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On the Statistical Properties of the Remkan Distribution

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Abstract

The Remkan distribution is a two-parameter lifetime distribution that has been introduced into the literature to meet the ever-growing demand for the development of new lifetime distributions to meet the goodness of fit demand of complex datasets. The mathematical properties of the Remkan distribution have been derived in the literature. This study therefore aims to derive important statistical properties including the mode, quantile function, order statistics, entropy, stochastic ordering, average absolute deviation, and mid-point and Reliability indices such as the survivorship or existence measurement function, risk measurement function, and average residual measurement lifetime function.

1. Introduction

In recent years, new two-parameter distributions have emerged in the literature. These new two-parameter distributions have been shown to provide a better fit to complex real life datasets than the one-parameter distributions. Some of the recently developed two-parameter distributions include the Darna distribution (Shraa and Al-Omari [18]), the Hamza distribution (Aijaz et al. [2]), the Samade distribution (Aderoju [1]), the Alzoubi distribution (Benrabia and Alzoubi [5]), and recently, the Copoun distribution (Uwaeme et al. [23]).

It is important to note that these distributions are a mixture of the Exponential and Gamma distributions. These two distributions are known to have their weaknesses. The weakness of the Exponential distribution is that the hazard rate function is constant;

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hence, it cannot handle datasets with monotone non-decreasing hazard rates (Elechi et al. [7], Epstein [8], Ronald et al. [10], Shukla [20], Shukla [19]). Furthermore, the weakness of the Gamma distribution is that the survival rate function cannot be expressed in closed form (Elechi et al. [7], Shanker [12, 13]). The weaknesses of these two distributions are what the aforementioned one-parameter and two-parameter distributions address, providing distributions whose survival rate function can be expressed in closed form and hazard rate functions capable of handling datasets with monotone non-decreasing hazard rates.

In contributing to this gap in the literature, Uwaeme and Akpan [22] proposed a new two-parameter distribution called the Remkan distribution. The Remkan distribution is a three-component density of an exponential (η), gamma $(3,\eta)$, and gamma $(4,\eta)$ distribution with mixing proportions $\tau_1 = \frac{\eta}{\eta+2\phi+6}$, $\tau_2 = \frac{2\phi}{\eta+2\phi+6}$ and $\tau_2 = \frac{6}{\eta+2\phi+6}$ such that

$$g(x;\eta,\phi) = \tau_1 g_1(x;\eta) + \tau_2 g_2(x;\eta) + \tau_2 g_3(x;\eta), \tag{1}$$

where

$$g_1(x;\eta) = \eta e^{-\eta x},\ g_2(x;\eta) = \frac{\eta^3 x^2 e^{-\eta x}}{\Gamma(3)},\ g_3(x;\eta) = \frac{\eta^4 x^3 e^{-\eta x}}{\Gamma(4)},\ \tau_1 = \frac{\eta}{(\eta+2\phi+6)},$$
$$\tau_2 = \frac{2\phi}{(\eta+2\phi+6)} \text{ and } \tau_3 = \frac{6}{(\eta+2\phi+6)}$$

therefore,

$$g(x_k;\eta,\phi) = \eta e^{-\eta x}\cdot\frac{\eta}{(\eta+2\phi+6)} + \frac{\eta^3 x^2 e^{-\eta x}}{\Gamma(3)}\cdot\frac{2\phi}{(\eta+2\phi+6)} + \frac{\eta^4 x^3 e^{-\eta x}}{\Gamma(4)}\cdot\frac{6}{(\eta+2\phi+6)}. \tag{2}$$

Solving equation 2 gives the probability density function of the new distribution

$$g(x_k;\eta,\phi) = \frac{\eta^2}{(\eta+2\phi+6)}[1+\phi\eta x^2+\eta^2 x^3]e^{-\eta x};\ x>0, \eta>0, \phi>0. \tag{3}$$

The corresponding cumulative distribution function (cdf) of (3) is obtained as

$$G(x;\eta,\phi) = 1-\left[1+\frac{\eta^3x^3+(3+\phi)\eta^2x^3+(6+2\phi)\eta x}{\eta+2\phi+6}\right]e^{-\eta x}. \tag{4}$$

The authors introduced some of the mathematical properties of the new distribution. They showed that the Remkan distribution exhibits shapes that are not bell-shaped, but positively skewed, unimodal, and right-tailed (Uwaeme and Akpan [22]).

In this study, we derive some of the important statistical properties of the Remkan distribution and examine its implications before concluding with some remarks.

2. Statistical Properties of the Remkan Distribution

In this section, we derive and present some statistical properties of the Remkan distribution. These include the survivorship or existence measurement function, risk measurement function, and average residual measurement lifetime function, stochastic ordering of random variate, absolute deviations from average and midpoints, Bonferroni curve, Lorenz curve, Bonferroni and Lorenz indices, entropy.

