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On Rayleigh-Ritz and Collocation Methods for Solving Second Order Boundary Value Problems of Ordinary Differential Equations

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Abstract

In this article, the Rayleigh-Ritz method is compared with the collocation method in solving second order boundary value problems of ordinary differential equations with the associated boundary conditions. The trial solution for the Rayleigh-Ritz method has to be chosen in such a way that the linearly independent functions must satisfy the boundary conditions. The collocation method, on the other hand make use of any basis function as the trial solution. The trial solution is then made to satisfy the differential equation and the boundary conditions at some interior points in the solution interval. Results obtained using the two methods show that the collocation method is simpler, easier and more accurate than the Rayleigh-Ritz method.

1 Introduction

The Finite Difference Method (FDM) has been in existence and was eminently used long time ago before the introduction of the Finite Element Method (FEM).

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While FDM provides solution to differential equations at some selected points in the solution domain, the FEM produces approximate solutions that are continuous over the interval. However, both Rayleigh-Ritz and collocation methods produce approximate solutions which are piecewise polynomials. Hence, the two techniques are classified as Finite Element Methods[1]. The Rayleigh-Ritz method is based on the idea of obtaining approximate solution to a differential equation using some arbitrary functions and see whether the functional can be minimized by a suitable choice of the parameters of the approximations [2]. There are two conditions to be met when choosing the trial functions in Rayleigh-Ritz method. The two conditions are that they must be chosen such that the trial solution is satisfied by the boundary conditions and the individual functions in the trial solution should be linearly independent [2, 3]. These two conditions make the use of Rayleigh-Ritz method to be complicated since we have no prior knowledge of the true solution of the problem being examined [4]. The likelihood to guess the functions that will provide a solution to closely resemble the true solution rarely exist. In view of the foregoing, researchers have relied on the use of polynomials as the trial solution for the method [5]. Unfortunately, the condition that the boundary conditions must be met is still a serious challenge [6]. Collocation method on the other hand uses an alternative approach in approximating the true solution of boundary value problems. The interval in which the solution is sought is partitioned into overlapping subintervals [7]. Let the interval be given by

$$a = x_1 < x_2 < \cdots < x_{l+1} = b$$

with

$$\mathrm{h} = \max_{1\leq j\leq l} h_j = \max_{1\leq j\leq l}(x_{j+1} - x_j)$$

and let the piecewise polynomial approximation in $[a, b]$ be

$$y(x) = \sum_{j=1}^{N} \alpha_j B_j(x)$$

The collocation method determines the unknown set $\{\alpha_j | j = 1, 2, ..., N\}$ by satisfying the ordinary differential equation at N points. If the two boundary conditions are satisfied, then the collocation points required are $N-2$. With this idea, the challenge of guessing the trial function as in Rayleigh-Ritz method is circumvented. Rather, any of the known basis functions, $B_j(x)$ such as power series, Legendre polynomials, Chebyshev polynomials and so on can be used as trial solution.

Two-point boundary value problems are usually difficult to solve analytically; as such numerical methods are often applied to solve them [8, 9]. Villadsen and Stewart proposed solution of boundary value problems by orthogonal collocation method [10], extended Adomian decomposition method by Jang [11], homotopy perturbation method proposed by Aminikhah and Hemmatnezhad [12] and shooting method by Burden [13]. Other methods propounded by researchers include Block method [14, 15, 16].

2 Methodology

The second order ordinary differential equation of boundary value problem of the form

$$y'' + py' + qy = f, \quad y(a) = y_a, \quad y(b) = y_b \tag{1}$$

where p, q, f may be constants or variables and y_a, y_b are constants is considered in this work.

2.1 Rayleigh-Ritz Method

In the Rayleigh-Ritz Method, the boundary value problem (1) is solved by approximating the solution using a linear approximation as basis function. The functional for equation (1) is written as

$$I[v] = \int_b^a [(v)^2 - pv^2 - 2qv + 2fv]dx \tag{2}$$

Equation (2) now have only first order derivatives. Since the solution of (2) is not known, then it is approximated by some arbitrary function. This functional is then minimized by a suitable choice of the parameters of the approximations. Let the approximation, $v(x)$ to the solution be of the form

$$v(x) = \sum_{i=0}^{n} a_i v_i(x) \tag{3}$$

For simplicity, ease of manipulation and the criterion of linear independence, $v_i(x)$ are chosen to be polynomials! and a_i are coefficients to be determined. Substituting (3) into the functional of (2), gives

$$I\left[\sum_{i=0}^{n} a_i\right] = \int_b^a \left[\left(\frac{d}{dx}\sum_{i=0}^{n} a_i v_i\right)^2 - p\left(\sum_{i=0}^{n} a_i v_i\right)^2 - 2q\sum_{i=0}^{n} a_i v_i + 2f\sum_{i=0}^{n} a_i v_i\right] dx \tag{4}$$

