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Blow-up of Solutions for a Problem with Balakrishnan-Taylor Damping and Nonlocal Singular Viscoelastic Equations

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Abstract

In this paper, we study the nonlinear one-dimensional viscoelastic nonlocal problem with Balakrishnan-Taylor damping terms and nonlinear source of polynomial type. We demonstrate that the nonlinear source of polynomial type is able to force solutions to blow up infinite time even in presence of stronger damping with non positive initial energy combined with a positive initial energy.

1 Introduction

In fluid dynamics, the blow-up problem of solutions has attracted much attention and challenge among physicists and mathematicians. In this works, we study blow-up of solutions of the following singular nonlinear nonlocal viscoelastic

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problem with Balakrishnan-Taylor damping and logarithmic nonlinearity source

$$
\begin{aligned}
& u_{tt}(t) - M(t)\frac{1}{x}(xu_x(t))_x + \int_0^t g(t-s)\frac{1}{x}(xu_x(x,s))_x\, ds + au_t(t) \\
= \quad & |u(t)|^{p-2} u(t), \text{ in } Q, \qquad (1.1)
\end{aligned}
$$

$$
\begin{cases}
u(x,0) = u_0(x), \; u_t(x,0) = u_1(x), \; x \in (0,\alpha), \\
\\
u_x(\alpha,t) = 0, \; \int_0^\alpha xu(x,t)\, dx = 0, \quad t \in [0,T],
\end{cases} \qquad (1.2)
$$

where $Q := (0,\alpha) \times (0,T)$, $\alpha < \infty$, $T < \infty$, $p > 4$, $g(.) : \mathbb{R}^+ \to \mathbb{R}^+$ are given functions which will be specified later, and $M(t) := \xi_0 + \xi_1 \|u_x(t)\|^2_{L^2_x(0,\alpha)} + \sigma(u_x(t), u_{xt}(t))_{L^2_x(0,\alpha)}$, where u is the plate transverse displacement, x is the spatial coordinate in the direction of the fluid flow, and t is time. The viscoelastic structural damping terms are denoted by ξ_1, σ is the nonlinear stiffness of the membrane, and ξ_0 is an in-plane tensile load. All quantities are physically non-dimensional zed and ξ_0, ξ_1 and σ are fixed positive. For more information on using Balakrishnan-Taylor terms, see $[22-24]$.

Over the past few decades, the uninterrupted, mixed problems of a wide range of partial and differential equations have been rewarded and the reason for this great concern is that these problems are particularly inspired by physics and physical science.

Generally speaking, nonlocal boundary conditions can be encountered in many scientific domains and are widely applied in heat transmission theory, and many engineering models population dynamics, and control theory, and chemical engineering, and medical science, and chemical reaction diffusion, and thermo elasticity, and heat conduction processes, and biological processes.

See in this regard the works by Cahlon and Shi [1], Mesloub and Lekrine [2], Ewing and Lin [3], Shi [4], Choi and Chan [5], Cannon [6], Capasso-Kunisch [7], Yurchuk [8], Shi and Shilor [9], Ionkin and Moiseev [10], Kamynin [11], Mesloub [12, 13], Ionkin [14], Mesloub and Messaoudi [15, 16], Kartynnik [17], Pulkina [18, 19], Mesloub and Bouziani [20, 21].

The motivation of our work is due to some results regarding the following research papers:

Mesloub and Mesloub [25] studied the solvability of a mixed nonlocal problem for a nonlinear singular viscoelastic equation

$$\begin{cases} u_{tt}(t) - \frac{1}{x}(xu_x(t))_x + \int_0^t g(t-s)\frac{1}{x}(xu_x(x,s))_x\,ds + au_t(t) \\ = f(x,t,u_x,u), \text{ in } Q, \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ x \in (0,1), \\ u_x(1,t) = 0,\ \int_0^1 xu(x,t)\,dx = 0, \quad t \in [0,T], \end{cases}$$

where $Q := (0,1) \times (0,T)$, $a > 0$ and for the relaxation function $g(t)$, we assume that $g : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a bounded C^2 function such that

$$g(s) \geq 0,\ g'(s) \leq 0 \text{ and } \int_0^\infty g(s)\,ds < 1.$$

Also, a result of local existence for problem $(1.1) - (1.2)$ for $\xi_1 = \sigma = 0$, has been proved in $[25]$, for $\xi_1, \sigma \neq 0$ and $a > 0$, in the same way as [25], we get the same basic results for the local existence of problem $(1.1) - (1.2)$ with a slight change in some calculations that do not affect the basic results.

Wu [26] studied the blow-up of solutions for a singular nonlocal viscoelastic equation

$$\begin{cases} u_{tt}(t) - \frac{1}{x}(xu_x(t))_x + \int_0^t g(t-s)\frac{1}{x}(xu_x(x,s))_x\,ds = |u|^{p-2}u, \text{ in } (0,\alpha)\times(0,T), \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ x \in (0,\alpha), \\ u(\alpha,t) = 0,\ \int_0^\alpha xu(x,t)\,dx = 0, \quad t \in [0,T], \end{cases}$$

where $\alpha < \infty$, $T < \infty$, $p > 2$, and $g : \mathbb{R}^+ \to \mathbb{R}^+$ represents the kernel of the memory term which is specified later (See [26]).

Liu et al. [27] studied the on decay and blow-up of solutions for a singular nonlocal viscoelastic problem with a nonlinear source term

$$\begin{cases} u_{tt}(t) - \frac{1}{x}(xu_x(t))_x + \int_0^t g(t-s)\frac{1}{x}(xu_x(x,s))_x ds + au_t = |u|^{p-2}u, \text{ in } (0,l)\times(0,T), \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), \quad x \in (0,l), \\ u(l,t) = 0,\ \int_0^l xu(x,t)\,dx = 0, \quad t \in [0,T], \end{cases}$$

where $a \geq 0$, $l < \infty$, $T < \infty$, $p > 2$, and $g : \mathbb{R}^+ \to \mathbb{R}^+$ represents the kernel of the memory term which is specified later (See [27]).

