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# A $(k,\mu)$-Paracontact Metric Manifolds satisfying Curvature Conditions

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**Abstract**

In the present paper, we have studied the curvature tensors of $(k,\mu)$-paracontact manifold satisfying the conditions $\widetilde{Z}\cdot\widetilde{C}=0$, $R\cdot\widetilde{C}=0$, $P\cdot\widetilde{C}=0$ and $\widetilde{C}\cdot\widetilde{C}=0$. According these cases, $(k,\mu)$-paracontact manifolds have been characterized.

## 1 Introduction

Following their introduction by Kaneyuki and Williams [10], Zamkovoy conducted a comprehensive investigation of paracontact metric manifolds and their subclasses. Subsequently, several geometers researched paracontact metric manifolds and discovered a variety of essential features of these manifolds [17]. Paracontact metric manifolds have been investigated from a variety of perspectives. Recently, Cappeletti-Montano and Di Terlizzi have introduced the notion of

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$(k,\mu)$-paracontact metric manifolds as those paracontact metric manifolds such that the underlying paracontact metrix structure $(\phi,\xi,\eta,g)$ satisfies the condition

$$R(X,Y)\xi = k(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY)$$

for some real numbers $k$ and $\mu$, where $2h$ denotes the Lie derivative of $\phi$ in the direction of $\xi$, giving several examples [5]. Para-Sasakian manifolds are included in the $(k,\mu)$-paracontact metric manifold class. Suppose that $k$ and $\mu$ are smooth functions, Küpeli Erken has studied the notion on generalized $(k,\mu)$-paracontact metric manifolds with $\xi(\mu) = 0$ [11].

Özgür and De researced some certain curvature conditions satisfying quasi-conformal curvature tensor in Kenmotsu manifolds [14]. Yano and Sawaki proposed the concept of quasi-conformal curvature tensor, which is an extension of the conformal curvature tensor [16]. It is crucial in differential geometry as well as in the theory of relativity.

Atçeken studied generalized Sasakian space form satisfying certain conditions on the concircular curvature tensor [2]. De et al. searched Sasakian manifolds with quasi-conformal curvature tensor [7]. Hosseinzadeh and Taleshian produced conformal and quasi-conformal curvature tensors of an $N(k)$-quasi Einstein manifold [9]. De and Sarkar studied properties of projective curvature tensor to generalized Sasakian space form [8]. Many geometers have studied these curvature tensors in different manifolds. [1, 3, 4, 12, 13, 18].

In this study, we characterize $(k,\mu)$-paracontact manifolds in response to the findings of the preceding writers, which satify the curvature conditions $\widetilde{Z}\cdot\widetilde{C} = 0$, $R\cdot\widetilde{C} = 0$, $P\cdot\widetilde{C} = 0$ and $\widetilde{C}\cdot\widetilde{C} = 0$ where, $\widetilde{C}$, $\widetilde{Z}$, $R$ and $P$ denote the quasi-conformal, concircular, projective and Riemannian tensors of manifold, respectively.

## 2 Preliminaries

A contact manifold is a $C^\infty - (2n+1)$ dimensional manifold $M^{2n+1}$ equipped with a global 1-form $\eta$ such that $\eta \wedge (d\eta)^n \neq 0$ everywhere on $M^{2n+1}$. Given such a form $\eta$, it is well known that there exists a unique vector field $\xi$, called the characteristic vector field, such that $\eta(\xi)=1$ and $d\eta(X,\xi)=0$ for every vector field $X$ on $M^{2n+1}$. A Riemannian metric $g$ is said to be associated metric if there exists a tensor field $\phi$ of type $(1,1)$ such that

$$\phi^2 X = X - \eta(X)\xi, \qquad \eta(\xi)=1, \qquad \eta\circ\phi = 0, \qquad \phi\xi = 0, \tag{2.1}$$

$$g(\phi X,\phi Y) = -g(X,Y) + \eta(X)\eta(Y), \quad g(X,\xi)=\eta(X) \tag{2.2}$$

for all vector fields $X,Y$ on $M$. Then the structure $(\phi,\xi,\eta,g)$ on $M$ is called a paracontact metric structure and the manifold equipped with such a structure is called a almost paracontact metric manifold [17].

We now define a $(1,1)$ tensor field $h$ by $h=\frac{1}{2}L_\xi\phi$, where $L$ denotes the Lie derivative. Then $h$ is symmetric and satisfies the conditions

$$h\phi = -\phi h, \qquad h\xi = 0, \qquad Tr.h = Tr.\phi h = 0. \tag{2.3}$$

If $\nabla$ denotes the Levi-Civita connection of $g$, then we have the following relation

$$\widetilde{\nabla}_X\xi = -\phi X + \phi h X \tag{2.4}$$

for any $X\in\chi(M)$ [17]. For a paracontact metric manifold $M^{2n+1}(\phi,\xi,\eta,g)$, if $\xi$ is a killing vector field or equivalently, $h=0$, then it is called a K-paracontact manifold.

