Earthline Journal of Mathematical Sciences
E-ISSN: 2581-8147
Volume 14, Number 2, 2024, Pages 243-256
https://doi.org/10.34198/ejms.14224.243256

Fixed Point Theory for (μ, ψ)-Generalized Weakly Reich Contraction Mapping in Partially Ordered Metric Spaces

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Abstract

In this paper we introduce a concept of (μ, ψ)-generalized weakly Reich contraction mapping, and prove a fixed point theorem. Some Corollaries are consequences of the main result.

1 Introduction and Preliminaries

Definition 1.1 ([1], [2], [3]). *Let (X, d) be a metric space. A map $T : X \mapsto X$ is called a weakly contractive mapping if for each $x, y \in X$*

$$d(Tx, Ty) \leq d(x, y) - \psi(d(x, y))$$

where $\psi : [0, \infty) \mapsto [0, \infty)$ is continuous and non-decreasing, $\psi(x) = 0$ if and only if $x = 0$ and $\lim \psi(x) = \infty$.

Remark 1.2. *If we take $\psi(x) = kx$, $0 < k < 1$, in the above definition, then a weakly contractive mapping is called a contraction.*

Received: November 29, 2023; Accepted: January 4, 2024; Published: January 8, 2024
2020 Mathematics Subject Classification: 41A50, 47H10, 54H25.
Keywords and phrases: metric space, fixed point theorem, (μ, ψ)-generalized weakly Reich contraction mapping.

Definition 1.3 ([4]). *Let (X, d) be a metric space. A map $T : X \mapsto X$ is called a f-weakly contractive mapping if for each $x, y \in X$,*

$$d(Tx, Ty) \leq d(fx, fy) - \psi(d(fx, fy))$$

where $f : X \mapsto X$ is a self-mapping, $\psi : [0, \infty) \mapsto [0, \infty)$ is continuous and non-decreasing, $\psi(x) = 0$ if and only if $x = 0$ and $\lim \psi(x) = \infty$.

Remark 1.4. *If we take $\psi(x) = (1 - k)x$, $0 < k < 1$ in the above definition, then a f-weakly contractive mapping is called a f-contraction. Further, if f is the identitiy mapping and $\psi(x) = (1 - k)x$, $0 < k < 1$, then a f-weakly contractive mapping is called a contraction.*

Definition 1.5 ([5]). *Let (X, d) be a metric space. A map $T : X \mapsto X$ is called a generalized f-weakly contractive mapping if for each $x, y \in X$,*

$$d(Tx, Ty) \leq \frac{1}{2}[d(fx, Ty) + d(fy, Tx)] - \psi(d(fx, Ty), d(fy, Tx))$$

where $f : X \mapsto X$ is a self-mapping, $\psi : [0, \infty)^2 \mapsto [0, \infty)$ is a continuous mapping such that $\psi(x, y) = 0$ if and only if $x = y = 0$.

Remark 1.6 ([2]). *If f is the identity mapping in the above definition, then generalized f-weakly contractive mapping is a generalized weakly contractive mapping.*

Definition 1.7 ([6]). *A function $\mu : [0, \infty) \mapsto [0, \infty)$ is called an altering distance function if the following properties are satisfied*

(a) *μ is monotone increasing and continuous;*

(b) *$\mu(t) = 0$ if and only if $t = 0$.*

Definition 1.8 ([7]). *Let (X, d) be a metric space. A map $T : X \mapsto X$ is called a (μ, ψ)-generalized f-weakly contractive mapping if for each $x, y \in X$*

$$\mu(d(Tx, Ty)) \leq \mu\left(\frac{1}{2}[d(fx, Ty) + d(fy, Tx)]\right) - \psi(d(fx, Ty), d(fy, Tx))$$

where $f : X \mapsto X$ is a self-mapping, $\mu : [0,\infty) \mapsto [0,\infty)$ is an altering distance function and $\psi : [0,\infty)^2 \mapsto [0,\infty)$ is a lower semi-continous mapping such that $\psi(x,y) = 0$ if and only if $x = y = 0$.