Zarai et al. [28] studied the blow-up of solutions for a system of nonlocal singular viscoelastic equations

$$\begin{cases} u_{tt} - \dfrac{1}{x}(xu_x)_x + \int_0^t g_1(t-s)\dfrac{1}{x}(xu_x(x,s))_x ds + u_t = |v|^{q+1}|u|^{p-1}u, \text{ in } Q, \\ v_{tt} - \dfrac{1}{x}(xv_x)_x + \int_0^t g_2(t-s)\dfrac{1}{x}(xv_x(x,s))_x ds + v_t = |u|^{p+1}|v|^{q-1}v, \text{ in } Q, \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ x \in (0,\alpha), \\ v(x,0) = v_0(x),\ v_t(x,0) = v_1(x),\ x \in (0,\alpha), \\ u(\alpha,t) = v(\alpha,t) = 0,\ \int_0^\alpha xu(x,t)\,dx = \int_0^\alpha xv(x,t)\,dx = 0, \end{cases}$$

where $Q := (0,\alpha)\times(0,T)$, $\alpha < \infty$, $T < \infty$, $p, q > 1$, and $g_1(.), g_2(.) : \mathbb{R}^+ \to \mathbb{R}^+$, given functions (See [28]).

However, among the literature, such as the work presented in $[1-31]$, which examines blow-up results of problem $(1.1)-(1.2)$, there is no blow-up result when $E(0) \geq 0$ and $E(0) < 0$. Motivated by the previous research papers, we will study the blow-up results when $E(0) \geq 0$ and $E(0) < 0$ of the model $(1.1)-(1.2)$.

The outline of the paper is as follows. In the second section we present some basic concepts as well as two lemmas to help solve the problem presented in $(1.1)-(1.2)$. In Section 3, we define an energy function $E(t)$ and show that it is a non-increasing function of t. Then we obtain **Theorem 1** , which gives the blow-up phenomena of solutions even for positive initial energy. Estimates for the blow-up time are also given.

2 Preliminaries

In this section, we introduce some functional spaces and establish two lemmas, which will be used for the remaining of the present paper.

Let $L_x^p := L_x^p((0,\alpha))$ be the weighted Banach space equipped with the norm

$$\|u(t)\|_{L_x^p(0,\alpha)} := \left[\int_0^\alpha x\,|u(t)|^p\,dx\right]^{\frac{1}{p}}.$$

In particular $L_x^2((0,\alpha))$, the Hilbert space of square integral functions having the finite norm

$$\|u(t)\|_{L_x^2(0,\alpha)} := \left[\int_0^\alpha xu^2(t)\,dx\right]^{\frac{1}{2}}.$$

Lemma 1. (*See* [29]) *Let $\delta > 0$ and $\beta(t) \in C^2(0,\infty)$ be a nonnegative function satisfying*

$$\beta''(t) - 4(\delta+1)\beta'(t) + 4(\delta+1)\beta(t) \geq 0.$$

If

$$\beta'(0) > r_2\beta(0) + k_0,$$

then

$$\beta'(t) > k_0,$$

for $t > 0$, where k_0 is a constant, $r_2 = 2(\delta+1) - 2\sqrt{(\delta+1)\delta}$ is the smallest root of the equation

$$r^2 - 4(\delta+1)r + 4(\delta+1) = 0.$$

Lemma 2. (*See* [29]) *If $J(t)$ is a non-increasing function on $[t_0,\infty)$, $t_0 \geq 0$ and satisfies the deferential inequality*

$$J'(t)^2 \geq \mu + bJ(t)^{2+\frac{1}{\delta}}, \quad \textit{for } t \geq t_0,$$

where $\mu > 0,\ b \in \mathbb{R}$, *then there exists a finite time* T^* *such that*

$$\lim_{t \to T^{*-}} J(t) = 0,$$

and the upper bound of T^* *is estimated respectively by the following cases:*
(i) *If* $b < 0$ *and* $J(t_0) < \min\left\{1, \sqrt{\frac{\mu}{-b}}\right\}$ *then*

$$T^* \leq t_0 + \frac{1}{\sqrt{-b}} \ln \frac{\sqrt{\frac{\mu}{-b}}}{\sqrt{\frac{\mu}{-b}} - J(t_0)}.$$

(ii) *If* $b = 0$, *then*

$$T^* \leq t_0 + \frac{J(t_0)}{\sqrt{\mu}}.$$

(iii) *If* $b > 0$, *then*

$$T^* \leq \frac{J(t_0)}{\sqrt{\mu}},$$

or

$$T^* \leq t_0 + 2^{\frac{3\delta+1}{2\delta}} \frac{\delta c}{\sqrt{\mu}} \left(1 - [1 + cJ(t_0)]^{\frac{1}{2\delta}}\right),$$

where $c := \left(\frac{b}{\mu}\right)^{\delta/(2+\delta)}$.

3 Blowing-up Property

In this section we shall discus the Blow-up phenomena of system $(1.1) - (1.2)$. In order to state our results we make further assumptions on g :

(A) $$g(s) \geq 0,\ g'(s) \leq 0 \text{ and } \int_0^\infty g(s)\,ds < \frac{p(p-2)}{(p-1)^2}\xi_0 < \xi_0, \quad \text{for } p > 4.$$

Remark 1. *The assumptions* $\xi_0 - \int_0^\infty g(s)\,ds > 0$ *is necessary to guarantee the hyperbolicity of the problem* $(1.1) - (1.2)$.

Definition 1. *A solution u of $(1.1)-(1.2)$ is called blow-up if there exists a finite time T^* such that*

$$\lim_{t\to T^{*-}}\left(\|u_x(t)\|^2_{L^2_x(0,\alpha)}\right)^{-1}=0. \tag{3.1}$$

Definite the energy function as

$$\begin{aligned} E(t) \quad : \quad &= \frac{1}{2}\|u_t(t)\|^2_{L^2_x(0,\alpha)}+\frac{1}{2}\left(\xi_0-\int_0^t g(s)\,ds\right)\|u_x(t)\|^2_{L^2_x(0,\alpha)} \\ &+\frac{\xi_1}{4}\|u_x(t)\|^4_{L^2_x(0,\alpha)}+\frac{1}{2}(g\circ u_x)(t)-\frac{1}{p}\left(\int_0^\alpha x\,|u(t)|^p\,dx\right), \end{aligned} \tag{3.2}$$

where

$$(g\circ u_x)(t):=\int_0^\alpha\int_0^t xg(t-s)\,|u_x(x,t)-u_x(x,s)|^2\,dsdx. \tag{3.3}$$