To minimize I, the partial derivatives with respect to each of the a_i's is set to zero, to obtain

$$\frac{\partial I}{\partial a_i} = 2\int_b^a \left[v'\frac{\partial v'}{\partial a_i} - pv\frac{\partial v}{\partial a_i} - 2q\frac{\partial v}{\partial a_i} + 2f\frac{\partial v}{\partial a_i}\right] dx \tag{5}$$

Equation (5) leads to a system of n by n algebraic linear equations. The system is then solved by Gaussian elimination method to obtain the values of a_i's which are then substituted into (2) to obtain the approximate solution.

2.2 Collocation method

This method uses the residual function, $r(x)$ of equation (1) defined as

$$r(x) = y'' + py' + qy - f \tag{6}$$

Let the approximate solution of (1) be given by

$$y(x) = \sum_{i=0}^{n} a_i T_i(x) \tag{7}$$

where $T_i(x)$ is Chebyshev polynomial of degree i defined in the interval $[a, b]$ as

$$T_{i+1}(x) = CosiCos^{-1}\left(\frac{2x - a - b}{b - a}\right) \tag{8}$$

which satisfies the recursive relation

$$T_{i+1}(x) = 2\left(\frac{2x-a-b}{b-a}\right)T_i(x) - T_{i-1}(x) \tag{9}$$

with the conditions

$$T_0(x) = 1$$
$$T_1(x) = \left(\frac{2x-a-b}{b-a}\right)$$

Substituting (7) into (6) and equating $r(x) = 0$ provides a suitable choice of the coefficients a_i's. In doing this, $r(x)$ is arbitrarily set to zero at a number of points inside the interval.

In this paper, the number of interior points chosen is 2 less than the number of the unknowns in the trial solution. With this, $(n-2)$ linear equations with n unknown constants are obtained. Two extra equations are obtained by ensuring that the trial solution meets the given boundary conditions. Hence, there are n linear equations with n unknowns which are then solved using Gaussian elimination method to obtain the values of the unknown constants. The values of the unknown constants are then substituted into (7) to obtain the approximate solution.

3 Numerical Examples

To compare the approximate solution with the exact solution, the absolute error between the two solutions is defined as

$$E_r = \mid y(x) - y_n(x) \mid, n = 1, 2, 3, ... \tag{10}$$

All computations and programmes are carried out with the aid of Mathematica software.

Example 1: Consider the second order Ordinary Differential Equation

$$y'' + y - 3x^2 = 0, \quad y(0) = 0, \quad y(2) = 3.5. \tag{11}$$

The exact solution is

$$y(x) = 6\cos x + 3(x^2 - 2) - 0.0034301\sin x$$

Using equations (8) and (9), the shifted Chebyshev polynomials for this problem are:

$$T_0(x) = 1$$

$$T_1(x) = x - 1$$

$$T_2(x) = 2x^2 - 4x + 1$$

$$T_3(x) = 4x^3 - 12x^2 + 9x - 1$$

$$T_4(x) = 8x^4 - 32x^3 + 40x^2 - 16x + 1$$

Table 1 shows the comparison between the Rayleigh-Ritz and collocation methods.

Gerald-Wheatley (2004)

Example 2: Consider the second order Ordinary Differential Equation of variable coefficients

$$y'' - \frac{2}{x}y' + \frac{4}{x^2}y = 1, \quad y(10) = 0, \; y(20) = 100. \tag{12}$$

Remark: Closed form solution does not exist.

Using equations (8) and (9), the shifted Chebyshev polynomials for this problem are:

$$T_0(x) = 1$$

$$T_1(x) = \tfrac{1}{5}x - 3$$

$$T_2(x) = \tfrac{2}{25}x^2 - \tfrac{12}{5}x + 17$$

$$T_3(x) = \tfrac{4}{125}x^3 - \tfrac{36}{25}x^2 + 21x - 99$$

$$T_4(x) = \frac{8}{625}x^4 - \frac{106}{125}x^3 + \frac{426}{25}x^2 - \frac{816}{5}x + 577$$

Table 2 shows the comparison between the Rayleigh-Ritz and collocation methods. **Zavalani (2015)**

Example 3: Consider the stiff second order Ordinary Differential Equation of variable coefficients

$$y'' - (x+1)y = -e^{-x}(x^2 - 2x + 2), \quad y(2) = 0,\ y(4) = 0.036631. \tag{13}$$

The exact solution is

$$y(x) = e^{-x}(x-2)$$

Using equations (8) and (9), the shifted Chebyshev polynomials for this problem are:

$$T_0(x) = 1$$

$$T_1(x) = x - 3$$

$$T_2(x) = 2x^2 - 12x + 17$$

$$T_3(x) = 4x^3 - 36x^2 + 105x - 99$$

$$T_4(x) = 8x^4 - 96x^3 + 424x^2 - 816x + 577$$

Table 3 shows the comparison between the Rayleigh-Ritz and collocation methods. **Gerald-Wheatley (2004)**.