Remark 1.9. *If f is the identity mapping in the above definition, then a (μ,ψ)-generalized f-weakly contractive mapping is a (μ,ψ)-generalized weakly contractive mapping.*

Definition 1.10 ([8]). *Let M be a nonempty subset of a metric space (X,d). A point $x \in M$ is a common fixed (coincidence) point of f and T if $x = fx = Tx$ ($fx = Tx$).*

Definition 1.11 ([8]). *Let M be a nonempty subset of a metric space (X,d). $T, f : M \mapsto M$ are called commuting if $Tfx = fTx$ for all $x \in M$.*

Definition 1.12 ([8]). *Let M be a nonempty subset of a metric space (X,d). $T, f : M \mapsto M$ are called compatible if $\lim d(Tfx_n, fTx_n) = 0$ whenever $\{x_n\}$ is a sequence such that $\lim Tx_n = \lim fx_n = t$ for some t in M.*

Definition 1.13 ([8]). *Let M be a nonempty subset of a metric space (X,d). $T, f : M \mapsto M$ are called weakly compatible if they commute at their coincidence points, that is, $Tfx = fTx$ whenever $fx = Tx$.*

Definition 1.14 ([8]). *Let $(X,\leq)$ be a partially ordered set and $T, f : X \mapsto X$. A mapping T is said to be monotone f-nondecreasing if for all $x, y \in X$, $fx \leq fy$ implies $Tx \leq Ty$.*

Remark 1.15. *If f is the identity mapping in the above definition, then T is monotone non-decreasing.*

Definition 1.16 ([8]). *A subset W of a partially ordered set X is said to be well-ordered if every two elements of W are comparable.*

2 Main Result

Definition 2.1. *A map $T : X \mapsto X$ will be called (μ, ψ)-generalized weakly Reich contractive if for each $x, y \in X$*

$$\mu(d(Tx, Ty)) \leq \mu\Big(\frac{1}{3}[d(fx, Tx) + d(fy, Ty) + d(fx, fy)]\Big) - \psi(d(fx, Tx), d(fy, Ty), d(fx, fy))$$

where

(a) $\mu : [0, \infty) \mapsto [0, \infty)$ *is an altering distance function;*

(b) $\psi : [0, \infty)^3 \mapsto [0, \infty)$ *is a lower semi-continuous function with* $\psi(x, y, z) = 0$ *if and only if* $x = y = z = 0$.

Theorem 2.2. *Let $(X, \leq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Suppose that T and f are self-mappings on X, $T(X) \subseteq f(X)$, T is a monotone f-nondecreasing mapping and*

$$\mu(d(Tx, Ty)) \leq \mu\Big(\frac{1}{3}[d(fx, Tx) + d(fy, Ty) + d(fx, fy)]\Big) - \psi(d(fx, Tx), d(fy, Ty), d(fx, fy))$$

where

(a) $\mu : [0, \infty) \mapsto [0, \infty)$ *is an altering distance function;*

(b) $\psi : [0, \infty)^3 \mapsto [0, \infty)$ *is a lower semi-continuous function with* $\psi(x, y, z) = 0$ *if and only if* $x = y = z = 0$.

If $\{fx_n\} \subset X$ is a nondecreasing sequence with $f(x_n) \to f(z)$ in $f(X)$, then $f(x_n) \leq f(z)$ and $f(z) \leq f(f(z))$ for every n. Also, suppoe that $f(X)$ is closed. If there exists an $x_0 \in X$ with $f(x_0) \leq T(x_0)$, then T and f have a coincidence *point. Further, if T and f are weakly compatible, then T and f have a common fixed point. Moreover, the set of common fixed points of T and f is well-ordered if and only if T and f have one and only one common fixed point.*

Proof. Let $x_0 \in X$ such that $f(x_0) \leq T(x_0)$. Since $T(X) \subseteq f(X)$ we can choose $x_1 \in X$ so that $fx_1 = Tx_0$. Since $Tx_1 \in f(X)$, there exists $x_2 \in X$ such that $fx_2 = Tx_1$. By induction, we can construct a sequence $\{x_n\} \in X$ such that $fx_{n+1} = Tx_n$, for every $n \geq 0$. Since $f(x_0) \leq T(x_0)$, $Tx_0 = fx_1$, $f(x_0) \leq f(x_1)$, T is monotone f-nondecreasing mapping, $T(x_0) \leq T(x_1)$. Similarly, $f(x_1) \leq f(x_2)$, $T(x_1) \leq T(x_2)$, $f(x_2) \leq f(x_3)$. Continuing, we obtain

$$T(x_0) \leq T(x_1) \leq T(x_2) \leq \cdots \leq T(x_n) \leq T(x_{n+1}) \leq \cdots .$$