Lemma 3. *Let (u,v) be the solution of problem $(1.1)-(1.2)$. Then $E(t)$ is a non-increasing function on $[0,t)$ and*

$$\begin{aligned} \frac{d}{dt}\{E(t)\} &= \frac{1}{2}\left(g'\circ u_x\right)(t)-\frac{1}{2}g(t)\|u_x(t)\|^2_{L^2_x(0,\alpha)} \\ &\quad -\|u_t(t)\|^2_{L^2_x(0,\alpha)}-\frac{\sigma}{4}\left(\frac{d}{dt}\left\{\|u_x(t)\|^2_{L^2_x(0,\alpha)}\right\}\right)^2 \\ &\le 0. \end{aligned} \tag{3.4}$$

Proof. Multiplying Eqs. (1.1) by $xu_t(t)$ integrating over $(0,\alpha)$ we obtain

$$\begin{aligned} &(u_{tt}(t),u_t(t))_{L^2_x(0,\alpha)}-M(t)\left((xu_x(t))_x,u_t(t)\right)_{L^2(0,\alpha)} \\ &+\left(\int_0^t g(t-s)\,(xu_x(x,s))_x\,ds,u_t(t)\right)_{L^2(0,\alpha)}+(u_t(t),u_t(t))_{L^2_x(0,\alpha)} \\ =\; &\left(|u(t)|^{p-2}u(t),u_t(t)\right)_{L^2_x(0,\alpha)}. \end{aligned} \tag{3.5}$$

Evaluation of term of (3.5), leads to

$$(u_{tt}(t),u_t(t))_{L^2_x(0,\alpha)}=\frac{1}{2}\frac{d}{dt}\left\{\|u_t(t)\|^2_{L^2_x(0,\alpha)}\right\}. \tag{3.6}$$

And

$$\begin{aligned}
& -M(t)\left((xu_x(t))_x, u_t(t)\right)_{L^2(0,\alpha)} \\
= & -\left(\xi_0 + \xi_1 \|u_x(t)\|^2_{L^2_x(0,\alpha)} + \sigma\left(u_x(t), u_{xt}(t)\right)_{L^2_x(0,\alpha)}\right) \int_0^\alpha (xu_x(t))_x u_t(t)\,dx \\
= & \frac{\xi_0}{2}\frac{d}{dt}\left\{\|u_x(t)\|^2_{L^2_x(0,\alpha)}\right\} + \frac{\xi_1}{4}\frac{d}{dt}\left\{\|u_x(t)\|^4_{L^2_x(0,\alpha)}\right\} \\
& +\frac{\sigma}{4}\left(\frac{d}{dt}\left\{\|u_x(t)\|^2_{L^2_x(0,\alpha)}\right\}\right)^2. \qquad (3.7)
\end{aligned}$$

Using integration by parts, we get

$$\begin{aligned}
& \left(\int_0^t g(t-s)\left(xu_x(x,s)\right)_x ds, u_t(t)\right)_{L^2(0,\alpha)} \\
= & \frac{1}{2}\frac{d}{dt}\left\{(g \circ u_x)(t) - \left(\int_0^t g(s)\,ds\right)\|u_x(t)\|^2_{L^2_x(0,\alpha)}\right\} \\
& -\frac{1}{2}\left(g' \circ u_x\right)(t) + \frac{1}{2}g(t)\|u_x(t)\|^2_{L^2_x(0,\alpha)} \qquad (3.8)
\end{aligned}$$

and

$$\left(|u(t)|^{p-2} u(t), u_t(t)\right)_{L^2_x(0,\alpha)} = \frac{d}{dt}\left\{\frac{1}{p}\|u(t)\|^p_{L^p_x(0,\alpha)}\right\}. \qquad (3.9)$$

By combining (3.6), (3.7), (3.11) and (3.12) into (3.5), we obtain

$$\begin{aligned}
& \frac{d}{dt}\left\{\frac{1}{2}\|u_t(t)\|^2_{L^2_x(0,\alpha)} + \frac{1}{2}\left(\xi_0 - \int_0^t g(s)\,ds\right)\|u_x(t)\|^2_{L^2_x(0,\alpha)}\right. \\
& \left. +\frac{\xi_1}{4}\|u_x(t)\|^4_{L^2_x(0,\alpha)} + \frac{1}{2}(g \circ u_x)(t) - \frac{1}{p}\|u(t)\|^p_{L^p_x(0,\alpha)}\right\} \\
= & \frac{1}{2}\left(g' \circ u_x\right)(t) - \frac{1}{2}g(t)\|u_x(t)\|^2_{L^2_x(0,\alpha)} \\
& -\|u_t(t)\|^2_{L^2_x(0,\alpha)} - \frac{\sigma}{4}\left(\frac{d}{dt}\left\{\|u_x(t)\|^2_{L^2_x(0,\alpha)}\right\}\right)^2, \qquad (3.10)
\end{aligned}$$

using (3.2) into (3.14), we get (3.4). □

Remark 2. *After integration* (3.4) *over* $(0,t)$, *we have*

$$\begin{aligned}
E(t) = & E(0) + \frac{1}{2}\int_0^t (g' \circ u_x)(s)ds - \frac{1}{2}\int_0^t g(s)\|u_x(s)\|^2_{L^2_x(0,\alpha)} ds \\
& -\int_0^t \|u_s(s)\|^2_{L^2_x(0,\alpha)} ds - \frac{\sigma}{4}\int_0^t \left(\frac{d}{dt}\left\{\|u_x(s)\|^2_{L^2_x(0,\alpha)}\right\}\right)^2 ds. \qquad (3.11)
\end{aligned}$$

Now, let (u, v) be a solution of $(1.1)-(1.2)$ and define

$$\begin{aligned} H(t) \quad : &= \|u(t)\|^2_{L^2_x(0,\alpha)} + \int_0^t \|u(s)\|^2_{L^2_x(0,\alpha)} ds \\ &+ \frac{\sigma}{2}\left\{\int_0^t \|u_x(s)\|^4_{L^2_x(0,\alpha)} ds + (T-t)\|u_x(0)\|^4_{L^2_x(0,\alpha)}\right\}. \end{aligned} \tag{3.12}$$