4 Tables of Results

Table 1: Numerical Results for Example 1: Comparison between the absolute errors in the Rayleigh-Ritz and Collocation methods.

x	Exact Solution	Chebyshev Collocation Method	Error	Ritz Method [2]	Error	Collocation Method [2]	Error
0.0	0.0000000000	0.00000000	0.00000000	0.0000000	0.00000000	0.0000000	0.00000000
0.2	-0.000281989	0.00761833	7.90032E-3	0.0204737	2.07557E-2	0.0884532	8.87350E-2
0.4	0.0050302200	0.01137040	6.34018E-3	0.0139649	8.93468E-3	0.1281520	1.23122E-1
0.6	0.0300769000	0.03236230	2.28540E-3	0.0180526	1.20243E-2	0.1591740	1.29097E-1
0.8	0.0977797000	0.09677200	1.00770E-3	0.0703158	2.74639E-2	0.2215990	1.23819E-1
1.0	0.2389280000	0.23584900	3.07900E-3	0.2083330	3.05950E-2	0.3555050	1.16577E-1
1.2	0.4909500000	0.48591500	5.03500E-3	0.4696840	2.12660E-2	0.6009700	1.10020E-1
1.4	0.8964230000	0.88836200	8.06100E-3	0.8919470	4.47600E-3	0.9980740	1.01651E-1
1.6	1.5013700000	1.48966000	1.17100E-2	1.5127000	1.11330E-2	1.5868900	8.55200E-2
1.8	2.3534500000	2.34133000	1.21200E-2	2.3695300	1.60800E-2	2.4075100	5.40600E-2
2.0	3.5000000000	3.50000000	0.00000000	3.5000000	0.00000000	3.5000000	0.00000000

Table 2: Numerical Results for Example 2: Comparison between the absolute errors in the Rayleigh-Ritz and Collocation methods.

x	Exact Solution Mathematica	True Solution by [17]	Error	Chebyshev Collocation Method	Error
10	$-1.11022 * 10^{-16}$	-0.01513	1.5130E-2	$1.06581 * 10^{-14}$	1.06692E-14
11	2.36996	0.632086	1.73787	2.33973	3.023E-2
12	6.22043	2.526800	3.69363	6.18761	3.282E-2
13	11.66800	5.927970	5.74003	11.6401	2.790E-2
14	18.80360	11.12550	7.67810	18.7802	2.340E-2
15	27.69750	18.43800	9.25950	27.6774	2.010E-2
16	38.40360	28.21160	10.1920	38.3873	1.630E-2
17	50.96200	40.81900	10.1430	50.9526	9.400E-2
18	65.40200	56.65880	8.74320	65.4018	2.000E-2
19	81.74370	76.15530	5.58840	81.7504	6.700E-2
20	100.0000	99.75790	2.4210E-1	100.0000	0.000000

Table 3: Numerical Results for Example 3: Comparison between the exact solution and the approximate solution by our method for $h = 0.0001$.

x	Exact Solution	Power Series Collocation Method N=7	Error	Chebyshev Collocation Method N=4	Error
2.0	0.0000000	-0.00000986	9.86E-6	0.00000624	6.24E-6
2.2	0.0221606	0.0221302	3.04E-5	0.0217249	4.357E-4
2.4	0.0362872	0.0362667	2.05E-5	0.0359400	3.472E-4
2.6	0.0445641	0.0445480	1.61E-5	0.0443943	1.698E-4
2.8	0.0486481	0.0486357	1.24E-5	0.0485827	6.540E-5
3.0	0.0497871	0.0497802	6.90E-6	0.0497527	3.440E-5
3.2	0.0489146	0.0489098	4.80E-6	0.0489038	1.080E-5
3.4	0.0467226	0.0467163	6.30E-6	0.0467879	6.530E-5
3.6	0.0437180	0.0437188	8.00E-7	0.0439092	1.912E-4
3.8	0.0402674	0.0402904	2.30E-5	0.0405241	2.567E-4
4.0	0.0366313	0.0366287	2.60E-6	0.0366414	1.010E-5

