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# Monotonicity and Convexity Properties and Some Inequalities Involving $E_{n,p}(x)$

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**Abstract**

In this paper, we established some monotonicity and convexity properties of the $p$-analogue of the exponential integral function. The increasing and decreasing, positive and negative, and convexity and concavity properties of the function were established and proved. Complete monotonicity of the function was also considered.

## 1 Introduction and Preliminaries

Some special functions have some formulae and identities which are employed by many mathematicians, engineers and physicists. These functions have several uses in pure mathematics and are applied in areas like fluid dynamics, solutions of wave equations, heat conduction, communication system, nonlinear wave propagation, electromagnetic theory, quantum mechanics, approximation theory, probability theory, and electric circuit theory, among others [1].

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The focus of this paper is on the classical exponential integral function defined by Schloemich in [2] as

$$E_n(x) = \int_1^\infty t^{-n} e^{-tx} \quad dt \qquad x > 0, \qquad n \in \mathbb{N}, \tag{1.1}$$

and the $i$-$th$ derivative of (1.1) is given by

$$E_n^{(i)}(x) = (-1)^i \int_1^\infty t^{i-n} e^{-xt} dt, \qquad i \in \mathbb{N}_0. \tag{1.2}$$

This special function has been investigated in diverse ways (see [3], [4], [5], [6], [7], [8], [9], [12] and the related references therein).

The $p$-analogue of the exponential integral function, $E_{n,p}(x)$ is defined for $x > 0$, $p > 1$ and $n \in \mathbb{N}_0$ by [10]

$$E_{n,p}(x) = \int_1^p t^{-n} A_p^{-xt} dt, \tag{1.3}$$

and the $i$-$th$ derivative of (1.3) is given by [11]

$$E_{n,p}^{(i)}(x) = \left(\ln A_p^{-1}\right)^i \int_1^p t^{i-n} A_p^{-xt} dt, \tag{1.4}$$

where, $E_{n,p}(x) \longrightarrow E_n(x)$ as $p \longrightarrow \infty$, $A_p = (1 + \frac{1}{p})^p$ and $E_{n,p}^{(i)}(x) \longrightarrow E_n^{(i)}(x)$ as $p \longrightarrow \infty$.

The objective of this paper is to establish some monotonicity and convexity properties of the $p$-analogue of the exponential integral function. The increasing and decreasing, positive and negative, and convexity and concavity properties of the function are established and proved. Complete monotonicity of the function is also considered. Additionally, some new inequalities which involve $E_{n,p}(x)$ are established.

We begin with the following well known results( see for instance [13], [14], [15] or [16]).

**Definition 1.1.** (Convexity) A function $f : \mathbb{I} \longrightarrow \mathbb{R}$ is said to be convex if

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y) \tag{1.5}$$

holds for all $x, y \in \mathbb{I}$ and $\lambda \in (0,1)$. If in (1.5) we have strict inequality, then $f$ is said to be strictly convex. If the inequalities are reversed, then $f$ is said to be concave [17].

**Lemma 1.2.** *(Convexity) Let $f : (a,b) \longrightarrow \mathbb{R}$ and for any $x \in (a,b)$ suppose there exists a second derivative $f^{''}(x)$. The function $f(x)$ is convex on $(a,b)$ if and only if for each $x \in (a,b)$ we have $f^{''}(x) \geq 0$. If $f^{''}(x) > 0$ for each $x \in (a,b)$, then $f$ is strictly convex on $(a,b)$.*

Clearly, according to Definition 1.1 and Lemma 1.2 we have that the function $f(x)$ is concave on $(a,b)$ if and only if $f^{''}(x) \leq 0$, for all $x \in (a,b)$.

**Definition 1.3.** (Log-convexity) A function $f : \mathbb{I} \longrightarrow \mathbb{R}^{+}$ is said to be logarithmic convex or in short log-convex if $\ln f$ is convex on $\mathbb{I}$. That is if

$$\ln f(\lambda x + (1-\lambda)y) \leq \lambda \ln f(x) + (1-\lambda) \ln f(y) \tag{1.6}$$

or equivalently

$$f(\lambda x + (1-\lambda)y) \leq (f(x))^{\lambda} (f(y))^{1-\lambda} \tag{1.7}$$

holds for each $x, y \in \mathbb{I}$ and $\lambda \in (0,1)$ [18].