Lemma 4. *Assume that* $(\mathbf{A})$ *hold, then we have*

$$\begin{aligned} & H''(t) - p\|u_t(t)\|^2_{L^2_x(0,\alpha)} \\ \geq \quad & 2p\left\{-E(0) + \int_0^t \|u_s(s)\|^2_{L^2_x(0,\alpha)} ds + \frac{\sigma}{4}\int_0^t \left(\frac{d}{ds}\left\{\|u_x(s)\|^2_{L^2_x(0,\alpha)}\right\}\right)^2 ds\right\}. \end{aligned} \tag{3.13}$$

Proof. From (3.16), we have

$$\begin{aligned} H'(t) \quad &= \quad 2\int_0^\alpha xu(t)\,u_t(t)\,dx + \|u(t)\|^2_{L^2_x(0,\alpha)} \\ &+ \frac{\sigma}{2}\int_0^t \frac{d}{ds}\left\{\|u_x(s)\|^4_{L^2_x(0,\alpha)}\right\} ds, \end{aligned} \tag{3.14}$$

and, we have

$$\begin{aligned} H''(t) \quad &= \quad 2\int_0^\alpha xu_t^2(t)\,dx + 2\int_0^\alpha xu(t)\,u_{tt}(t)\,dx + 2\int_0^\alpha xu(t)\,u_t(t)\,dx \\ &+ \frac{\sigma}{2}\frac{d}{dt}\left\{\|u_x(t)\|^4_{L^2_x(0,\alpha)}\right\}. \end{aligned} \tag{3.15}$$

Using Eqs. $(1.1)-(1.2)$, we get

$$\begin{aligned} & 2\int_0^\alpha xu(t)\,u_{tt}(t)\,dx \\ = \quad & -2\xi_0\|u_x(t)\|^2_{L^2_x(0,\alpha)} - 2\xi_1\|u_x(t)\|^4_{L^2_x(0,\alpha)} - \frac{\sigma}{2}\frac{d}{dt}\left\{\|u_x(t)\|^4_{L^2_x(0,\alpha)}\right\} \\ & + 2\int_0^\alpha\int_0^t xg(t-s)\,u_x(x,t)u_x(x,s)ds dx \\ & - 2\int_0^\alpha xu(t)\,u_t(t)\,dx + 2\|u(t)\|^p_{L^p_x(0,\alpha)}. \end{aligned} \tag{3.16}$$

By replacement (3.20) into (3.19) , we get

$$\begin{aligned} H''(t) &= 2\|u_t(t)\|^2_{L^2_x(0,\alpha)} - 2\xi_0 \|u_x(t)\|^2_{L^2_x(0,\alpha)} - 2\xi_1 \|u_x(t)\|^4_{L^2_x(0,\alpha)} \\ &+2\int_0^\alpha \int_0^t xg(t-s)\, u_x(x,t)u_x(x,s)dsdx + 2\|u(t)\|^p_{L^p_x(0,\alpha)} . \end{aligned} \tag{3.17}$$

And multiplying Eqs. (3.15) by $2p$ summing up in (3.21), we obtain

$$\begin{aligned} & H''(t) - (2+p)\|u_t(t)\|^2_{L^2_x(0,\alpha)} \\ = \;& 2p\left\{-E(0) + \int_0^t \|u_s(s)\|^2_{L^2_x(0,\alpha)} ds + \frac{\sigma}{4}\int_0^t \left(\frac{d}{ds}\left\{\|u_x(s)\|^2_{L^2_x(0,\alpha)}\right\}\right)^2 ds\right\} \\ & + \left\{\xi_0(p-2) - p\left(\int_0^t g(s)\,ds\right)\right\}\|u_x(t)\|^2_{L^2_x(0,\alpha)} + \frac{\xi_1(p-4)}{2}\|u_x(t)\|^4_{L^2_x(0,\alpha)} \\ & +2\int_0^\alpha\int_0^t xg(t-s)\,u_x(x,t)u_x(x,s)dsdx + p(g\circ u_x)(t) \\ & -p\int_0^t (g'\circ u_x)(s)ds + p\int_0^t g(s)\|u_x(s)\|^2_{L^2_x(0,\alpha)} ds. \end{aligned}$$

And using (**A**), we get

$$\begin{aligned} & H''(t) - p\|u_t(t)\|^2_{L^2_x(0,\alpha)} \\ \geq \;& 2p\left\{-E(0) + \int_0^t \|u_s(s)\|^2_{L^2_x(0,\alpha)} ds + \frac{\sigma}{4}\int_0^t \left(\frac{d}{ds}\left\{\|u_x(s)\|^2_{L^2_x(0,\alpha)}\right\}\right)^2 ds\right\} \\ & + \left\{\xi_0(p-2) - p\left(\int_0^t g(s)\,ds\right)\right\}\|u_x(t)\|^2_{L^2_x(0,\alpha)} + \frac{\xi_1(p-4)}{2}\|u_x(t)\|^4_{L^2_x(0,\alpha)} \\ & +2\int_0^\alpha\int_0^t xg(t-s)\,u_x(x,t)u_x(x,s)dsdx + p(g\circ u_x)(t). \end{aligned} \tag{3.18}$$

Using Young's inequality (for $\varepsilon = p$) , we have

$$\begin{aligned} & 2\int_0^t\int_0^\alpha xg(t-s)\,u_x(x,s)\,u_x(x,t)\,dxds \\ \geq \;& \left(2-\frac{1}{p}\right)\left(\int_0^t g(s)\,ds\right)\|u_x(t)\|^2_{L^2_x(0,\alpha)} - p(g\circ u_x)(t). \end{aligned} \tag{3.19}$$