2.1. Mode

Theorem 1. *Given a continuous random variable X which follows the Remkan distribution, the mode of X, is given as*

$$Mode = -\frac{\eta^3\left(1+x^2(\phi\eta-3)-2\phi x+\eta x^3\right)e^{-\eta x}}{\eta+2\phi+6}. \tag{5}$$

Proof. From given a continuous random variable X, the mode of X, is obtained by

$$Mode = \frac{d}{dx}g(x;\eta,\phi) = 0, \tag{6}$$

$$\frac{d}{dx}\left(\frac{\eta^2}{(\eta+2\phi+6)}[1+\phi\eta x^2+\eta^2x^3]e^{-\eta x}\right) = 0, \tag{7}$$

$$\frac{\eta^2}{(\phi+\eta)}\left[\frac{d}{dx}([1+\phi\eta x^2+\eta^2x^3]e^{-\eta x})\right] = 0, \tag{8}$$

$$\therefore \frac{d}{dx}g(x;\eta,\phi) = -\frac{\eta^3\left(1+x^2(\phi\eta-3)-2\phi x+\eta x^3\right)e^{-\eta x}}{\eta+2\phi+6} = 0. \tag{9}$$

2.2. Quantile function

Theorem 2. *Given a continuous random variable X, then the quantile function is obtained by*

$$Q(x) = \eta^{-1}\ln\left[\eta^3x_q^3+(3+\phi)\eta^2x_q^2+(6+2\phi)\eta x_q-(1-q)\right]. \tag{10}$$

Proof. Given a continuous random variable X, then the quantile function is obtained by

$$Q(x) = G^{-1}(x;\eta,\phi), \tag{11}$$

$$q = 1-\left[1+\frac{\eta^3x_q^3+(3+\phi)\eta^2x_q^3+(6+2\phi)\eta x_q}{\eta+2\phi+6}\right]e^{-\eta x_q}, \tag{12}$$

$$(1-q) = \left[\frac{\eta^3x_q^3+(3+\phi)\eta^2x_q^3+(6+2\phi)\eta x_q+(\eta+2\phi+6)}{\eta+2\phi+6}\right]e^{-\eta x_q} = 0. \tag{13}$$

Taking the log of both sides and simplifying, we have

$$\therefore x_q = \eta^{-1} \ln\left[\eta^3 x_q^3 + (3+\phi)\eta^2 x_q^2 + (6+2\phi)\eta x_q - (1-q)\right]. \tag{14}$$

Which completes the proof.

2.3. Order statistics

Theorem 3. *Given a continuous random variable X, pdf and cdf of the pth order statistics, say $X = X_{(p)}$, is given respectively by*

$$g_p(x) = \frac{n![1+\phi\eta x^2+\eta^2 x^3]\eta^2 e^{-\eta x}}{(\eta+2\phi+6)(p-1)!(n-p)!} \sum_{i=0}^{n-p} \binom{n-p}{i} (-1)^i \left[1 - \left[1 + \frac{\eta^3 x^3 + (3+\phi)\eta^2 x^3 + (6+2\phi)\eta x}{\eta+2\phi+6}\right] e^{-\eta x}\right]^{p+i-1} \tag{15}$$

and

$$G_p(x) = \sum_{j=p}^{n} \sum_{i=0}^{n-j} \binom{n}{j} \binom{n-j}{i} (-1)^i \left[1 - \left[1 + \frac{\eta^3 x^3 + (3+\phi)\eta^2 x^3 + (6+2\phi)\phi\eta}{\eta+2\phi+6}\right] e^{-\eta x}\right]^{j+1}. \tag{16}$$

Proof. Given a continuous random variable X, the pdf of the pth order statistics, say $X = X_{(p)}$, is obtained by

$$g_p(x) = \frac{n! g(x_k; \Phi)}{(p-1)!(n-p)!} \sum_{i=0}^{n-p} \binom{n-p}{i} (-1)^i G(x_k; \Phi)^{p+i-1}, \tag{17}$$

$$g_p(x) = \frac{n!\left(\frac{\eta^2}{(\eta+2\phi+6)}[1+\phi\eta x^2+\eta^2 x^3]e^{-\eta x}\right)}{(p-1)!(n-p)!} \sum_{i=0}^{n-p} \binom{n-p}{i} (-1)^i \left[1 - \left[1 + \frac{\eta^3 x^3 + (3+\phi)\eta^2 x^3 + (6+2\phi)\eta x}{\eta+2\phi+6}\right] e^{-\eta x}\right]^{p+i-1}, \tag{18}$$

$$g_p(x) = \frac{n![1+\phi\eta x^2+\eta^2 x^3]\eta^2 e^{-\eta x}}{(\eta+2\phi+6)(p-1)!(n-p)!} \sum_{i=0}^{n-p} \binom{n-p}{i} (-1)^i \left[1 - \left[1 + \frac{\eta^3 x^3 + (3+\phi)\eta^2 x^3 + (6+2\phi)\eta x}{\eta+2\phi+6}\right] e^{-\eta x}\right]^{p+i-1}. \tag{19}$$

Which completes the proof.