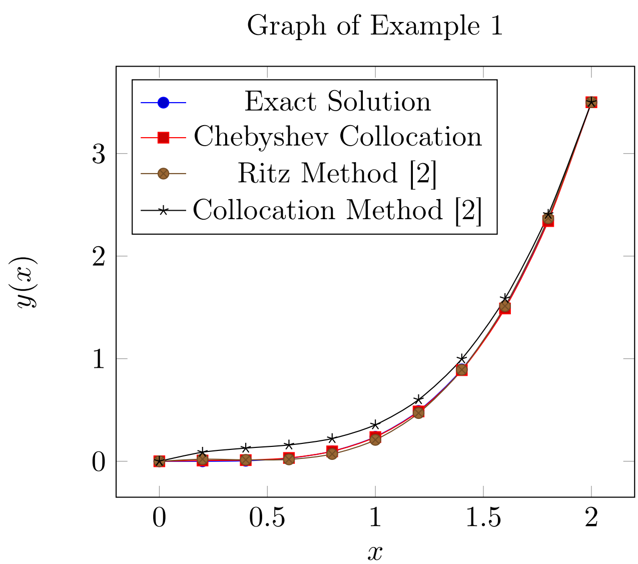

Figure 1: The behaviour of the exact solution compared with the solutions by Ritz method [2], Collocation method [2] and the new method.

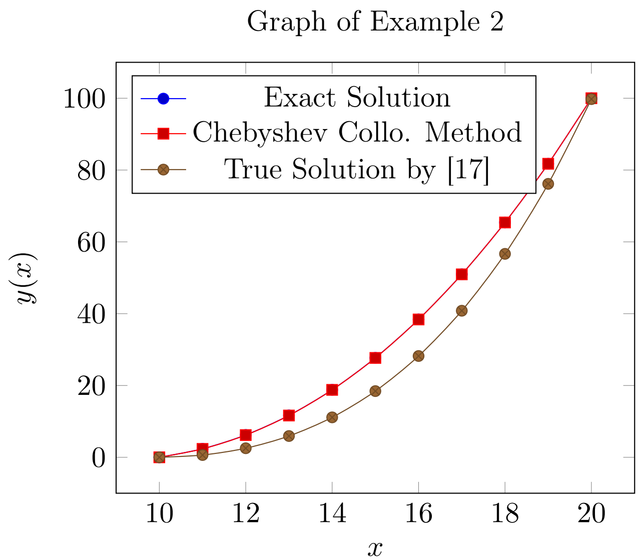

Figure 2: The behaviour of the exact solution compared with the solutions by Chebyshev Collocation method and True solution obtained by [17].

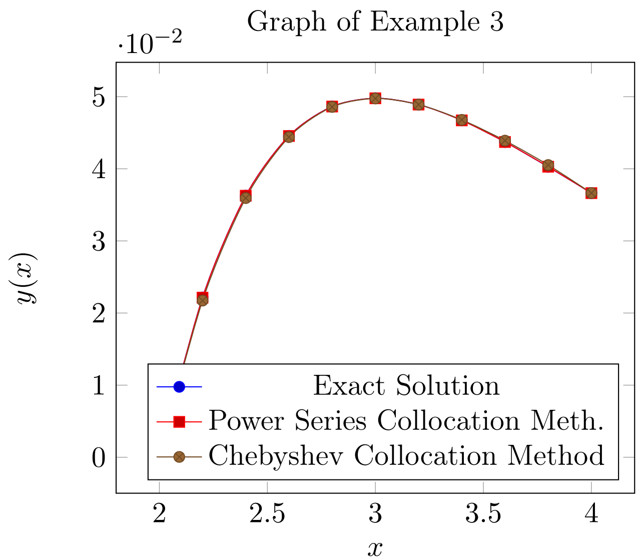

Figure 3: The behaviour of the exact solution compared with the solutions by Power series collocation method and Chebyshev collocation method.

5 Discussion of Results and Conclusion

In this paper, the Rayleigh-Ritz and Collocation methods using shifted Chebyshev polynomials for solving second order boundary value problems of ordinary differential equations are discussed. The trial solutions for the Rayleigh-Ritz method are carefully chosen to satisfy the two conditions of compatibility and linear independence. In the method, the interval $[a, b]$ is divided into sub-intervals and a linear approximation as basis function is chosen as a trial solution. For the collocation method, shifted Chebyshev polynomials are used as basis functions in the trial solution. The trial solution and its derivatives of appropriate orders are substituted into the problem being considered and the resulting equation is collocated at equally spaced interior points.

A comparison of the two methods in terms of the absolute errors produced by the examples considered shows that collocation method is easier and more accurate than the Rayleigh-Ritz method. This is evident in the tables of results presented.

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