Using (3.24) into (3.23), we get

$$\begin{aligned}
& H''(t) - p\|u_t(t)\|^2_{L^2_x(0,\alpha)} \\
\geq\ & 2p\left\{-E(0) + \int_0^t \|u_s(s)\|^2_{L^2_x(0,\alpha)}\,ds + \frac{\sigma}{4}\int_0^t \left(\frac{d}{ds}\left\{\|u_x(s)\|^2_{L^2_x(0,\alpha)}\right\}\right)^2 ds\right\} \\
& + \left\{\xi_0(p-2) - \frac{(p-1)^2}{p}\left(\int_0^t g(s)\,ds\right)\right\}\|u_x(t)\|^2_{L^2_x(0,\alpha)} \\
& + \frac{\xi_1(p-4)}{2}\|u_x(t)\|^4_{L^2_x(0,\alpha)},
\end{aligned}$$

using $(\mathbf{A})$, we get

$$\xi_0(p-2) - \frac{(p-1)^2}{p}\left(\int_0^t g(s)\,ds\right) > 0,$$

and for $p > 4$, we get (3.17). Then the proof is complete. □

Lemma 5. *Assume that* $(\mathbf{A})$ *hold and that either one of the following condition is satisfied*

$(\mathbf{i})$ $E(0) < 0$,

$(\mathbf{ii})$ $E(0) = 0$, *and*

$$H'(0) > \|u_0\|^2_{L^2_x(0,\alpha)}. \tag{3.20}$$

$(\mathbf{iii})$ $E(0) > 0$ *and*

$$H'(0) > r_2\left[H(0) + \frac{k_0}{p}\right] + \|u_0\|^2_{L^2_x(0,\alpha)}, \tag{3.21}$$

where

$$r_2 := \frac{p - \sqrt{p(p-4)}}{2},$$

and

$$k_0 := 2pE(0) + p\|u_0\|^2_{L^2_x(0,\alpha)}. \tag{3.22}$$

Then

$$H'(t) \geq \|u_0\|^2_{L^2_x(0,\alpha)}, \tag{3.23}$$

for $t > t_0$, *and*

$$t^* := \max\left\{0, \frac{H'(0) - \|u_0\|^2_{L^2_x(0,\alpha)}}{2pE(0)}\right\}, \tag{3.24}$$

where $t_0 = t^*$*in case* (**i**) *and* $t_0 = 0$ *in case* (**ii**) *and* (**iii**).

Proof. (**i**) If $E(0) < 0$, then

$$-2(p+q+2)E(0) > 0,$$

using (3.17), we get

$$H''(t) \geq -2pE(0), \tag{3.25}$$

by integration (3.30) for $(0,t)$, we get

$$\int_0^t H''(s)\,ds \geq \int_0^t -2pE(0)\,ds,$$

then

$$H'(t) - H'(0) \geq -2pE(0)\,t, \tag{3.26}$$

then (3.31) is write en forme

$$\begin{aligned} & H'(t) \\ \geq\ & \|u_0\|^2_{L^2_x(0,\alpha)} + \left\{-2pE(0)\,t + H'(0) - \|u_0\|^2_{L^2_x(0,\alpha)}\right\}. \end{aligned} \tag{3.27}$$

Let

$$t \geq \frac{H'(0) - \|u_0\|^2_{L^2_x(0,\alpha)}}{2pE(0)}, \tag{3.28}$$

and using (3.33) into (3.32), and using $t \geq 0$, then for $t \geq t^*$, of t^* definite in (3.29), we get (3.28).

(**ii**) If $E(0) = 0$, then by using (3.17), we get for all $t \geq 0$

$$H''(t) \geq 0, \tag{3.29}$$

and by integration (3.34) for $(0,t)$, we get

$$\int_0^t H''(s)\,ds \geq 0,$$

then

$$H'(t) - H'(0) \geq 0, \tag{3.30}$$

then (3.35) is writen in form

$$H'(t) \geq \|u_0\|^2_{L^2_x(0,\alpha)} + \left\{H'(0) - \|u_0\|^2_{L^2_x(0,\alpha)}\right\}.$$

Furthermore, if (3.25) hold, then for $t \geq 0$ we get (3.28).

(**iii**) For the case that $E(0) > 0$, we first note that

$$\|u(t)\|^2_{L^2_x(0,\alpha)} = 2\int_0^t (u(s), u_s(s))_{L^2_x(0,\alpha)}\, ds + \|u_0\|^2_{L^2_x(0,\alpha)}. \tag{3.31}$$

Using Young's inequality (for $\varepsilon = 1$) in (3.36), we have

$$\begin{aligned} &\|u(t)\|^2_{L^2_x(0,\alpha)} \\ = \;& 2\int_0^t \int_0^\alpha \left[\sqrt{x}u(s)\right]\left[\sqrt{x}u_s(s)\right] dxds + \|u_0\|^2_{L^2_x(0,\alpha)} \\ \leq \;& \int_0^t \|u(s)\|^2_{L^2_x(0,\alpha)}\, ds + \int_0^t \|u_s(s)\|^2_{L^2_x(0,\alpha)}\, ds + \|u_0\|^2_{L^2_x(0,\alpha)}. \end{aligned} \tag{3.32}$$

Using Young's inequality (for $\varepsilon = 1$), we have

$$2(u(t), u_t(t))_{L^2_x(0,\alpha)} \leq \|u(t)\|^2_{L^2_x(0,\alpha)} + \|u_t(t)\|^2_{L^2_x(0,\alpha)}. \tag{3.33}$$

Using (3.37) and (3.38) into (3.18) we get

$$\begin{aligned} H'(t) \leq \;& \|u(t)\|^2_{L^2_x(0,\alpha)} + \int_0^t \|u(s)\|^2_{L^2_x(0,\alpha)}\, ds \\ &+ \frac{\sigma}{2}\left\{\int_0^t \|u_x(s)\|^4_{L^2_x(0,\alpha)}\, ds + (T-t)\|u_x(0)\|^4_{L^2_x(0,\alpha)}\right\} \\ &+ \|u_t(t)\|^2_{L^2_x(0,\alpha)} + \int_0^t \|u_s(s)\|^2_{L^2_x(0,\alpha)}\, ds + \|u_0\|^2_{L^2_x(0,\alpha)} \\ &+ \frac{\sigma}{2}\int_0^t \frac{d}{ds}\left\{\|u_x(s)\|^4_{L^2_x(0,\alpha)}\right\} ds \\ &- \frac{\sigma}{2}\left\{\int_0^t \|u_x(s)\|^4_{L^2_x(0,\alpha)}\, ds + (T-t)\|u_x(0)\|^4_{L^2_x(0,\alpha)}\right\}. \end{aligned} \tag{3.34}$$