Correspondingly, given a continuous random variable X, the cdf of the pth order statistics, say $X = X_{(p)}$, is obtained by

$$G_p(x) = \sum_{j=p}^{n} \sum_{i=0}^{n-j} \binom{n}{j} \binom{n-j}{i} (-1)^i G(x_k; \Phi)^{j+1}, \tag{20}$$

$$G_p(x) = \sum_{j=p}^{n} \sum_{i=0}^{n-j} \binom{n}{j} \binom{n-j}{i} (-1)^i \left[1 - \left[1 + \frac{\eta^3 x^3 + (3+\phi)\eta^2 x^3 + (6+2\phi)\eta x}{\eta+2\phi+6}\right] e^{-\eta x}\right]^{j+1}. \tag{21}$$

Which completes the proof.

2.4. Entropy

Entropy measures the uncertainties associated with a random variable of a probability distribution. Shannon (Shannon [17]) and Rényi's entropy (Rényi [9]) is widely used in the literature.

Theorem 4. *Given a random variable X, which follows the Remkan distribution* $g(x;\eta,\phi)$. *The Rényi entropy is given by*

$$T_R(\lambda) = \frac{1}{1-\lambda}\log\left\{\frac{\eta^{2\lambda}}{(\eta+2\phi+6)^{\lambda}}\sum_{m=0}^{\infty}\binom{\lambda}{m}\cdot\left[\frac{(\eta\phi)^m\Gamma(2m+1)}{(\eta\lambda)^{2m+1}}+\frac{\eta^{2n}\Gamma(3m+1)}{(\eta\lambda)^{3m+1}}\right]\right\}. \tag{22}$$

Proof. The Rényi entropy is given by

$$T_R(\lambda) = \frac{1}{1-\lambda}\log\left\{\int_0^{\infty} g_n^{\lambda}(x_k;\Phi)dx\right\}, \tag{23}$$

$$T_R(\lambda) = \frac{1}{1-\lambda}\log\left\{\int_0^{\infty}\left(\frac{\eta^2}{(\eta+2\phi+6)}[1+\phi\eta x^2+\eta^2x^3]e^{-\eta x}\right)^{\lambda}dx\right\}, \tag{24}$$

$$T_R(\lambda) = \frac{1}{1-\lambda}\log\left\{\frac{\eta^{2\lambda}}{(\eta+2\phi+6)^{\lambda}}\int_0^{\infty}[1+\phi\eta x^2+\eta^2x^3]^{\lambda}e^{-\lambda\eta x}dx\right\}. \tag{25}$$

Recall $(1+q^2)^{\vartheta} = \sum_{n=0}^{\infty}\binom{\vartheta}{n}(q^2)^n$ and $\int_0^{\infty} z^w e^{-\varrho z}\,dz = \frac{\Gamma(w+1)}{\varrho^{w+1}}$. Substituting,

$$T_R(\lambda) = \frac{1}{1-\lambda}\log\left\{\frac{\eta^{2\lambda}}{(\eta+2\phi+6)^{\lambda}}\sum_{m=0}^{\infty}\binom{\lambda}{m}\left[(\phi\eta)^m\int_0^{\infty}x^{2m}e^{-\lambda\eta x}dx+\eta^{2m}\int_0^{\infty}x^{3m}e^{-\lambda\eta x}dx\right]\right\}, \tag{26}$$

$$\therefore T_R(\lambda) = \frac{1}{1-\lambda}\log\left\{\frac{\eta^{2\lambda}}{(\eta+2\phi+6)^{\lambda}}\sum_{m=0}^{\infty}\binom{\lambda}{m}\cdot\left[\frac{(\eta\phi)^m\Gamma(2m+1)}{(\eta\lambda)^{2m+1}}+\frac{\eta^{2n}\Gamma(3m+1)}{(\eta\lambda)^{3m+1}}\right]\right\}. \tag{27}$$

Which completes the proof.

2.5. Reliability indices

Given any probability distribution, the reliability analysis is always considered based on the Survivorship or Existence Measurement Function, Risk Measurement Function and Average Residual Measurement Life-Time Function. Hence, for the Remkan distribution, the Survivorship or Existence Measurement Function, Risk Measurement Function, and Average Residual Measurement Life-Time Function is given below.

2.5.1. Survivorship or existence measurement function

The survivorship or existence measurement function (also known as survival function) is defined as the probability that an item does not fail prior to sometime *t* (Elechi et al. [7], Epstein [8], Ronald et al. [10], Shanker and Shukla [16]).