**Definition 1.4.** (Complete Monotonicity) A function $f : \mathbb{I} \longrightarrow \mathbb{R}$ is said to be completely monotonic on $\mathbb{I}$ if $f$ has a derivative of all order on $I$ and

$$(-1)^k f^{(k)}(x) \geq 0 \tag{1.8}$$

holds for $x \in \mathbb{I}$ and $k \in \mathbb{N}$ [19].

**Definition 1.5.** (Arithmetic-mean/Geometric-mean Inequality) The AM-GM inequality is sometimes called the Cauchy inequality (1821):

$$\frac{x_1 + ... + x_n}{n} \geq (x_1 ... x_n)^{\frac{1}{n}} \tag{1.9}$$

for all $x_k > 0$ [20].

The results are presented in the following section.

## 2 Main Results

**Theorem 2.1.** *Let $n \in \mathbb{N}_0$, $p > 1$ and $i \in \mathbb{N}$. Then the function $E_{n,p}(x)$ has the properties:*
*(a) $E_{n,p}(x)$ is strictly decreasing;*
*(b) $E_{n,p}^{(i)}(x)$ is positive and strictly decreasing if $i$ is even;*
*(c) $E_{n,p}^{(i)}(x)$ is negative and strictly increasing if $i$ is odd;*
*(d) $E_{n,p}^{(i)}(x)$ is strictly convex if $i$ is even;*
*(e) $E_{n,p}^{(i)}(x)$ is strictly concave if $i$ is odd;*
*for $x > 0$.*

*Proof.* Using (1.3), we have

$$E'_{n,p}(x) = -\ln A_p \int_1^p t^{1-n} A_p^{-xt} dt < 0, \tag{2.1}$$

which completes the proof of (a). Similarly, using (1.4) for even $i$, we have

$$E_{n,p}^{(i)}(x) = (-1)^i (\ln A_p)^i \int_1^p t^{i-n} A_p^{-xt} dt > 0 \tag{2.2}$$

which shows that $E_{n,p}^{(i)}(x)$ is positive for even $i$. Next for even $i$, we have

$$(E_{n,p}^{(i)}(x))' = E_{n,p}^{(i+1)}(x) = (-1)^{i+1} (\ln A_p)^{i+1} \int_1^p t^{i+1-n} A_p^{-xt} dt < 0 \tag{2.3}$$

which shows that $E_{n,p}^{(i)}(x)$ is strictly decreasing. This completes the proof of (b). By similar procedure the results yields (c). Furthermore by (1.4), we have

$$(E_{n,p}^{(i)}(x))'' = E_{n,p}^{(i+2)}(x) = (-1)^{i+2} (\ln A_p)^{i+2} \int_1^p t^{i+2-n} A_p^{-xt} dt > 0 \tag{2.4}$$

for even $i$. This yields (d). By a similarly procedure also yields (e). □

**Theorem 2.2.** *The function $E_{n,p}(x)$ is strictly completely monotonic for all $n \in \mathbb{N}_0$, $p > 1$, $i \in \mathbb{N}$ and $x > 0$.*

*Proof.* Using (1.4), we have

$$\begin{aligned}(-1)^i E_{n,p}^{(i)}(x) &= (-1)^i \left(\ln A_p^{-1}\right)^i \int_1^p t^{i-n} A_p^{-xt} dt \\ &= (-1)^{2i} (\ln A_p)^i \int_1^p t^{i-n} A_p^{-xt} dt \\ &> 0.\end{aligned}$$

This completes the proof. □

**Theorem 2.3.** *The function $E_{n,p}^{(r)}(x)$ is strictly completely monotonic if $r \in \mathbb{N}_0$ is even and $-E_{n,p}^{(r)}(x)$ is strictly completely monotonic if $r \in \mathbb{N}_0$ is odd respectively for $x > 0$.*

*Proof.* Using (1.4), we have

$$\begin{aligned}(-1)^i E_{n,p}^{(r+i)}(x) &= (-1)^i \left(\ln A_p^{-1}\right)^{r+i} \int_1^p t^{r+i-n} A_p^{-xt} dt \\ &= (-1)^{r+2i} (\ln A_p)^{r+i} \int_1^p t^{r+i-n} A_p^{-xt} dt \\ &= Q(x).\end{aligned}$$

$Q(x) > 0$ if $r$ is even and $Q(x) < 0$ if $r$ is odd. This completes the proof. □