And using (3.16) into (3.39), we get

$$
\begin{aligned}
H'(t) &\leq H(t) + \|u_t(t)\|^2_{L^2_x(0,\alpha)} + \int_0^t \|u_s(s)\|^2_{L^2_x(0,\alpha)} ds + \|u_0\|^2_{L^2_x(0,\alpha)} \\
&\quad + \frac{\sigma}{2}\int_0^t \frac{d}{ds}\left\{\|u_x(s)\|^4_{L^2_x(0,\alpha)}\right\} ds - \frac{\sigma}{2}\int_0^t \|u_x(s)\|^4_{L^2_x(0,\alpha)} ds.
\end{aligned}
$$

Then

$$
\begin{aligned}
& p\left\{H'(t) - H(t)\right\} \\
\leq\ & p\|u_t(t)\|^2_{L^2_x(0,\alpha)} + p\int_0^t \|u_s(s)\|^2_{L^2_x(0,\alpha)} ds + p\|u_0\|^2_{L^2_x(0,\alpha)} \\
& + \frac{p\sigma}{2}\int_0^t \frac{d}{ds}\left\{\|u_x(s)\|^4_{L^2_x(0,\alpha)}\right\} ds - \frac{p\sigma}{2}\int_0^t \|u_x(s)\|^4_{L^2_x(0,\alpha)} ds, \quad (3.35)
\end{aligned}
$$

and using (3.17) into (3.40), we have

$$
\begin{aligned}
& p\left\{H'(t) - H(t)\right\} \\
\leq\ & H''(t) + 2pE(0) - \frac{p\sigma}{2}\int_0^t \left(\frac{d}{dt}\left\{\|u_x(s)\|^2_{L^2_x(0,\alpha)}\right\}\right)^2 ds \\
& - p\int_0^t \|u_s(s)\|^2_{L^2_x(0,\alpha)} ds \\
& + p\|u_0\|^2_{L^2_x(0,\alpha)} + \frac{p\sigma}{2}\int_0^t \frac{d}{ds}\left\{\|u_x(s)\|^4_{L^2_x(0,\alpha)}\right\} ds \\
& - \frac{p\sigma}{2}\int_0^t \|u_x(s)\|^4_{L^2_x(0,\alpha)} ds. \quad (3.36)
\end{aligned}
$$

Using

$$
\begin{aligned}
& -\frac{p\sigma}{2}\int_0^t \left(\frac{d}{dt}\left\{\|u_x(s)\|^2_{L^2_x(0,\alpha)}\right\}\right)^2 ds + \frac{p\sigma}{2}\int_0^t \frac{d}{ds}\left\{\|u_x(s)\|^4_{L^2_x(0,\alpha)}\right\} ds \\
& -\frac{p\sigma}{2}\int_0^t \|u_x(s)\|^4_{L^2_x(0,\alpha)} ds \\
=\ & -\frac{p\sigma}{2}\int_0^t \left(\frac{d}{dt}\left\{\|u_x(s)\|^2_{L^2_x(0,\alpha)}\right\} - \|u_x(s)\|^2_{L^2_x(0,\alpha)}\right)^2 ds \\
\leq\ & 0,
\end{aligned}
$$

into (3.41), we get

$$p\left\{H'(t)-H(t)\right\} \leq H''(t)+2pE(0)+p\|u_0\|^2_{L^2_x(0,\alpha)},$$

then

$$H''(t)-pH'(t)+pH(t)+k_0 \geq 0,$$

where k_0 is definite in (3.27). Let

$$B(t) := H(t)+\frac{k_0}{p},$$

then $B'(t)=H'(t)$, $B''(t)=H''(t)$, and $B(t)$ satisfy

$$B''(t)-pB'(t)+pB(t) \geq 0. \tag{3.37}$$

Using **Lemma 1** in (3.42) and (3.26), then

$$H'(t) \geq \|u_0\|^2_{L^2_x(0,\alpha)}, \quad t \geq 0.$$

Then the proof of **Lemma 5** is completed. □

Theorem 1. *Assume that* $(\mathbf{A})$ *holds and that either one of the following conditions is satisfied*

$(\mathbf{i})$ $E(0)<0$,

$(\mathbf{ii})$ $E(0)=0$ *and* (3.25) *holds,*

$(\mathbf{iii})$ $0 < E(0) < \dfrac{\left(\frac{p-4}{4}\right)^2\left[H'(t_0)-\|u_0\|^2_{L^2_x(0,\alpha)}\right]^2 \times J(t_0)^{\frac{1}{\gamma}}}{\left((p-2)^2-4\right)\left(\frac{1}{2}-\frac{1}{p-2}\right)}$, *and* (3.26) *holds.*

Then the solution u *blows-up at finite time* T^* *in the sense of* (3.1).

In case $(\mathbf{i})$,

$$T^* \leq t_0-\frac{J(t_0)}{J'(t_0)}.$$

Furthermore, if $J(t_0) < \min\left\{1, \sqrt{\frac{\mu}{-\beta}}\right\}$, *then we have*

$$T^* \leq t_0+\frac{1}{\sqrt{-\beta}}\ln\frac{\sqrt{\frac{\mu}{-\beta}}}{\sqrt{\frac{\mu}{-\beta}}-J(t_0)}.$$

In case (**ii**)

$$T^* \leq t_0 - \frac{J(t_0)}{J'(t_0)},$$

or

$$T^* \leq t_0 + \frac{J(t_0)}{J'(t_0)}.$$

In case (**iii**)

$$T^* \leq \frac{J(t_0)}{\sqrt{\alpha}},$$

or

$$T^* \leq t_0 + 3^{\frac{3\gamma+1}{2\gamma}} \frac{\gamma c}{\sqrt{\mu}} \left\{ 1 - \left[1 + cJ(t_0)\right]^{\frac{1}{2\gamma}} \right\},$$

where $c := \left(\frac{\beta}{\mu}\right)^{\frac{\gamma}{2+\gamma}}$, $\gamma := \frac{p-4}{4}$, *and* $J(t)$, μ *and* β *are given in* (3.43), (3.48) *and* (3.49) *respectively.*
*Note that in **case*** (**i**), $t_0 = t^*$ *is given in* (3.29) *and* $t_0 = 0$ *in **case*** (**ii**) *and* (**iii**).