The survivorship or existence measurement function of the Remkan distribution is given by

$$s(x) = 1 - G(x;\eta,\phi), \tag{28}$$

$$s(x) = 1 - \left[1 - \left[1 + \frac{\eta^3x^3+(3+\phi)\eta^2x^3+(6+2\phi)\eta x}{\eta+2\phi+6}\right]e^{-\eta x}\right], \tag{29}$$

$$\therefore s(x) = \left[1 + \frac{\eta^3x^3+(3+\phi)\eta^2x^3+(6+2\phi)\eta x}{\eta+2\phi+6}\right]e^{-\eta x}. \tag{30}$$

2.5.2. Risk measurement function

The risk measurement function (also known as hazard rate function) on the other hand can be seen as the conditional probability of failure given that it has survived to the time *t* (Elechi et al. [7], Ronald et al. [10], Shanker [15], Umeh and Ibenegbu [21]).

The risk measurement function of the Remkan distribution is given by

$$h(x) = \frac{g(x_k;\Phi)}{1-G(x_k;\Phi)}, \tag{32}$$

$$h(x) = \frac{\frac{\eta^2}{(\eta+2\phi+6)}\left[1+\phi\eta x^2+\eta^2x^3\right]e^{-\eta x}}{\left[1-\left[1+\frac{\eta^3x^3+(3+\phi)\eta^2x^3+(6+2\phi)\phi\eta}{\eta+2\phi+6}\right]e^{-\eta x}\right]}, \tag{33}$$

$$h(x) = \frac{\eta^2\left[1+\phi\eta x^2+\eta^2x^3\right]}{\eta^3x^3+(3+\phi)\eta^2x^3+(6+2\phi)\phi\eta+(\eta+2\phi+6)}. \tag{34}$$

2.5.3. Average residual measurement life-time function

The average residual measurement lifetime function of the new distribution is given by

$$m(x) = E[X-x|X>x] = \frac{1}{1-G(x)}\int_x^\infty 1-G(h)\ dh, \tag{35}$$

$$m(x) = \frac{\left[\eta^3x^3+\eta^2x^2(3+3\eta+\phi\eta)+\eta x(6+120\eta+4\phi\eta)+\left(6+18\eta+6\phi\eta+\eta^2\right)\right]}{\eta^2\left[\eta^3x^3+(3+\phi)\eta^2x^3+(6+2\phi)\phi\eta+(\eta+2\phi+6)\right]}. \tag{36}$$

Remark 2. It can easily be observed that for $x = 0$,

$$S(0) = 1,\ h(0) = \frac{\eta^2}{(\eta+2\phi+6)} = g(0), \text{ and } m(0) = \frac{6+18\eta+6\phi\eta+\eta^2}{\eta^2(\eta+2\phi+6)}.$$

Figure 1 and Figure 2 show the graphical plots of $h(x)$ and $M(x)$ (for some selected but different real points of η and ϕ) of a Remkan distribution.

