - \eta) q \tilde{B}(t) - \frac{\alpha\gamma u_L \beta v}{s^{\beta-1}} \tilde{B}(t) \\ & - \frac{u_L^2}{2k^2} v + \frac{k^2 s^4 \beta^2 q^2}{2z^2} v^2 \tilde{B}^2(t) = 0. \end{aligned} \quad (6.2.18)$$

**Some cases may arise** ( $\beta = 0$  or  $\beta < 0$ )

**Case 1:**  $\beta = 0$  (i.e., the Geometric Brownian Motion (GBM)), (6.2.18) becomes

$$\begin{aligned} & \tilde{A}_t + \tilde{B}_t v + \frac{rz}{q} + \frac{c_i q^4}{z^2 b^2(t)} - \frac{(\alpha\gamma)^2 ks(a_{i+1} - \eta)}{2} qv \tilde{B}(t) + \alpha\gamma k(a_{i+1} - \eta) q \tilde{B}(t) \\ & - \frac{u_L^2}{2k^2} v = 0. \end{aligned} \quad (6.2.19)$$

Factoring out terms that depend on  $v$  and splitting (6.2.19)

$$\begin{aligned} & \tilde{A}_t + \frac{rz}{q} + \frac{c_i q^4}{z^2 b^2(t)} + \alpha\gamma k(a_{i+1} - \eta) q \tilde{B}(t) = 0, \\ & \tilde{A}(T) = 0, \tilde{B}(T) = 0, b(T) = 1, q > 0 \end{aligned} \quad (6.2.20)$$

and

$$\begin{aligned} & \left( \tilde{B}_t - \frac{(\alpha\gamma)^2 ks(a_{i+1} - \eta)}{2} q \tilde{B}(t) - \frac{u_L^2}{2k^2} \right) v = 0 \\ & \Rightarrow \tilde{B}_t - \frac{(\alpha\gamma)^2 ks(a_{i+1} - \eta)}{2} q \tilde{B}(t) - \frac{u_L^2}{2k^2} = 0, q > 0, \tilde{B}(T) = 0, s = s(t). \end{aligned} \quad (6.2.21)$$

Equivalently

$$\frac{d\tilde{B}}{dt} - P(t)\tilde{B} = Q(t), \quad q > 0, \quad \tilde{B}(T) = 0, \quad s = s(t) \quad (6.2.22)$$

where

$$P(t) = \frac{(\alpha\gamma)^2 k s(t)(a_{i+1} - \eta)}{2} q; \quad Q(t) = \frac{u_L^2}{2k^2}. \quad (6.2.23)$$

Solving (6.2.22), at  $\tilde{B}(T) = 0$ , using integrating factor method

$$\begin{aligned} \tilde{B}(t) = e^{-\frac{(\alpha\gamma)^2 k(a_{i+1}-\eta)}{2} q \int s(T) dt} & \frac{u_L^2}{2k^2} \left[ \int e^{\frac{(\alpha\gamma)^2 k(a_{i+1}-\eta)}{2} q \int s(t) dt} dt \right. \\ & \left. - \int e^{\frac{(\alpha\gamma)^2 k(a_{i+1}-\eta)}{2} q \int s(T) dT} dT \right]. \end{aligned} \quad (6.2.24)$$

Putting back (6.2.24) into (6.2.20) and solving at  $\tilde{A}(T) = 0$ , gives

$$\tilde{A} = \int w(T) dT + \int w(t) dt. \quad (6.2.25)$$

**Theorem 6.2.** *Let (6.2.9), (6.2.10) and (6.2.3) hold. Then the optimal wealth invested in stock for the CARA utility function is given by*

$$\begin{aligned} u_s^* = & \frac{\alpha\gamma(a_{i+1} - \eta)}{2Y(t)kS^\beta} + u_L \ell^{r(T-t)} \\ & - \frac{\alpha\gamma}{2Y(t)S^{\beta-2}} \frac{1}{q} \frac{d}{ds} \left( \ell^{-(\alpha\gamma)^2 k(a_{i+1}-\eta) \int s(t) dt} \int \ell^{(\alpha\gamma)^2 k(a_{i+1}-\eta) \int s(T) dT} dT \right). \end{aligned}$$

**Proof.** Taking into (6.2.9), (6.2.10) and (6.2.3), we obtain our expected utility function, thus

$$\begin{aligned} h(t, s, z) = & \frac{-1}{q} \left[ e^{r(T-t)} (\ln z + m(t, s)) \right] \\ & + e^{-rt} \left[ \int e^{rt} \frac{[2k(\theta_i - r_d)C_{i+1} + \alpha\gamma u_L(a_{i+1} - \eta)]}{2k} dt \right] \end{aligned}$$

$$- \int e^{rT} \frac{[2k(\theta_i - r_d)C_{i+1} + \alpha\gamma u_L(a_{i+1} - \eta)]}{2k} dT \Big]. \quad (6.2.26)$$