We suppose that $d(Tx_n, Tx_{n+1}) > 0$ for all n. If not, then $Tx_{n+1} = Tx_n$ for spme n, $Tx_{n+1} = fx_{n+1}$, that is, T and f have a coincidence point x_{n+1}, and so we have the result. Now we have

$$\begin{aligned}
\mu(d(Tx_{n+1}, Tx_n)) &\leq \mu\left(\frac{1}{3}[d(fx_{n+1}, Tx_{n+1}) + d(fx_n, Tx_n) + d(fx_{n+1}, fx_n)]\right) \\
&\quad - \psi(d(fx_{n+1}, Tx_{n+1}), d(fx_n, Tx_n), d(fx_{n+1}, fx_n)) \\
&\leq \mu\left(\frac{1}{3}[d(Tx_n, Tx_{n+1}) + d(Tx_{n-1}, Tx_n) + d(Tx_n, Tx_{n-1})\right) \\
&\quad - \psi(d(Tx_n, Tx_{n+1}), d(Tx_{n-1}, Tx_n), d(Tx_n, Tx_{n-1})) \\
&\leq \mu\left(\frac{1}{3}[d(Tx_n, Tx_{n+1}) + d(Tx_{n-1}, Tx_n) + d(Tx_n, Tx_{n-1})\right) \\
&\leq \mu\left(\frac{1}{3}[3d(Tx_n, Tx_{n-1})]\right) \\
&= \mu(d(Tx_n, Tx_{n-1})).
\end{aligned}$$

Since μ is a non-decreasing function for all $n = 1, 2, \cdots$, we have $d(Tx_{n+1}, Tx_n) \leq d(Tx_n, Tx_{n-1})$. Thus, $\{d(Tx_{n+1}, Tx_n)\}$ is a monotone decreasing sequence of non-negative real numbers and hence is convergent. Hence there exists $r \geq 0$ such that $d(Tx_{n+1}, Tx_n) \to r$. Now, since

$$\mu(d(Tx_{n+1}, Tx_n)) \leq \mu\left(\frac{1}{3}[d(Tx_n, Tx_{n+1}) + d(Tx_{n-1}, Tx_n) + d(Tx_n, Tx_{n-1})\right) - \psi(d(Tx_n, Tx_{n+1}), d(Tx_{n-1}, Tx_n), d(Tx_n, Tx_{n-1})).$$

If we take limits in the above inequality as $n \to \infty$ we get that

$$\mu(r) \leq \mu\left(\frac{1}{3}(r + r + r)\right) - \psi(r, r, r) = \mu(r) - \psi(r, r, r)$$

which implies that $\psi(r, r, r) \leq 0$. Thus $r = 0$, and hence $\lim_{n\to\infty} d(Tx_{n+1}, Tx_n) = 0$. Now we show that $\{Tx_n\}$ is a Cauchy sequence. If otherwise, then there exist $\epsilon > 0$ for which we can find subsequences $\{Tx_{m(k)}\}$ and $\{Tx_{n(k)}\}$ of $\{Tx_n\}$ with $n(k) > m(k) > k$ such that for every k, $d(Tx_{m(k)}, Tx_{n(k)}) \geq \epsilon$,$d(Tx_{m(k)}, Tx_{n(k)-1}) < \epsilon$. So we have,

$$\begin{aligned}\epsilon &\leq d(Tx_{m(k)}, Tx_{n(k)}) \\ &\leq d(Tx_{m(k)}, Tx_{n(k)-1}) + d(Tx_{n(k)-1}, Tx_{n(k)}) \\ &\leq \epsilon + d(Tx_{n(k)-1}, Tx_{n(k)}).\end{aligned}$$

Letting $n \to \infty$ and using $d(Tx_{n-1}, Tx_n) \to 0$, we have, $\lim d(Tx_{m(k)}, Tx_{n(k)}) = \epsilon = \lim d(Tx_{m(k)}, Tx_{n(k)-1})$. Now we have

$$\begin{aligned}\mu(\epsilon) &\leq \mu(d(Tx_{m(k)}, Tx_{n(k)})) \\ &\leq \mu\left(\frac{1}{3}[d(fx_{m(k)}, Tx_{m(k)}) + d(fx_{n(k)}, Tx_{n(k)}) + d(fx_{m(k)}, fx_{n(k)})\right) \\ &\quad - \psi(d(fx_{m(k)}, Tx_{m(k)}), d(fx_{n(k)}, Tx_{n(k)}), d(fx_{m(k)}, fx_{n(k)}))\end{aligned}$$