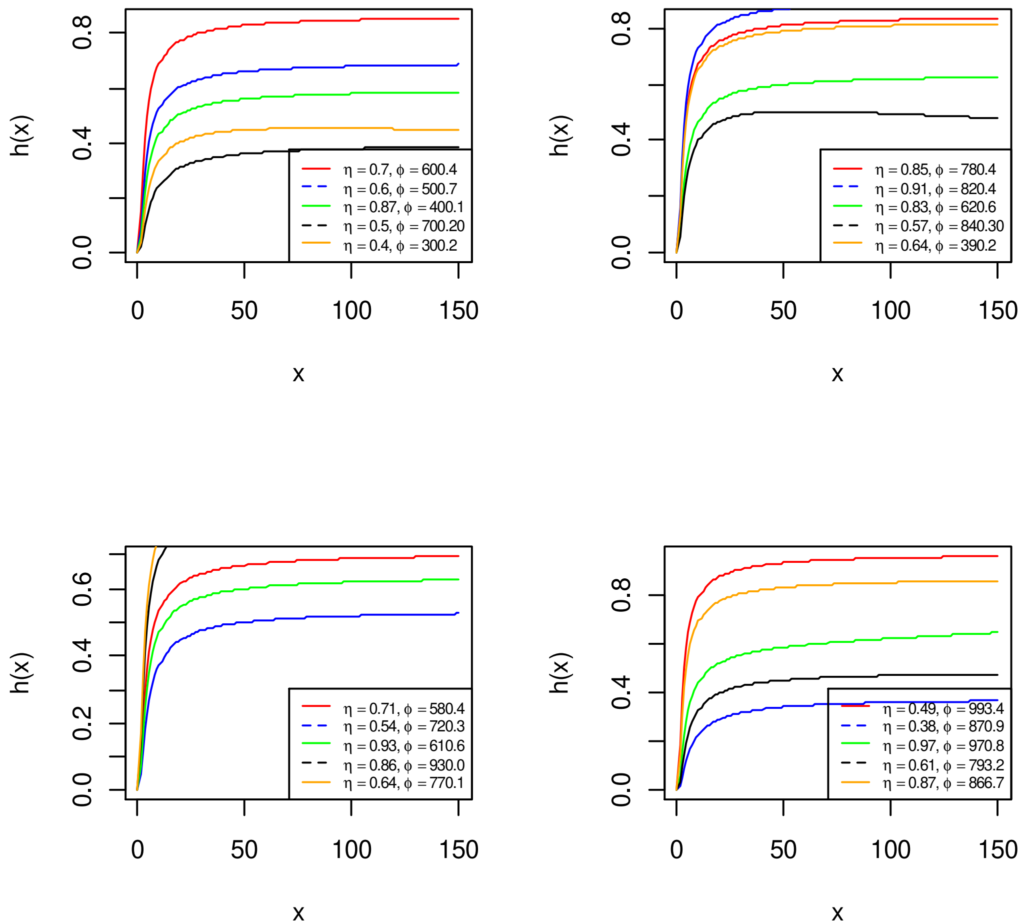

Figure 1: The graphical plot of the risk measurement function of the Remkan distribution showing the different values of the parameters.

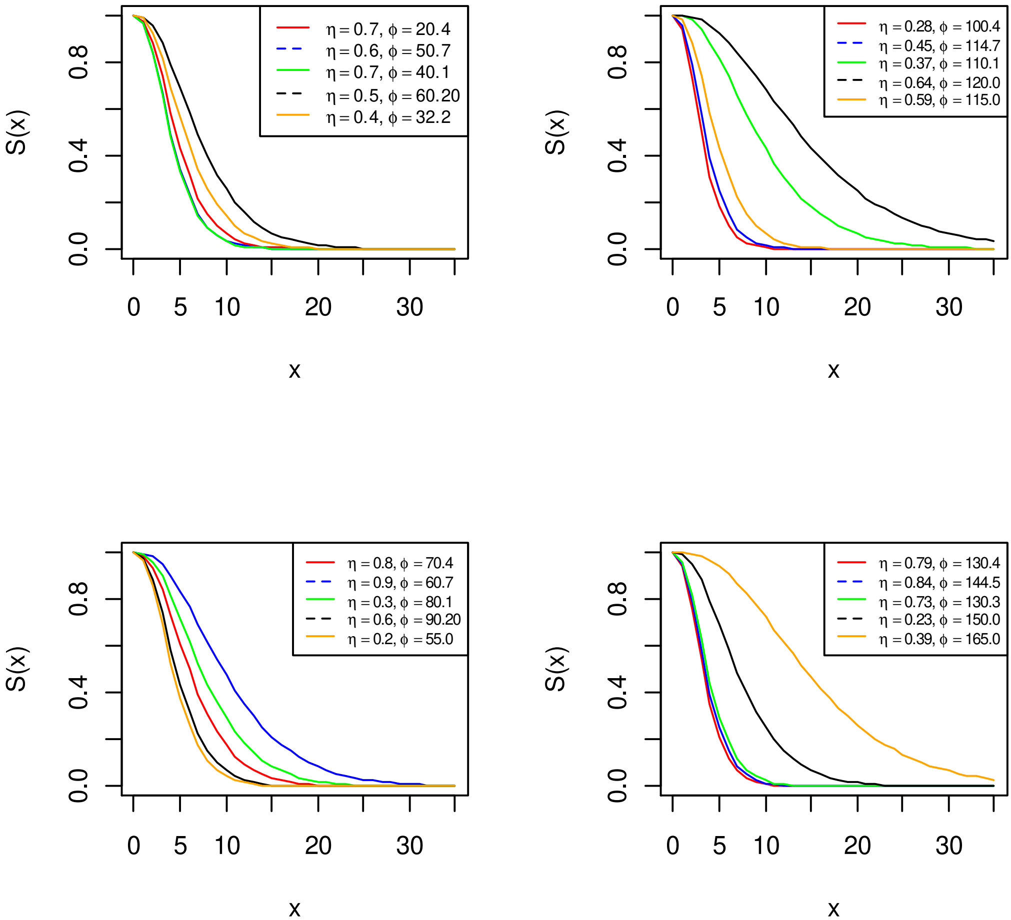

Figure 2: The graphical plot of the survivorship or existence measurement function of the Remkan distribution showing the different values of the parameters.