But,

$$m(t, s) = f(t, v) = \tilde{A}(t) + \tilde{B}(t)v; \quad v = s^{-2\beta} = 1, \quad (6.2.27)$$

since  $\beta = 0$ , satisfied by the assumption of case 1 and by combining, (6.2.24), (6.2.25) and (6.2.27), we have

$$m(t, s) = \int w(T) dT - \int w(t) dt + e^{-\frac{(\alpha\gamma)^2 k(a_{i+1} - \eta)}{2} q \int s(t) dt} \cdot \frac{u_L^2}{2k^2} \left[ \int e^{\frac{(\alpha\gamma)^2 k(a_{i+1} - \eta)}{2} q \int s(t) dt} dt - \int e^{\frac{(\alpha\gamma)^2 k(a_{i+1} - \eta)}{2} q \int s(T) dT} dT \right]. \quad (6.2.28)$$

Now, returning (6.2.28) to (6.2.26) yields

$$h(t, s, z) = \frac{-1}{q} \left[ e^{r(T-t)} \left( \ln z + \int w(T) dT - \int w(t) dt + e^{-\frac{(\alpha\gamma)^2 k(a_{i+1} - \eta)}{2} q \int s(t) dt} \cdot \frac{u_L^2}{2k^2} \left[ \int e^{\frac{(\alpha\gamma)^2 k(a_{i+1} - \eta)}{2} q \int s(t) dt} dt - \int e^{\frac{(\alpha\gamma)^2 k(a_{i+1} - \eta)}{2} q \int s(T) dT} dT \right] \right) \right. \\ \left. + e^{-rt} \left[ \int e^{rt} \frac{[2k(\theta_i - r_d)C_{i+1} + \alpha\gamma u_L(a_{i+1} - \eta)]}{2k} dt - \int e^{rt} \frac{[2k(\theta_i - r_d)C_{i+1} + \alpha\gamma u_L(a_{i+1} - \eta)]}{2k} dT \right] \right]. \quad (6.2.29)$$

In order to obtain the optimal strategy in (5.5), we shall firstly, obtain  $h_s$  and  $h_z$ , and this yields

$$u_s^* = \frac{\alpha\gamma(a_{i+1} - \eta)}{2Y(t)kS^\beta} + (\mu - r)\ell^{r(T-t)} - \frac{\alpha\gamma}{2Y(t)S^{\beta-2}} \cdot \frac{1}{q} \frac{d}{ds} \left( \ell^{-(\alpha\gamma)^2 k(a_{i+1} - \eta) q \int S(t) dt} \int \ell^{(\alpha\gamma)^2 k(a_{i+1} - \eta) q \int S(T) dT} dT \right), \quad (6.2.30)$$

where,  $q > 0$ .

Equivalently,

$$u_s^* = \frac{\alpha\gamma(a_{i+1} - \eta)}{2Y(t)kS^\beta} + u_L \cdot \ell^{r(T-t)} - \frac{\alpha\gamma}{2Y(t)S^{\beta-2}} \cdot \frac{1}{q} \frac{d}{ds} \left( \ell^{-(\alpha\gamma)^2 k(a_{i+1} - \eta)q} \int S(t) dt \int \ell^{(\alpha\gamma)^2 k(a_{i+1} - \eta)q} \int S(T) dT dT \right), \quad (6.2.31)$$

where  $q > 0$ ,  $\mu = u_L + r$ .

## 7. Sensitivity Analysis

From (6.2.29) and (6.2.31), if we set  $\alpha\gamma = 0$ , that is, saying that stock and time have orthogonal relationship, then the satisfaction of the contributors is reduced to (7.1) below, and the optimal investment made in stock will reduce to the product of instantaneous stock returns and the instantaneous continuous technical rate, we have, respectively

$$h(t, s, z) = \frac{-1}{q} \left[ e^{r(T-t)} \left( \ln z + \int w(T) dT - \int w(t) dt + \frac{u_L^2}{2k^2} \left[ \int dt - \int dT \right] \right) \right] + e^{-rt} \left[ \int e^{rt} \frac{[2k(\theta_i - r_d) C_{i+1}]}{2k} dt - \int e^{rT} \frac{[2k(\theta_i - r_d) C_{i+1}]}{2k} dT \right] \quad (7.1)$$