which implies that

$$\begin{aligned}\mu(\epsilon) &\leq \mu(d(Tx_{m(k)}, Tx_{n(k)})) \\ &\leq \mu\left(\frac{1}{3}[d(Tx_{m(k)-1}, Tx_{m(k)}) + d(Tx_{n(k)-1}, Tx_{n(k)}) + d(Tx_{m(k)-1}, Tx_{n(k)-1})\right) \\ &\quad - \psi(d(Tx_{m(k)-1}, Tx_{m(k)}), d(Tx_{n(k)-1}, Tx_{n(k)}), d(Tx_{m(k)-1}, Tx_{n(k)-1})).\end{aligned}$$

If we take limits in the above inequality and using the fact that $\lim_{n\to\infty} d(Tx_{n+1}, Tx_n) = 0$ we deduce the following

$$\begin{aligned}\mu(\epsilon) &\leq \mu\left(\frac{1}{3}[0+0+\epsilon]\right) - \psi(0,0,\epsilon)\\ &= \mu\left(\frac{1}{3}\epsilon\right) - \psi(0,0,\epsilon)\\ &\leq \mu(\epsilon) - \psi(0,0,\epsilon)\end{aligned}$$

which implies that $\psi(0,0,\epsilon) \leq 0$, which is a contradiction since $\epsilon > 0$. Hence $\{Tx_n\}$ is a Cauchy sequence and therefore is convergent in the complete metric space (X, d). As $f(X)$ is closed and $fx_n = Tx_{n-1}$, $\{fx_n\}$ is also a Cauchy sequence, there is some $z \in X$ such that $\lim fx_{n+1} = \lim Tx_n = fz$. Since $\{fx_n\}$ is a non-decreasing sequence and $\lim fx_{n+1} = fz$, $f(x_n) \leq f(z)$ and $f(z) \leq f(f(z))$ for every n. Now we have

$$\begin{aligned}\mu(d(Tz, fx_{n+1})) &= \mu(d(Tz, Tx_n))\\ &\leq \mu\left(\frac{1}{3}[d(fz,Tz) + d(fx_n, Tx_n) + d(fz, fx_n)]\right)\\ &\quad - \psi(d(fz,Tz), d(fx_n,Tx_n), d(fz,fx_n))\\ &= \mu\left(\frac{1}{3}[d(fz,Tz) + d(fx_n, fx_{n+1}) + d(fz, fx_n)]\right)\\ &\quad - \psi(d(fz,Tz), d(fx_n,fx_{n+1}), d(fz,fx_n)).\end{aligned}$$

Now taking limits as $n \to \infty$ we deduce the following

$$\begin{aligned}\mu(d(Tz, fz)) &\leq \mu\left(\frac{1}{3}d(fz,Tz)\right) - \psi(d(fz,Tz),0,0)\\ &\leq \mu\Big(d(fz,Tz)\Big) - \psi(d(fz,Tz),0,0)\end{aligned}$$

which implies $\psi(d(fz,Tz),0,0) \leq 0$. Hence, $d(fz,Tz) = 0$, thus, $fz = Tz$, and hence z is a coincidence point of T and f. Now suppose that T and f are weakly compatible. Let $w = T(z) = f(z)$, then $T(w) = T(f(z)) = f(T(z)) = f(w)$ and

$f(z) \leq f(f(z)) = f(w)$. Now we have,

$$\begin{aligned}\mu(d(Tz, Tw)) &\leq \mu\left(\frac{1}{3}[d(fz, Tz) + d(fw, Tw) + d(fz, fw)]\right) \\ &\quad - \psi(d(fz, Tz), d(fw, Tw), d(fz, fw)) \\ &= \mu\left(\frac{1}{3}[d(Tz, Tz) + d(Tw, Tw) + d(Tz, Tw)]\right) \\ &\quad - \psi(d(Tz, Tz), d(Tw, Tw), d(Tz, Tw)) \\ &= \mu\left(\frac{1}{3}d(Tz, Tw)\right) - \psi(0, 0, d(Tz, Tw)) \\ &\leq \mu\Big(d(Tz, Tw)\Big) - \psi(0, 0, d(Tz, Tw))\end{aligned}$$