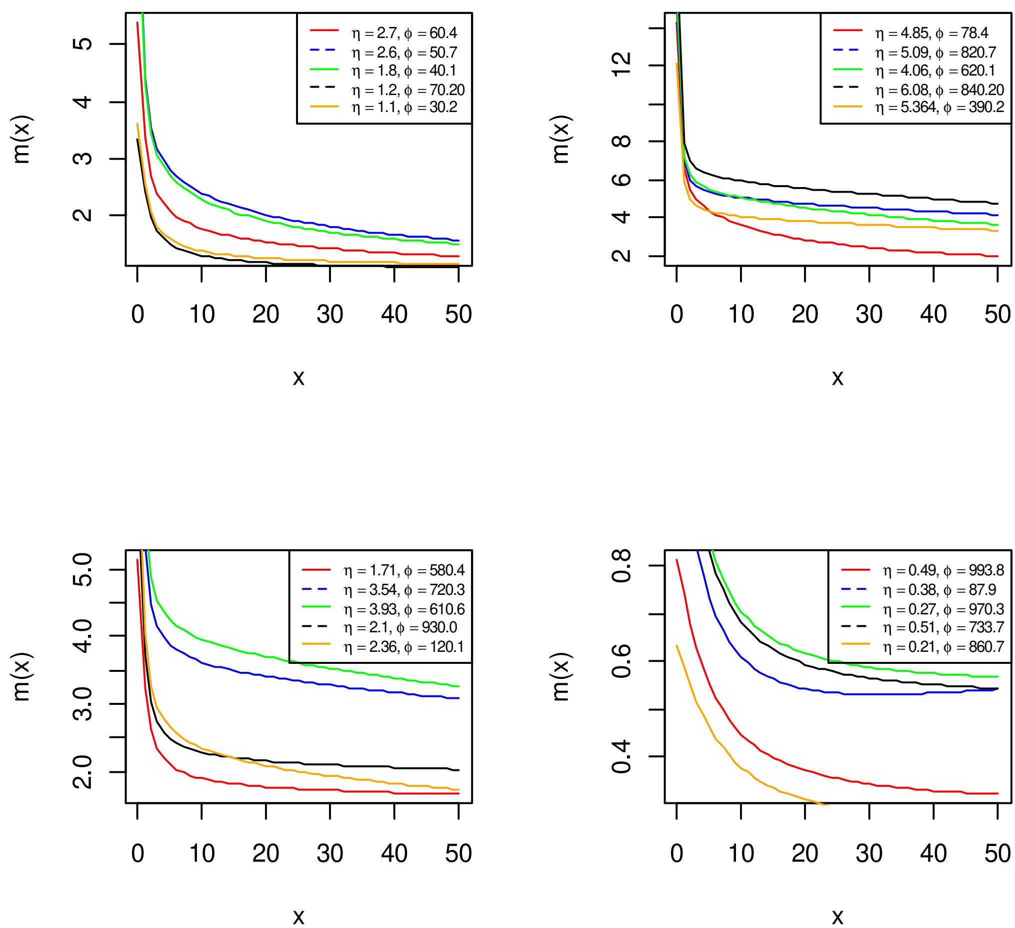

Figure 3: The graphical plots of average residual measurement lifetime function of the Remkan distribution showing the different values of the parameters.

Figure 1 shows that the graphical plots of $h(x)$ show that $h(x)$ display an increasing failure rate (IFR) function, increasing failure rate average (IFRA), new better than used (NBU), and new better than used in expectation (NBUE), respectively. (See (Barlow and Proschan [4])).

Figures 2 and 3 show that the graphical plots of $S(x)$ and $M(x)$ display a monotone non-increasing function respectively.

2.6. Stochastic ordering

Theorem 5. *Let* $S\sim$ CD(η_1,ϕ_1) *and* $Q\sim$ CD(η_2,ϕ_2). *If* $\eta_1=\eta_2$ *and* $\phi_1=\phi_2$ *and* $\phi_1\geq\phi_2$ *(or if* $\phi_2=\phi_2$ *and* $\eta_1=\eta_2$ *and* $\eta_1\geq\eta_2$*), then* $S\leq_{stor}Q$ *and hence* $S\leq_{Ror}Q$, $S\leq_{mrl}Q$ *and* $S\leq_{lor}Q$.

Proof. Let $S\sim$ CD(η_1,ϕ_1) and Q~ CD(η_2,ϕ_2). We obtain that

$$\frac{f_S(\varphi)}{f_Q(\varphi)}=\frac{\eta_1^2(\eta_2+\phi_2+6)}{\eta_2^2(\eta_1+2\phi_1+6)}\cdot e^{-(\eta_1-\eta_2)\varphi} \tag{37}$$

and

$$\log_\varphi\frac{f_S(\varphi)}{f_Q(\varphi)}=\log_\varphi\left[\frac{\eta_1^2(\eta_2+\phi_2+6)}{\eta_2^2(\eta_1+2\phi_1+6)}\right]-(\eta_1-\eta_2)\varphi. \tag{38}$$

Hence,

$$\frac{d}{d\varphi}\log_\varphi\frac{f_S(\varphi)}{f_Q(\varphi)}=-(\eta_2-\eta_1). \tag{39}$$

Which completes the proof.