Proof. Let

$$J(t) := \left\{ H(t) + (T-t) \|u_0\|^2_{L^2_x(0,\alpha)} \right\}^{-\gamma}. \tag{3.38}$$

Differentiating $J(t)$ twice, we obtain

$$J'(t) = -\gamma J(t)^{1+\frac{1}{\gamma}} \left[H'(t) - \|u_0\|^2_{L^2_x(0,\alpha)} \right].$$

Then

$$J''(t) = -\gamma J(t)^{1+\frac{2}{\gamma}} Q(t), \tag{3.39}$$

where

$$\begin{aligned} Q(t) \; : \; &= H''(t) \left[H(t) + (T-t) \|u_0\|^2_{L^2_x(0,\alpha)} \right] \\ &- (1+\gamma) \left\{ H'(t) - \|u_0\|^2_{L^2_x(0,\alpha)} \right\}^2. \end{aligned}$$

From (3.17), we have

$$\begin{aligned} & H''(t) \\ \geq\ & -2pE(0)+p\left\{\frac{\sigma}{2}\int_0^t\left(\frac{d}{dt}\left\{\|u_x(s)\|^2_{L^2_x(0,\alpha)}\right\}\right)^2 ds\right. \\ & \left.+\|u_t(t)\|^2_{L^2_x(0,\alpha)}+\int_0^t\|u_s(s)\|^2_{L^2_x(0,\alpha)}\,ds\right\}. \end{aligned}$$

From (3.18), we get

$$\begin{aligned} & H'(t)-\|u_0\|^2_{L^2_x(0,\alpha)} \\ =\ & 2\left(u(t),u_t(t)\right)_{L^2_x(0,\alpha)}+\|u(t)\|^2_{L^2_x(0,\alpha)}-\|u_0\|^2_{L^2_x(0,\alpha)} \\ & +\frac{\sigma}{2}\int_0^t\frac{d}{ds}\left\{\|u_x(s)\|^4_{L^2_x(0,\alpha)}\right\}ds. \end{aligned} \tag{3.40}$$

Using

$$\|u(t)\|^2_{L^2_x(0,\alpha)}-\|u_0\|^2_{L^2_x(0,\alpha)}=2\int_0^t\left(u(s),u_s(s)\right)_{L^2_x(0,\alpha)}ds,$$

we get

$$\begin{aligned} & H'(t)-\|u_0\|^2_{L^2_x(0,\alpha)} \\ =\ & 2\left(u(t),u_t(t)\right)_{L^2_x(0,\alpha)}+2\int_0^t\left(u(s),u_s(s)\right)_{L^2_x(0,\alpha)}ds \\ & +\frac{\sigma}{2}\int_0^t\frac{d}{ds}\left\{\|u_x(s)\|^4_{L^2_x(0,\alpha)}\right\}ds. \end{aligned} \tag{3.41}$$

Clearly, we can write

$$\begin{aligned} & H''(t)\left[H(t)+(T-t)\|u_0\|^2_{L^2_x(0,\alpha)}\right] \\ \geq\ & -2pE(0)\left[H(t)+(T-t)\|u_0\|^2_{L^2_x(0,\alpha)}\right] \\ & +p\left\{\frac{\sigma}{2}\int_0^t\left(\frac{d}{dt}\left\{\|u_x(s)\|^2_{L^2_x(0,\alpha)}\right\}\right)^2 ds+\|u_t(t)\|^2_{L^2_x(0,\alpha)}+\int_0^t\|u_s(s)\|^2_{L^2_x(0,\alpha)}\,ds\right\} \\ & \times\left[\|u(t)\|^2_{L^2_x(0,\alpha)}+\int_0^t\|u(s)\|^2_{L^2_x(0,\alpha)}\,ds\right. \\ & \left.+\frac{\sigma}{2}\left\{\int_0^t\|u_x(s)\|^4_{L^2_x(0,\alpha)}\,ds+(T-t)\|u_x(0)\|^4_{L^2_x(0,\alpha)}\right\}\right], \end{aligned} \tag{3.42}$$

and

$$\begin{aligned} & -(1+\gamma)\left\{H'(t)-\|u_0\|^2_{L^2_x(0,\alpha)}\right\}^2 \\ = & -4(1+\gamma)\left\{(u(t),u_t(t))_{L^2_x(0,\alpha)}+\int_0^t (u(s),u_s(s))_{L^2_x(0,\alpha)}\,ds\right. \\ & \left.+\frac{\sigma}{4}\int_0^t \frac{d}{ds}\left\{\|u_x(s)\|^4_{L^2_x(0,\alpha)}\right\}ds\right\}^2, \end{aligned}$$

then

$$\begin{aligned} Q(t) \geq & -2pE(0)\left[H(t)+(T-t)\|u_0\|^2_{L^2_x(0,\alpha)}\right] \\ & +p\left\{\|u_t(t)\|^2_{L^2_x(0,\alpha)}+\int_0^t \|u_s(s)\|^2_{L^2_x(0,\alpha)}\,ds+\frac{\sigma}{2}\int_0^t\left(\frac{d}{dt}\left\{\|u_x(s)\|^2_{L^2_x(0,\alpha)}\right\}\right)^2 ds\right\} \\ & \times\left\{\|u(t)\|^2_{L^2_x(0,\alpha)}+\int_0^t \|u(s)\|^2_{L^2_x(0,\alpha)}\,ds+\frac{\sigma}{2}\int_0^t \|u_x(s)\|^4_{L^2_x(0,\alpha)}\,ds\right\} \\ & -4(1+\gamma)\left\{(u(t),u_t(t))_{L^2_x(0,\alpha)}+\int_0^t (u(s),u_s(s))_{L^2_x(0,\alpha)}\,ds\right. \\ & \left.+\frac{\sigma}{4}\int_0^t \frac{d}{ds}\left\{\|u_x(s)\|^4_{L^2_x(0,\alpha)}\right\}ds\right\}^2. \end{aligned}$$