which implies that $\psi(0, 0, d(Tz, Tw)) \leq 0$. Hence, $d(Tz, Tw) = 0$. Therefore, $Tw = fw = w$. Now suppose that the set of common fixed points of T and f is well-ordered. We claim that the common fixed points of T and f is unique. Assume on contrary that, $Tu = fu = u$ and $Tv = fv = v$ but $u \neq v$. Now observe we have the following

$$\begin{aligned}\mu(d(u, v)) &= \mu(d(Tu, Tv)) \\ &\leq \mu\left(\frac{1}{3}[d(fu, Tu) + d(fv, Tv) + d(fu, fv)]\right) \\ &\quad - \psi(d(fu, Tu), d(fv, Tv), d(fu, fv)) \\ &\leq \mu\left(\frac{1}{3}[d(u, u) + d(v, v) + d(u, v)]\right) \\ &\quad - \psi(d(u, u), d(v, v), d(u, v)) \\ &= \mu\left(\frac{1}{3}[0 + 0 + d(u, v)]\right) - \psi(0, 0, d(u, v)) \\ &\leq \mu(d(u, v)) - \psi(0, 0, d(u, v))\end{aligned}$$

which implies that $\psi(0, 0, d(u, v)) \leq 0$. Therefore $d(u, v) = 0$, and hence $u = v$. Conversely, if T and f have only one common fixed point, then the set of common fixed point of T and f being a singleton is well-ordered, and the proof is finished. □

Corollary 2.3. *Let $(X, \leq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Suppose that T is a self-mappings on X, T is a monotone nondecreasing mapping and*

$$\mu(d(Tx,Ty)) \leq \mu\left(\frac{1}{3}[d(x,Tx)+d(y,Ty)+d(x,y)]\right) - \psi(d(x,Tx), d(y,Ty), d(x,y))$$

for all $x, y \in X$ for which $x \geq y$ where

(*a*) $\mu : [0,\infty) \mapsto [0,\infty)$ *is an altering distance function;*

(*b*) $\psi : [0,\infty)^3 \mapsto [0,\infty)$ *is a lower semi-continuous function with $\psi(x,y,z) = 0$ if and only if $x = y = z = 0$.*

If either

(*i*) *$\{x_n\} \subset X$ is a nondecreasing sequence with $x_n \to z$ in X, then $x_n \leq z$ for every n or*

(*ii*) *T is continuous.*

If there exists an $x_0 \in X$ with $x_0 \leq T(x_0)$, then T has a fixed point. Moreover, for arbitrary two points $x, y \in X$, there exists $w \in X$ such that w is comparable with both x and y, then the fixed point of T is unique.

Proof. If (i) holds, then taking f to be the identity mapping in the above theorem, we get the result. If (ii) holds, then proceeding as in the above theorem we can prove that $\{Tx_n\}$ is a Cauchy sequence, $z = \lim x_{n+1} = \lim T(x_n) = T(\lim x_n) = Tz$, and hence T has a fixed point. Let u and v be two fixed points of T such that $u \neq v$. We consider two cases (*a*) If u and v are comparable. We have

$$\begin{aligned}
\mu(d(u,v)) &= \mu(d(Tu,Tv)) \\
&\le \mu\left(\frac{1}{3}[d(u,Tu)+d(v,Tv)+d(u,v)]\right) \\
&\quad - \psi(d(u,Tu),d(v,Tv),d(u,fv)) \\
&\le \mu\left(\frac{1}{3}[d(u,u)+d(v,v)+d(u,v)]\right) \\
&\quad - \psi(d(u,u),d(v,v),d(u,v)) \\
&= \mu\left(\frac{1}{3}[0+0+d(u,v)]\right) - \psi(0,0,d(u,v)) \\
&\le \mu(d(u,v)) - \psi(0,0,d(u,v))
\end{aligned}$$

which implies that $\psi(0,0,d(u,v)) \le 0$. Thus, $d(u,v)=0$, hence $u=v$.