This implies that $S\leq_{lor}Q\Rightarrow S\leq_{Ror}Q\Rightarrow\begin{cases}S\leq_{sor}Q\\S\leq_{mrl}Q\end{cases}$. This clearly indicates that the Remkan distribution is ordered in the likelihood ratio and consequently has risk measurement, average residual measurement life, and stochastic orderings. These results have been established in the literature for stochastic ordering of distributions (Shaked and Shanthikumar [11], Shanker [14]).

2.7. Average absolute deviations and midpoints

The attitude of variation inherent in a group of observations can be determined by the absolute deviations about the average and midpoints respectively. The absolute deviations about the average point (denoted $\psi_1(x_k)$) and that about the mid-point (denoted $\psi_2(x_k)$) are defined by

$$\psi_1(x_k)=E(|X-\mu^*|)=\int_0^\infty|X-\mu^*|\,g(x;\eta,\phi)dx, \tag{40}$$

$$\psi_2(x_k)=E(|X-M^*|)=\int_0^\infty|X-M^*|\,g(x;\eta,\phi)dx. \tag{41}$$

The absolute deviation about the average point can be obtained by using

$$\psi_1(x_k)=\int_0^\infty|x_k-\mu^*|\,g(x;\eta,\phi)dx$$

$$= \int_0^{\mu^*} |x_k - \mu^*| \, g(x_k; \Phi) dx + \int_{\mu^*}^{\infty} |x_k - \mu^*| \, g(x; \eta, \phi) dx \qquad (42)$$

$$= 2\mu^* G(\mu^*) - 2 \int_0^{\mu^*} x_k g(x; \eta, \phi) dx. \qquad (43)$$

In addition, absolute deviation about the mid-point can be obtained by using

$$\psi_2(x_k) = \int_0^{\infty} |x_k - M^*| \, g(x; \eta, \phi) dx$$

$$= \int_0^{M^*} |x_k - M^*| \, g(x; \eta, \phi) dx + \int_{M^*}^{\infty} |x_k - M^*| \, g(x; \eta, \phi) dx \qquad (44)$$

$$= -\mu^* + 2 \int_{M^*}^{\infty} x_k g(x; \eta, \phi) dx \qquad (45)$$

$$= \mu^* - 2 \int_0^{M^*} x_k g(x; \eta, \phi) dx \qquad (46)$$

where $\mu^* = E(X)$ is the average of the random variable X, M^* is the mid-point of the random variable X and $G(\mu^*) = \int_0^{\mu^*} g(x; \eta, \phi) dx$.

By using the pdf in equation 3, we obtain the following equations

$$\int_0^{\mu^*} x g(x; \eta, \phi) dx = \mu - \frac{[\eta^4 \mu^{*4} + \eta^3 \mu^{*3}(4 + \phi\eta) + \eta^2 \mu^{*2}(12 + 3\phi\eta) + \eta\mu^*(24 + 6\phi\eta + \eta^2) + (24 + 6\phi\eta + \eta^2)] e^{-\eta\mu^*}}{\eta(\eta + 2\phi + 6)}, \qquad (47)$$

$$\int_0^{M^*} x g(x; \eta, \phi) dx = \mu^* - \frac{[\eta^4 M^{*4} + \eta^3 M^{*3}(4 + \phi\eta) + \eta^2 M^{*2}(12 + 3\phi\eta) + \eta M^*(24 + 6\phi\eta + \eta^2) + (24 + 6\phi\eta + \eta^2)] e^{-\eta M^*}}{\eta(\eta + 2\phi + 6)}, \qquad (48)$$

$$\int_{M^*}^{\infty} x_k g(x; \eta, \phi) dx = \frac{[\eta^4 M^{*4} + \eta^3 M^{*3}(4 + \phi\eta) + \eta^2 M^{*2}(12 + 3\phi\eta) + \eta M^*(24 + 6\phi\eta + \eta^2) + (24 + 6\phi\eta + \eta^2)] e^{-\eta M^*}}{\eta(\eta + 2\phi + 6)}, \qquad (49)$$

$$\int_0^{\mu^*} g(x; \eta, \phi) dx = 1 - \frac{[\phi\eta^3 \mu^{*3} + (3 + \phi)\eta^2 \mu^{*3} + (6 + 2\phi)\eta\mu^* + (\eta + 2\phi + 6)] e^{-\eta\mu^*}}{(\eta + 2\phi + 6)}. \qquad (50)$$