Let us designate by

$$\begin{aligned} \mathbf{A} &: = \|u(t)\|^2_{L^2_x(0,\alpha)}+\int_0^t \|u(s)\|^2_{L^2_x(0,\alpha)}\,ds+\frac{\sigma}{2}\int_0^t \|u_x(s)\|^4_{L^2_x(0,\alpha)}\,ds, \\ \mathbf{B} &: = (u(t),u_t(t))_{L^2_x(0,\alpha)}+\int_0^t (u(s),u_s(s))_{L^2_x(0,\alpha)}\,ds+\frac{\sigma}{4}\int_0^t \frac{d}{ds}\left\{\|u_x(s)\|^4_{L^2_x(0,\alpha)}\right\}ds, \\ \mathbf{C} &: = \|u_t(t)\|^2_{L^2_x(0,\alpha)}+\int_0^t \|u_s(s)\|^2_{L^2_x(0,\alpha)}\,ds+\frac{\sigma}{2}\int_0^t\left(\frac{d}{dt}\left\{\|u_x(s)\|^2_{L^2_x(0,\alpha)}\right\}\right)^2 ds. \end{aligned}$$

Thus, we get

$$Q(t) \geq -2pE(0)\,J(t)^{-\frac{1}{\gamma}}+p\left\{\mathbf{AC}-\mathbf{B}^2\right\}. \tag{3.43}$$

Now we observe that, for all $(\rho,\eta)\in\mathbb{R}^2$ and $t>0$,

$$\begin{aligned}
&\mathbf{A}\rho^2+2\mathbf{B}\rho\eta+\mathbf{C}\eta^2\\
=\;&\rho^2\left\|u(t)\right\|^2_{L^2_x(0,\alpha)}+\rho^2\int_0^t\left\|u(s)\right\|^2_{L^2_x(0,\alpha)}ds+\frac{\sigma}{2}\rho^2\int_0^t\left\|u_x(s)\right\|^4_{L^2_x(0,\alpha)}ds\\
&+2\rho\eta(u,u_t)_H+2\rho\eta\int_0^t\left(u(s),u_s(s)\right)_{L^2_x(0,\alpha)}ds+\frac{2\rho\eta\sigma}{4}\int_0^t\frac{d}{ds}\left\{\left\|u_x(s)\right\|^4_{L^2_x(0,\alpha)}\right\}ds\\
&+\eta^2\left\|u_t(t)\right\|^2_{L^2_x(0,\alpha)}+\eta^2\int_0^t\left\|u_s(s)\right\|^2_{L^2_x(0,\alpha)}ds+\frac{\eta^2\sigma}{2}\int_0^t\left(\frac{d}{dt}\left\{\left\|u_x(s)\right\|^2_{L^2_x(0,\alpha)}\right\}\right)^2ds.
\end{aligned}$$

Then

$$\begin{aligned}
&\mathbf{A}\rho^2+2\mathbf{B}\rho\eta+\mathbf{C}\eta^2\\
=\;&\int_0^\alpha x\left(\rho^2u^2(t)+2\rho u(t)\,\eta u_t(t)+\eta^2u_t^2(t)\right)dx\\
&+\int_0^t\int_0^\alpha x\left(\rho^2u^2(s)+2\rho u(s)\,\eta u_s(s)+\eta^2u_s^2(s)\right)dxds\\
&+\frac{\sigma}{2}\int_0^t\Bigg\{\rho^2\left\|u_x(s)\right\|^4_{L^2_x(0,\alpha)}+2\left(\rho\left\|u_x(s)\right\|^2_{L^2_x(0,\alpha)},\eta\frac{d}{ds}\left\{\left\|u_x(s)\right\|^2_{L^2_x(0,\alpha)}\right\}\right)_{L^2_x(0,\alpha)}\\
&+\eta^2\left(\frac{d}{dt}\left\{\left\|u_x(s)\right\|^2_{L^2_x(0,\alpha)}\right\}\right)^2\Bigg\}ds.
\end{aligned}$$

This identify camber written in the form

$$\begin{aligned}
&\mathbf{A}\rho^2+2\mathbf{B}\rho\eta+\mathbf{C}\eta^2\\
=\;&\int_0^\alpha x\left(\rho u(t)+\eta u_t(t)\right)^2dx+\int_0^t\int_0^\alpha x\left(\rho u(s)+\eta u_s(s)\right)^2dxds\\
&+\frac{\sigma}{2}\int_0^t\left\{\rho\left\|u_x(s)\right\|^2_{L^2_x(0,\alpha)}+\eta\left(\frac{d}{dt}\left\{\left\|u_x(s)\right\|^2_{L^2_x(0,\alpha)}\right\}\right)\right\}^2ds,
\end{aligned}$$

it is easy to see that

$$\mathbf{A}\rho^2+2\mathbf{B}\rho\eta+\mathbf{C}\eta^2\geq 0,$$

and

$$\mathbf{B}^2-\mathbf{AC}\leq 0.$$

Hence, we obtain from (3.45) that

$$Q(t)\geq -2pE(0)\,J(t)^{-\frac{1}{\gamma}},\quad t\geq t_0. \tag{3.44}$$

Therefore by using (3.46) into (3.44), we get

$$J''(t) \leq \left(\frac{(p-2)^2}{2} - 2\right) E(0)\, J(t)^{1+\frac{1}{\gamma}}, \quad t \geq t_0. \tag{3.45}$$

Note that by **Lemma 5** $J'(t) < 0$ for $t \geq t_0$. Multiplying (3.47) by $J'(t)$ and integrating it from t_0 to t, we have

$$J'(t)^2 \geq \mu + \beta J(t)^{2+\frac{1}{\gamma}},$$

where

$$\begin{aligned} \mu \quad : \quad &= \left\{ \left(\frac{p-4}{4}\right)^2 \left[H'(t_0) - \|u_0\|^2_{L^2_x(0,\alpha)}\right]^2 \right. \\ &\left. -((p-2)^2 - 4)\left(\frac{1}{2} - \frac{1}{p-2}\right) E(0)\, J(t_0)^{-\frac{1}{\gamma}} \right\} \times J(t_0)^{2+\frac{2}{\gamma}} \\ > \quad & 0, \end{aligned} \tag{3.46}$$

and

$$\beta := \left((p-2)^2 - 4\right)\left(\frac{1}{2} - \frac{1}{p-2}\right) E(0). \tag{3.47}$$

Then by **Lemma 2** the proof of theorem is completed.

Hence there exist a finite time t such that $\lim_{t\to T^{*-}} \{J(t)\} = 0$ and the upper bounds of T^* are estimation respectively according to the sign of $E(0)$ this will imply that (3.1). □

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