(*b*) If u and v are not comparable. Choose an element $w \in X$ comparable with both of them. Then also $u = T^n u$ is comparable to $T^n w$ for each n. Now we have

$$\begin{aligned}
\mu(d(u,T^n w)) &= \mu(d(T^n u, T^n w)) \\
&= \mu(d(TT^{n-1}u, TT^{n-1}w)) \\
&\le \mu\left(\frac{1}{3}[d(T^{n-1}u,T^n u)+d(T^{n-1}w,T^n w)+d(T^{n-1}u,T^{n-1}w)]\right) \\
&\quad - \psi(d(T^{n-1}u,T^n u),d(T^{n-1}w,T^n w),d(T^{n-1}u,T^{n-1}w)) \\
&\le \mu\left(\frac{1}{3}[d(u,u)+d(T^{n-1}w,T^n w)+d(u,T^{n-1}w)]\right) \\
&\quad - \psi(d(u,u),d(T^{n-1}w,T^n w),d(u,T^{n-1}w))
\end{aligned}$$

$$
\begin{aligned}
&\leq \mu\left(\frac{1}{3}[0 + d(T^{n-1}w, T^n w) + d(u, T^{n-1}w)]\right) \\
&\quad - \psi(0, d(T^{n-1}w, T^n w), d(u, T^{n-1}w)) \\
&\leq \mu\left(\frac{1}{3}[d(T^{n-1}w, u) + d(u, T^n w) + d(u, T^{n-1}w)]\right) \\
&\leq \mu\left(\frac{1}{3}[3d(u, T^{n-1}w)]\right) \\
&= \mu(d(u, T^{n-1}w))
\end{aligned}
$$

and hence we get $d(u, T^n w) \leq d(u, T^{n-1}w)$. This proves that the nonnegative decreasing sequence $\{d(u, T^n w)\}$ is convergent. Let $d(u, T^n w) \to r$. Since,

$$
\begin{aligned}
\mu(d(u, T^n w)) &= \mu(d(T^n u, T^n w)) \\
&\leq \mu\left(\frac{1}{3}[0 + d(T^{n-1}w, T^n w) + d(u, T^{n-1}w)]\right) \\
&\quad - \psi(0, d(T^{n-1}w, T^n w), d(u, T^{n-1}w)).
\end{aligned}
$$

If we take limits in the above inequality as $n \to \infty$ we get that

$$
\begin{aligned}
\mu(r) &\leq \mu\left(\frac{1}{3}(0 + 2r + r)\right) - \psi(0, 2r, r) \\
&= \mu(r) - \psi(0, 2r, r)
\end{aligned}
$$

which implies that $\psi(0, 2r, r) \leq 0$. Thus $r = 0$, and hence $d(u, T^n w) \to 0$.. Analogously, it can be proved that $d(v, T^n w) \to 0$. Since the limit is unique we get that $u = v$.

$\square$

If $\mu(t) = t$, then we have the following result

Corollary 2.4. *Let $(X, \leq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Suppose that T is a self-mappings on X, T is a monotone nondecreasing mapping and*

$$d(Tx, Ty) \leq \frac{1}{3}[d(x, Tx) + d(y, Ty) + d(x, y)] - \psi(d(x, Tx), d(y, Ty), d(x, y))$$

for all $x, y \in X$ for which $x \geq y$ where

(a) *$\mu : [0, \infty) \mapsto [0, \infty)$ is an altering distance function;*

(b) *$\psi : [0, \infty)^3 \mapsto [0, \infty)$ is a lower semi-continuous function with $\psi(x, y, z) = 0$ if and only if $x = y = z = 0$.*

If either

(i) *$\{x_n\} \subset X$ is a nondecreasing sequence with $x_n \to z$ in X, then $x_n \leq z$ for every n or*

(ii) *T is continuous.*

If there exists an $x_0 \in X$ with $x_0 \leq T(x_0)$, then T has a fixed point. Moreover, for arbitrary two points $x, y \in X$, there exists $w \in X$ such that w is comparable with both x and y, then the fixed point of T is unique.

If $\psi(x, y, z) = \left(\frac{1}{3} - k\right)(x + y + z)$, then we have the following result

Corollary 2.5. *Let $(X, \leq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Suppose that T is a non-decreasing self-mapping of X, and T satisfies*

$$d(Tx, Ty) \leq k[d(x, Tx) + d(y, Ty) + d(x, y)]$$

for all $x, y \in X$ for which $x \geq y$ where $0 < k < \frac{1}{3}$. If either (*i*) $\{x_n\} \subset X$ *is a nondecreasing sequence with* $x_n \to z$ *in* X*, then* $x_n \leq z$ *for every* n *or*

(*ii*) T *is continuous.*

If there exists an $x_0 \in X$ *with* $x_0 \leq T(x_0)$*, then* T *has a fixed point.*

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