**Theorem 2.4.** *The function $E_{n,p}(x)$ satisfies the inequality*

$$\left| E_{n,p}^{\left(\frac{i}{\eta}+\frac{j}{\mu}\right)}\left(\frac{x}{\eta}+\frac{y}{\mu}\right) \right| \leq \left| E_{n,p}^{(i)}(x) \right|^{\frac{1}{\eta}} \left| E_{n,p}^{(j)}(y) \right|^{\frac{1}{\mu}}, \tag{2.5}$$

*for $\eta > 1$, $x, y > 0$, $i, j \in \mathbb{N}$ and $\frac{1}{\eta} + \frac{1}{\mu} = 1$.*

*Proof.* Using (1.4) and Hölder's inequality for integrals, we have

$$
\begin{aligned}
&\left|E_{n,p}^{(\frac{i}{\eta}+\frac{j}{\mu})}\left(\frac{x}{\eta}+\frac{y}{\mu}\right)\right| \\
&= \left(\ln A_p^{-1}\right)^{\frac{i}{\eta}+\frac{j}{\mu}} \int_1^p t^{(\frac{i}{\eta}+\frac{j}{\mu})-n} A_p^{-(\frac{x}{\eta}+\frac{y}{\mu})t} dt \\
&= \left(\ln A_p^{-1}\right)^{\frac{i}{\eta}+\frac{j}{\mu}} \int_1^p t^{(\frac{i}{\eta}+\frac{j}{\mu})-n(\frac{1}{\eta}+\frac{1}{\mu})} A_p^{-(\frac{x}{\eta}+\frac{y}{\mu})t} dt \\
&= \left(\ln A_p^{-1}\right)^{\frac{i}{\eta}+\frac{j}{\mu}} \int_1^p t^{\frac{i}{\eta}-\frac{n}{\eta}} A_p^{-\frac{xt}{\eta}} t^{\frac{j}{\mu}-\frac{n}{\mu}} A_p^{-\frac{yt}{\mu}} dt \\
&\leq \left(\ln A_p^{-1}\right)^{\frac{i}{\eta}} \left(\ln A_p^{-1}\right)^{\frac{j}{\mu}} \left(\int_1^p \left(t^{\frac{i}{\eta}-\frac{n}{\eta}} A_p^{-\frac{xt}{\eta}}\right)^{\eta} dt\right)^{\frac{1}{\eta}} \left(\int_1^p \left(t^{\frac{j}{\mu}-\frac{n}{\mu}} A_p^{-\frac{yt}{\mu}}\right)^{\mu} dt\right)^{\frac{1}{\mu}} \\
&= \left(\left(\ln A_p^{-1}\right)^{i} \int_1^p t^{i-n} A_p^{-xt} dt\right)^{\frac{1}{\eta}} \left(\left(\ln A_p^{-1}\right)^{j} \int_1^p t^{j-n} A_p^{-yt} dt\right)^{\frac{1}{\mu}} \\
&= \left|E_{n,p}^{(i)}(x)\right|^{\frac{1}{\eta}} \left|E_{n,p}^{(j)}(y)\right|^{\frac{1}{\mu}}.
\end{aligned}
$$

This completes the proof. □

**Remark.** *When $i = j$ is even in (2.5), then $E_{n,p}(x)$ satisfies the inequality*

$$E_{n,p}^{(i)}\left(\frac{x}{\eta}+\frac{y}{\mu}\right) \leq \left(E_{n,p}^{(i)}(x)\right)^{\frac{1}{\eta}} \left(E_{n,p}^{(i)}(y)\right)^{\frac{1}{\mu}}, \tag{2.6}$$

*which implies that the function $E_{n,p}^{(i)}(x)$ is logarithmically convex for even $i$. If $i = 0$ in (2.6), then we have*

$$E_{n,p}\left(\frac{x}{\eta}+\frac{y}{\mu}\right) \leq \left(E_{n,p}(x)\right)^{\frac{1}{\eta}} \left(E_{n,p}(y)\right)^{\frac{1}{\mu}}, \tag{2.7}$$

*which implies that the function $E_{n,p}(x)$ is logarithmically convex.*

*Substituting $\eta = \mu = 2$, $x = y$ and $j = i + 2$ in (2.5), we have the Turan-type inequality*

$$\left|E_{n,p}^{(i+1)}(x)\right|^2 \leq \left|E_{n,p}^{(i+2)}(x)\right| \left|E_{n,p}^{(i)}(x)\right|. \tag{2.8}$$