Substituting equation 47 and equation 48 into equations 43 and equation 46 respectively, we obtain

$$\psi_1(x_k) = \frac{2[-\eta^4 \mu^{*3}\phi + \eta^3 \mu^{*3}(\phi - 1) - 3\eta^3 \mu^{*2}\phi + \eta^2 \mu^{*2}(2\phi - 6) + \eta\mu^*(24 + 6\phi\eta + \eta^2) - 6\eta(\phi + \eta) - 24] e^{-\eta\mu^*}}{\eta(\eta + 2\phi + 6)} \qquad (51)$$

and

$$\psi_2(x_k) = \frac{[\eta^4 M^{*4} + \eta^3 M^{*3}(4+\phi\eta) + \eta^2 M^{*2}(12+3\phi\eta) + \eta M^*(24+6\phi\eta+\eta^2) + (24+6\phi\eta+\eta^2)]e^{-\eta M^*}}{\eta(\eta+2\phi+6)} - \mu^*. \quad (52)$$

2.8. Bonferroni and Lorenz curves

The Bonferroni and Lorenz curves (Arcagni and Porro [3], Bonferroni [6]) have applications not only in economics to study income and poverty, but also in other fields like reliability, demography, insurance, and medicine. The Bonferroni and Lorenz curves are defined as

$$B(h) = \frac{1}{h\mu^*}\int_0^h xg(x;\eta,\phi)dx = \frac{1}{h\mu^*}\left[\int_0^\infty xg(x;\eta,\phi)dx - \int_h^\infty xg(x;\eta,\phi)dx\right]. \quad (53)$$

$$\therefore B(h) = \frac{1}{h\mu^*}\left[\mu^* - \int_h^\infty xg(x;\eta,\phi)dx\right]. \quad (54)$$

and

$$L(h) = \frac{1}{\mu^*}\int_0^h xg(x;\eta,\phi)dx = \frac{1}{\mu^*}\left[\int_0^\infty xg(x;\eta,\phi)dx - \int_h^\infty xg(x;\eta,\phi)dx\right]. \quad (55)$$

$$\therefore L(h) = \frac{1}{\mu^*}\left[\mu^* - \int_h^\infty xg(x;\eta,\phi)dx\right]. \quad (56)$$

Respectively or equivalently

$$B(h) = \frac{1}{h\mu^*}\int_0^h G^{-1}(x;\eta,\phi)dx \quad (57)$$

and

$$L(h) = \frac{1}{\mu^*}\int_0^h G^{-1}(x;\eta,\phi)dx \quad (58)$$

respectively, where $\mu^* = E(X)$ and $h = G^{-1}(h)$.

By using the pdf in equation 3, we obtain the following equations

$$\int_h^\infty xg(x;\eta,\phi)dx = \frac{[\eta^4 h^4 + \eta^3 h^3(4+\phi\eta) + \eta^2 h^2(12+3\phi\eta) + \eta h(24+6\phi\eta+\eta^2) + (24+6\phi\eta+\eta^2)]e^{-\eta h}}{\eta(\eta+2\phi+6)}. \quad (59)$$

On substituting equation 59 in equations 54, we obtain

$$\therefore B(h) = \frac{1}{h}\left[1 - \frac{\eta^2(\eta+6\phi+24) - \eta^3[\eta^4 h^4 + \eta^3 h^3(4+\phi\eta) + \eta^2 h^2(12+3\phi\eta) + \eta h(24+6\phi\eta+\eta^2) + (24+6\phi\eta+\eta^2)]e^{-\eta h}}{\eta^4(\eta+2\phi+6)}\right]. \quad (60)$$

Also, substituting equation 59 into equation 56 the Lorenz curve is obtained by

$$L(h) = \frac{1}{\mu^*}\left[\mu^* - \int_h^\infty x_k g(x_k; \Phi) dx\right].$$

$$\therefore L(h) = 1 - \frac{\left[\eta^4 h^4 + \eta^3 h^3 (4+\phi\eta) + \eta^2 h^2 (12+3\phi\eta) + \eta h\left(24+6\phi\eta+\eta^2\right) + \left(24+6\phi\eta+\eta^2\right)\right] e^{-\eta h}}{(\eta+6\phi+24)}. \quad (61)$$

3. Conclusion

This paper examined some of the properties of the Remkan distribution and derived some important statistical properties of the Remkan distribution such as the mode, quantile function, entropy, order statistics and reliability indices. The reliability indices from the derived statistical properties of the Remkan distribution showed that the risk measurement function of the Remkan distribution can model datasets with increasing failure rate (IFR), increasing failure rate average (IFRA), new better than used (NBU), and new better than used in expectation (NBUE) in survival analysis. In addition, the survivorship or existence measurement function and the average residual measurement lifetime function are monotone non-increasing functions respectively.

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