A para-contact metric structure $(\phi,\xi,\eta,g)$ is normal, that is, satisfies $[\phi,\phi]+2d\eta\otimes\xi=0$, which is equivalent to

$$(\widetilde{\nabla}_X\phi)Y = -g(X,Y)\xi + \eta(Y)X$$

for all $X,Y\in\chi(M)$ [17]. If an almost paracontact metric manifold is normal, then it called paracontact metric manifold. Any para-Sasakian manifold is K-paracontact, and the converse holds when $n=1$, that is, for 3-dimensional spaces. Any para-Sasakian manifold satisfies

$$R(X,Y)\xi = -(\eta(Y)X - \eta(X)Y) \tag{2.5}$$

for all $X,Y \in \chi(M)$, but this is not a sufficient condition for a paracontact manifold to be para-Sasakian. It is clear that every para-Sasakian manifold is K-paracontact. But the converse is not always true [4].

A paracontact manifold $M$ is said to be $\eta$-Einstein if its Ricci tensor $S$ of type $(0,2)$ is of the from $S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y)$, where $a,b$ are smooth functions on $M$. If $b = 0$, then the manifold is also called Einstein [15].

A paracontact metric manifold is said to be a $(k,\mu)$-paracontact manifold if the curvature tensor $R$ satisfies

$$\widetilde{R}(X,Y)\xi = k\left[\eta(Y)X - \eta(X)Y\right] + \mu\left[\eta(Y)hX - \eta(X)hY\right] \tag{2.6}$$

for all $X,Y \in \chi(M)$, where $k$ and $\mu$ are real constants.

This class is very wide containing the para-Sasakian manifolds as well as the paracontact metric manifolds satisfying $R(X,Y)\xi = 0$ [18].

In particular, if $\mu = 0$, then the paracontact metric $(k,\mu)$-manifold is called paracontact metric $N(k)$-manifold . Thus for a paracontact metric $N(k)$-manifold the curvature tensor satisfies the following relation

$$R(X,Y)\xi = k(\eta(Y)X - \eta(X)Y) \tag{2.7}$$

for all $X,Y \in \chi(M)$. Though the geometric behavior of paracontact metric $(k,\mu)$-spaces is different according as $k < -1$, or $k > -1$, but there are also some common results for $k < -1$ and $k > -1$.

**Lemma 2.1.** *There does not exist any paracontact $(k,\mu)$-manifold of dimension greater than 3 with $k > -1$ which is Einstein whereas there exits such manifolds for $k < -1$ [6].*

In a paracontact metric $(k,\mu)$-manifold $(M^{2n+1}\phi,\xi,\eta,g)$, $n > 1$, the following relation hold:

$$h^2 = (k+1)\phi^2, \text{ for } k \neq -1, \tag{2.8}$$

$$(\widetilde{\nabla}_X\phi)Y = -g(X - hX, Y)\xi + \eta(Y)(X - hX), \tag{2.9}$$

$$\begin{aligned} S(X,Y) &= [2(1-n)+n\mu]g(X,Y) + [2(n-1)+\mu]g(hX,Y) \\ &\quad +[2(n-1)+n(2k-\mu)]\eta(X)\eta(Y), \end{aligned} \tag{2.10}$$

$$S(X,\xi) = 2nk\eta(X), \tag{2.11}$$

$$\begin{aligned} QY &= [2(1-n)+n\mu]Y + [2(n-1)+\mu]hY \\ &\quad +[2(n-1)+n(2k-\mu)]\eta(Y)\xi, \end{aligned} \tag{2.12}$$

$$Q\xi = 2nk\xi, \tag{2.13}$$

$$Q\phi - \phi Q = 2[2(n-1)+\mu]h\phi \tag{2.14}$$

for any vector fields $X, Y$ on $M^{2n+1}$, where $Q$ and $S$ denotes the Ricci operator and Ricci tensor of $(M^{2n+1}, g)$, respectively [6].

The concept of quasi-conformal curvature tensor was defined by Yano and Sawaki [16]. Quasi-conformal curvature tensor of a $(2n+1)$-dimensional Riemannian manifold is defined as

$$\begin{aligned} \widetilde{C}(X,Y)Z &= aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y \\ &\quad +g(Y,Z)QX - g(X,Z)QY] \\ &\quad - \frac{\tau}{2n+1}[\frac{a}{2n} + 2b][g(Y,Z)X - g(X,Z)Y] \end{aligned} \tag{2.15}$$

where $a$ and $b$ are arbitrary scalars, and $r$ is the scalar curvature of the manifold, $Q$, $S$ and $r$ denote the Ricci operator, Ricci tensor and scalar curvature of manifold, respectively.

Let $(M, g)$ be an $(2n+1)$-dimensional Riemannian manifold. Then the concircular curvature tensor $\widetilde{Z}$ is defined by [15].

$$\widetilde{Z}(X,Y)Z = R(X,Y)Z - \frac{\tau}{2n(2n+1)}[g(Y,Z)X - g(X,Z)Y], \tag{2.16}$$

for all $X, Y, Z \in \chi(M)$. On the other hand, projective curvature tensor $P$ is defined by

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{2n}[S(Y,Z)X - S(X,Z)Y], \tag{2.17}$$

for all $X, Y, Z \in \chi(M)$, where $r$ is the scalar curvature of $M$ and $Q$ is the Ricci operator given by $g(QX, Y) = S(X, Y)$ [15].