**Remark.** *The log-convexity of $E_{n,p}(x)$ implies that*

*(a) $E_{n,p}''(x)\, E_{n,p}(x) > [E_{n,p}'(x)]^2$, $x > 0$*

*(b) The function $\frac{E_{n,p}'(x)}{E_{n,p}(x)}$ is increasing for $x > 0$.*

**Corollary 2.5.** *The function $E_{n,p}(x)$ satisfies the inequalities*

$$[E_{n,p}(x+y)]^2 \leq E_{n,p}(x)\, E_{n,p}(y)\,, \tag{2.9}$$

*and*

$$E_{n,p}(x+y) \leq E_{n,p}(x) + E_{n,p}(y)\,, \tag{2.10}$$

*hold for $x, y > 0$ and $n \in \mathbb{N}_0$.*

*Proof.* Since $E_{n,p}(x)$ is decreasing, we have

$$E_{n,p}(x+y) \leq E_{n,p}\left(\frac{x+y}{2}\right).$$

Substituting $\eta = \mu = 2$ in (2.7) , we have

$$E_{n,p}\left(\frac{x+y}{2}\right) \leq \sqrt{E_{n,p}(x)\, E_{n,p}(y)} \tag{2.11}$$

which implies

$$E_{n,p}(x+y) \leq \sqrt{E_{n,p}(x)\, E_{n,p}(y)}$$

and that completes the proof of (2.9). Next, by the Arithmetic Meam-Geometric Mean inequality, we have

$$E_{n,p}(x+y) \leq \sqrt{E_{n,p}(x)\, E_{n,p}(y)} \leq \frac{E_{n,p}(x)}{2} + \frac{E_{n,p}(y)}{2} \leq E_{n,p}(x) + E_{n,p}(y),$$

which completes the proof of (2.10). □

**Theorem 2.6.** *The function $E_{n,p}(x)$ satisfies the inequality*

$$1 < \frac{E_{n,p}(\omega)}{E_{n,p}(\omega+1)} < \frac{E_{n,p}(\omega-1)}{E_{n,p}(\omega)}, \tag{2.12}$$

*for $\omega > 1$ and $n \in \mathbb{N}_0$.*

*Proof.* Since $E_{n,p}(x)$ is decreasing, we have

$$E_{n,p}(\omega+1) < E_{n,p}(\omega)$$

which implies

$$1 < \frac{E_{n,p}(\omega)}{E_{n,p}(\omega+1)}.$$

Substituting $x = \omega - 1$ and $y = \omega + 1$ in (2.11) gives

$$E_{n,p}^2(\omega) < E_{n,p}(\omega-1)\, E_{n,p}(\omega+1),$$

which can be written as

$$\frac{E_{n,p}(\omega)}{E_{n,p}(\omega+1)} < \frac{E_{n,p}(\omega-1)}{E_{n,p}(\omega)},$$

and this completes the proof for (2.12). □

**Theorem 2.7.** *Let $a > 0$, $p > 1$, $n \in \mathbb{N}_0$. Then the function*

$$\psi(x) = a^x E_{n,p}(x) \tag{2.13}$$

*is log-convex for $x > 0$.*

*Proof.* Let $x > 0$, $y > 0$, $\eta > 1$ and $\frac{1}{\eta} + \frac{1}{\mu} = 1$. Since the function $E_{n,p}(x)$ is log-convex, we have

$$\begin{aligned}\psi\left(\frac{x}{\eta}+\frac{y}{\mu}\right) &= a^{\frac{x}{\eta}+\frac{y}{\mu}} E_{n,p}\left(\frac{x}{\eta}+\frac{y}{\mu}\right)\\ &\le a^{\frac{x}{\eta}+\frac{y}{\mu}} \left[E_{n,p}(x)\right]^{\frac{1}{\eta}} \left[E_{n,p}(y)\right]^{\frac{1}{\mu}}\\ &= \left[a^x E_{n,p}(x)\right]^{\frac{1}{\eta}} \left[a^y E_{n,p}(y)\right]^{\frac{1}{\mu}}\\ &= \left[\psi(x)\right]^{\frac{1}{\eta}} \left[\psi(y)\right]^{\frac{1}{\mu}}.\end{aligned}$$