and,

$$u_s^* = u_L \ell^{r(T-t)}. \quad (7.2)$$

Again, suppose, no money is paid to the next-of-kin of the dead contributors, that is, setting  $a_{i+1} - \eta = 0$ , and this yields, respectively

$$h(t, s, z) = \frac{-1}{q} \left[ e^{r(T-t)} \left( \ln z + \int w(T) dT - \int w(t) dt + \frac{u_L^2}{2k^2} \left[ \int dt - \int dT \right] \right) \right] + e^{-rt} \left[ \int e^{rt} \frac{[2k(\theta_i - r_d) C_{i+1}]}{2k} dt - \int e^{rT} \frac{[2k(\theta_i - r_d) C_{i+1}]}{2k} dT \right] \quad (7.3)$$

and,

$$u_s^* = u_L \cdot \ell^{r(T-t)} - \frac{\alpha\gamma}{2Y(t)S^{\beta-2}} \cdot \frac{1}{q} \frac{d}{ds} \left( \int dT \right) \quad (7.4)$$

Equivalently,

$$u_s^* = u_L \cdot \ell^{r(T-t)} - \frac{\alpha\gamma}{2Y(t)S^{\beta-2}} \cdot \frac{1}{q} F'(s). \tag{7.5}$$

Then, from (7.1) and (7.3), we see that when stock and time have orthogonal relationship and when no money was paid to the next-of-kin of the dead contributors, we have the same level of satisfaction on the contributors. However, both have significant effect on the satisfaction of the contributors when there is orthogonal relationship between stock and time, and also when payments are made to the next-of-kin of the dead contributors. More so, there is a significant effect on the investment made in stock.

From (6.1.21) and (6.1.24), if we set  $\alpha\gamma = 0$ , that is, saying that stock and time have orthogonal relationship, then the satisfaction of the contributors will reduce to (7.6), but has significant negative effect on the optimal investment made in stock, and we have

$$h(t, s, z) = z^{\frac{1}{p-1}} \ell^{\left[ \frac{r}{1-p} - r + \frac{1}{c_1(2-p)} - \frac{u_s^2}{k^2(1-p)} + \frac{u_s^2(2-p)j}{2k^2(1-p)} \right] [T-t]} - \ell^{-rt} [2k(\theta_i - r_d) C_{i+1}] [T-t] \tag{7.6}$$

and,

$$u_s^* = -u_s(p-1) z^{\frac{2p-3}{p-1}} \ell^{\left[ \frac{rp}{1-p} - r + \frac{1}{c_1(2-p)} - \frac{u_s^2}{k^2(1-p)} + \frac{u_s^2(2-p)j}{2k^2(1-p)} \right] [T-t]}. \tag{7.7}$$

Again, suppose, no money is paid to the next-of-kin of the dead contributors, that is, setting  $a_{i+1} - \eta = 0$ , and this yields, respectively

$$h(t, s, z) = z^{\frac{1}{p-1}} \ell^{\left[ \frac{r}{1-p} - r + \frac{1}{c_1(2-p)} - \frac{u_s^2}{k^2(1-p)} + \frac{u_s^2(2-p)j}{2k^2(1-p)} \right] [T-t]} - \ell^{-rt} [2k(\theta_i - r_d) C_{i+1}] [T-t] \tag{7.8}$$

and

$$u_s^* = -u_s(p-1) z^{\frac{2p-3}{p-1}} \ell^{\left[ \frac{rp}{1-p} - r + \frac{1}{c_1(2-p)} - \frac{u_s^2}{k^2(1-p)} + \frac{u_s^2(2-p)j}{2k^2(1-p)} \right] [T-t]}. \tag{7.9}$$

Then, from (7.6) and (7.8), we see that when stock and time have orthogonal relationship and when no money was paid to the next-of-kin of the dead contributors, we have the same level of satisfaction on the contributors. However, both have significant effect on the satisfaction of the contributors when there is orthogonal relationship between stock and time, and also when payments are made to the next-of-kin of the dead contributors.

From (7.7) and (7.9), we observe that the introduction of the orthogonal relationship between stock and time, and the nonpayment of benefits to the next-of-kin of the dead contributors have a negative significant effect on the money invested in stock. However, both introductions have the same negative effect (i.e., a decline in stock investment) on the investment made in stock.

## 8. Numerical Illustration

A numerical example of the proposed model was given to demonstrate the dynamic behaviour of a DC pension fund and optimal investment strategy. Nigeria-National Pension Fund Administration (NNPFA) real data was used to illustrate the efficiency of the proposed model. The parameters used are summarized in Table 1, for  $T = 35$ .  $t = 0, 5, 10, 15, 20, 25, 30, 35$  with  $i = 4, 5, \dots, n - 1$  and  $\theta_4 = 4, \theta_5 = 5, \theta_6 = 6, \dots, \theta_n = n$ .