This completes the proof. □

**Remark.** *Theorem 2.7 was motivated by Theorem 2.9 of [21].*

**Theorem 2.8.** *The function $E_{n,p}(x)$ satisfies the inequalities*

$$[E_{n,p}^{(i)}(xy)]^2 \leq E_{n,p}^{(i)}(x)\, E_{n,p}^{(i)}(y)\,, \tag{2.14}$$

*for $x \geq 1$, $y \geq 1$, and $n \in \mathbb{N}_0$.*

*Proof.* Since $x \geq 1$ and $y \geq 1$, then $xy \geq x$ and $xy \geq y$. If $i$ is even, by Theorem 2.1, $E_{n,p}^{(i)}(x)$ is positive and decreasing. Then we have

$$0 < E_{n,p}^{(i)}(xy) \leq E_{n,p}^{(i)}(x)$$

and

$$0 < E_{n,p}^{(i)}(xy) \leq E_{n,p}^{(i)}(y)$$

for $x, y \geq 1$, so

$$[E_{n,p}^{(i)}(xy)]^2 \leq E_{n,p}^{(i)}(x)\, E_{n,p}^{(i)}(y)\,.$$

Also, if $i$ is odd, $E_{n,p}^{(i)}(x)$ is negative and increasing by Theorem 2.1, then

$$E_{n,p}^{(i)}(x) \leq E_{n,p}^{(i)}(xy) < 0$$

and

$$E_{n,p}^{(i)}(y) \leq E_{n,p}^{(i)}(xy) < 0$$

for $x, y \geq 1$, so

$$[E_{n,p}^{(i)}(xy)]^2 \leq E_{n,p}^{(i)}(x)\, E_{n,p}^{(i)}(y)\,.$$

This completes the proof. □

**Theorem 2.9.** *Let $a > 1$, $p > 1$, $n \in \mathbb{N}_0$. Then the function*

$$h(x) = \frac{[E_{n,p}(x+1)]^a}{E_{n,p}(ax+1)} \tag{2.15}$$

*is decreasing for $x > 0$ and*

$$\frac{E_{n,p}(ay+1)}{E_{n,p}(ax+1)} \geq \left[\frac{E_{n,p}(y+1)}{E_{n,p}(x+1)}\right]^a \tag{2.16}$$

*holds for $0 < x \leq y$.*

*Proof.* Let $0 < x \leq y$ and

$$f(x) = \ln h(x) = a \ln E_{n,p}(x+1) - \ln E_{n,p}(ax+1).$$

Then

$$\begin{aligned} f'(x) &= a\frac{E'_{n,p}(x+1)}{E_{n,p}(x+1)} - a\frac{E'_{n,p}(ax+1)}{E_{n,p}(ax+1)} \\ &= a\left[\frac{E'_{n,p}(x+1)}{E_{n,p}(x+1)} - \frac{E'_{n,p}(ax+1)}{E_{n,p}(ax+1)}\right] \leq 0 \end{aligned}$$

which means $f(x)$ is decreasing and as a result, $h(x)$ is also decreasing. Then for $0 < x \leq y$, we have $h(x) \geq h(y)$ which when rearranged gives (2.16). □

**Theorem 2.10.** *Let $p > 1$, $n \in \mathbb{N}_0$. Then the inequality,*

$$E_{n,p}(x)\,E_{n,p}(y) \leq (1-n)^{-1}(p^{1-n}-1)E_{n,p}(x+y). \tag{2.17}$$

*for $x > 0$, $y > 0$.*

*Proof.* Let $g(x,y) = \frac{E_{n,p}(x)E_{n,p}(y)}{E_{n,p}(x+y)}$ and $f(x,y) = \ln g(x,y)$. That is

$$f(x,y) = \ln E_{n,p}(x) + \ln E_{n,p}(y) - \ln E_{n,p}(x+y).$$

Then, for a fixed $y$, we have

$$\frac{\delta}{\delta x} f(x,y) = \frac{E'_{n,p}(x)}{E_{n,p}(x)} - \frac{E'_{n,p}(x+y)}{E_{n,p}(x+y)} \leq 0$$

which means $f(x,y)$ is decreasing in terms of $x$ and as a result, $g(x,y)$ is also decreasing in terms of $x$. Then for $x = 0$, we have

$$g(x,y) \leq g(0,y) = E_{n,p}(0) = (1-n)^{-1}(p^{1-n}-1),$$

this completes the proof. □

## 3 Conclusion

Using inequalities (1.5), (1.6), (1.7), (1.8) and (1.9), we established some monotonicity and convexity properties of the $p$-analogue of the exponential integral function. The increasing and decreasing, positive and negative, and convexity and concavity properties of the function were established and proved. Complete monotonicity of the function was also considered. It is our hope that the findings will contribute greatly to knowledge in the area of mathematical analysis.

## References

[1] Bell, W. W. (2004). *Special functions for scientists and engineers*. Mineola, NY: Dover Publications.

[2] Kotani, M., Amerniy, A., Ishiguro, E., & Kimura, T. (1955). *Table of molecular integrals*. Tokyo, Japan: Maruzen Co. Ltd.

[3] Chiccoli, C., Lorenzutta, S., & Maino, G. (1989). Recent results for generalized exponential integrals. *Computers in Mathematics and Applications*, 19(5), 21-29. https://doi.org/10.1016/0898-1221(90)90098-5

[4] Milgram, M. A. (1985). The generalized integro-exponential function. *Mathematics of Computation*, 44, 441-458. https://doi.org/10.1090/S0025-5718-1985-0777276-4

[5] Nantomah, K., Merovci, F., & Nasiru, S. (2017). A generalization of the exponential integral and some associated inequalities. *Honan Mathematical Journal*, 39(1), 49-59. https://doi.org/10.5831/HMJ.2017.39.1.49

[6] Salem, A. (2011). A $q$-analogue of the exponential integral. *Afr. Mat.*, 24, 117-125. https://doi.org/10.1007/s13370-011-0046-6

[7] Sroysang, B. (2014). The $k$-th derivative of the incomplete exponential integral function. *Mathematica Aeterna*, 2, 141-144.

[8] Sulaiman, W. T. (2012). Turan inequalities of the exponential integral function. *Communications in Optimization Theory*, 1, 35-41.

[9] Sroysang, B. (2013). Inequalities for the incomplete exponential integral function. *Communications in Mathematical Applications*, 4(2), 145-148.

[10] Yakubu, A., Nantomah, K., & Iddrisu, M. M. (2020). A $p$-analogue of the exponential integral function and some properties. *Advances in Inequalities and Applications*, 2020(7), 1-9.

[11] Yakubu, A., Nantomah, K., & Iddrisu, M. M. (2020). The *i-th* derivative of the $p$-analogue of the exponential integral function and some properties. *Journal of Mathematical and Computational Science*, 10(5), 1801-1807.

[12] Yakubu, A., Abubakari, B., & Assan, F. K. (2023). Some properties and inequalities for a two-parameter generalization of the incomplete exponential integral function. *Asian Research Journal of Mathematics*, 19(10), 154-160. https://doi.org/10.9734/arjom/2023/v19i10737

[13] Nantomah, K. (2017). Generalized Holder's and Minkowski's inequalities for Jackson's $q$-integral and some applications to the incomplete $q$-gamma function. *Abstract and Applied Analysis*, 6(2017), Article ID 9796873.

[14] Kazarinoff, N. D. (1961). *Analytic inequalities.* New York: Holt, Rinehart and Winston.

[15] Mitrinovic, D. S. (1970). *Analytic inequalities.* New York: Springer-Verlag.

[16] Monica, M. A., & Nantomah, K. (2019). Some inequalities for the Chaudhry-Zubair Extension of the gamma function. *Asian Research Journal of Mathematics*, 14(1), 1-9. https://doi.org/10.9734/arjom/2019/v14i130117

[17] Widder, D. V. (1946). *The Laplace transform.* Princeton Mathematical Series 6. Princeton University Press.

[18] Widder, D. V. (1934). The inversion of the Laplace integral and the related moment problem. *Transactions of the American Mathematical Society*, 36, 107-200. https://doi.org/10.1090/S0002-9947-1934-1501737-7

[19] Widder, D. V. (1931). Necessary and sufficient conditions for the representation of a function as a Laplace integral. *Transactions of the American Mathematical Society*, 33, 851-892. https://doi.org/10.1090/S0002-9947-1931-1501621-6

[20] Maligranda, L. (2012). The AM-GM inequality is equivalent to the Bernoulli inequality. *Math Intelligencer*, 34(1), 2 pages. https://doi.org/10.1007/s00283-011-9266-8

[21] Nantomah, K. (2018). A generalization of Nielsen's beta-function. *International Journal of Open Problems in Computer Mathematics*, 11(2), 16-26. https://doi.org/10.12816/0049058