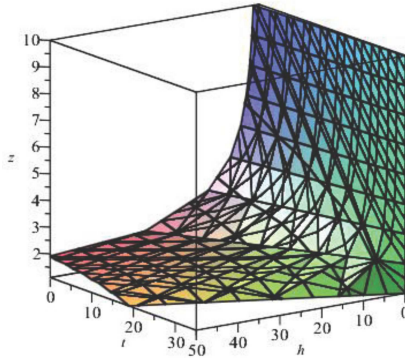
**Table 1.** Parameters and their respective values.

Name of parameter	Symbol used	Values
Constant rate of interest	$r$	0.02
Expected stock returns	$\mu$	0.10
Instantaneous stock returns	$u_S = u_L$	0.07
Stock volatility	$k$	0.55
Risk aversion	$p = q$	0.50
Rate of contribution	$r_d$	0.075
Management fee	$\eta$	0.025



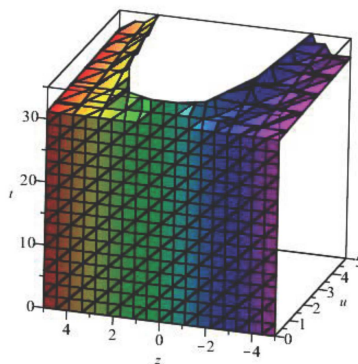
**Results**

Figure 1 describes the investment strategy for the CRRA utility function under the constant elasticity of variance (CEV) model using (6.1.21) when  $c_1 = \frac{1}{2}(a_{i+1} - \eta)^2 + \frac{3}{8}(\alpha\gamma)^2(a_{i+1} - \eta)^2$ ,  $j = \ell^{-2\beta}$ ,  $\beta = 0 \Rightarrow j = 1$ . It shows that the stock price has significant effect on the optimal investment made in stock.



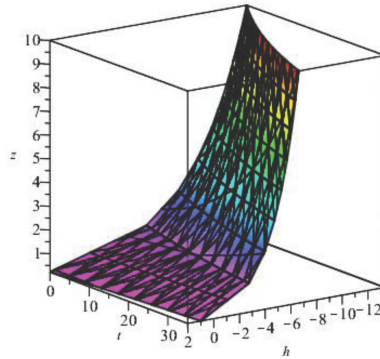
**Figure 1.** Investment strategy under CEV using (6.1.21).

Figure 2 shows the influence instantaneous stock returns on the optimal investment strategies using (6.1.24) with  $c_1 = \frac{1}{2}(a_{i+1} - \eta)^2 + \frac{3}{8}(\alpha\gamma)^2(a_{i+1} - \eta)^2$ ,  $Y(t) = 1000$ . It reveals that the optimal investment policies increase with time. That is, as the time passes on, investment in riskless asset decreases. Results suggest that the pension fund manager maintains diversifying the portfolio by investing more in stock since the optimal investment strategies in risky assets increase with time.



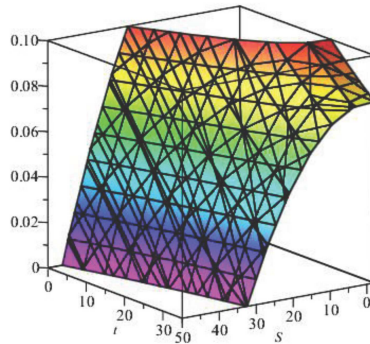
**Figure 2.** Influence of instantaneous stock returns on the optimal investment strategies.

Figure 3 describes the investment strategy for the CARA utility function under the constant elasticity of variance (CEV) model using (6.2.26). It shows that increase in the interest rate volatility decreases the optimal investment strategy; this tells the pension fund manager to invest more in a riskless asset to hedge risk.



**Figure 3.** Investment strategy under CEV using (6.1.26).

Figure 4 presents the impact of instantaneous stock returns on the optimal investment strategies. It points out that as instantaneous stock returns increase the optimal investment strategies decreases.



**Figure 4.** Influence of instantaneous stock returns on the optimal investment strategies using (6.2.30).

## 9. Conclusion

We studied and modified the optimal investment strategy for annuity contract under the constant elasticity of variance as in literature to show that the elastic parameter takes values other than unity. We also constructed the optimal pension wealth investment

strategy during the decumulation phase, in a defined contribution (DC) pension scheme and obtained the explicit solution of the constant relative risk aversion (CRRA) and constant absolute risk aversion (CARA) utility functions.

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