## 3 A $(k, \mu)$-Paracontact Metric Manifolds satisfying Certain Curvature Conditions

In this section, we will give the main results for this paper.

Let $M$ be $(2n+1)$-dimensional $(k, \mu)$-paracontact metric manifold and we denote the Riemannian curvature tensor of $R$, from (2.6), we have for later

$$R(\xi, Y)Z = k(g(Y,Z)\xi - \eta(Z)Y) + \mu(g(hY,Z)\xi - \eta(Z)hY). \tag{3.1}$$

In (3.1), choosing $Z = \xi$ and taking into account (2.3), we obtain

$$R(\xi, Y)\xi = k(\eta(Y)\xi - Y) - \mu hY \tag{3.2}$$

In the same way, choosing $Z = \xi$ in (2.15) and using (2.6), we have

$$\begin{aligned} \widetilde{C}(X,Y)\xi &= (ak + 2nkb - \frac{r}{2n(2n+1)}(\frac{a}{2n} + 2b)(\eta(Y)X - \eta(X)Y) \\ &\quad + a\mu(\eta(Y)hX - \eta(X)hY) + b(\eta(Y)QX - \eta(X)QY) \end{aligned} \tag{3.3}$$

In (3.3), choosing $X = \xi$ and using (2.11), we obtain

$$\begin{aligned} \widetilde{C}(\xi,Y)\xi &= (ak + 2nkb - \frac{r}{2n(2n+1)}(\frac{a}{2n} + 2b)(\eta(Y)\xi - Y) \\ &\quad - a\mu hY + b(2nk\eta(Y)\xi - QY). \end{aligned} \tag{3.4}$$

In same way from (3.1) and (2.16), we get

$$\widetilde{Z}(\xi,Y)Z = (k - \frac{r}{2n(2n+1)})(g(Y,Z)\xi - \eta(Z)Y) + \mu(g(hY,Z)\xi - \eta(Z)hY), \tag{3.5}$$

from which

$$\widetilde{Z}(\xi,Y)\xi=(k-\frac{r}{2n(2n+1)})(\eta(Y)\xi-Y)-\mu hY. \tag{3.6}$$

From (3.1) and (2.17), we have

$$P(\xi,Y)Z=kg(Y,Z)\xi+\mu(g(hY,Z)\xi-\eta(Z)hY)-\frac{1}{2n}S(Y,Z)\xi. \tag{3.7}$$

Choosing $Z=\xi$ in (3.7), we obtain

$$P(\xi,Y)\xi=-\mu hY. \tag{3.8}$$

**Theorem 3.1.** *Let $M^{2n+1}(\phi,\xi,\eta,g)$ be a $(k,\mu)$-paracontact space. Then $\widetilde{Z}\cdot\widetilde{C}=0$ if and only if $M$ is an Einstein manifold.*

*Proof.* Suppose that $\widetilde{Z}\cdot\widetilde{C}=0$. This implies that

$$\begin{aligned}(\widetilde{Z}(X,Y)\widetilde{C})(U,W)Z &= \widetilde{Z}(X,Y)\widetilde{C}(U,W)Z-\widetilde{C}(\widetilde{Z}(X,Y)U,W)Z\\ &\quad -\widetilde{C}(U,\widetilde{Z}(X,Y)W)Z-\widetilde{C}(U,W)\widetilde{Z}(X,Y)Z\\ &= 0,\end{aligned} \tag{3.9}$$

for any $X,Y,U,W,Z\in\chi(M)$. Taking $X=Z=\xi$ in (3.9), making use of (3.3), (3.5) and (3.6) we have

$$\begin{aligned}(\widetilde{Z}(\xi,Y)\widetilde{C})(U,W)\xi &= \widetilde{Z}(\xi,Y)(A(\eta(W)U-\eta(U)W)+a\mu(\eta(W)hU\\ &\quad -\eta(U)hW)+b(\eta(W)QU-\eta(U)QW))\\ &\quad -\widetilde{C}(B(g(Y,U)\xi-\eta(U)Y)+\mu(g(hY,U)\xi\\ &\quad -\eta(U)hY),W)\xi-\widetilde{C}(U,B(g(Y,W)\xi-\eta(W)Y)\\ &\quad +\mu(g(hY,W)\xi-\eta(W)hY)\xi\\ &\quad -\widetilde{C}(U,W)(B(\eta(Y)\xi-Y)-\mu hY)=0,\end{aligned} \tag{3.10}$$

where $A=[ak+2nkb-\frac{r}{(2n+1)}(\frac{a}{2n}+2b)]$ and $B=k-\frac{r}{2n(2n+1)}$. Taking into account

(3.3), (3.4), (3.5) and inner product both sides of (3.10) by $Z \in \chi(M)$, we obtain

$$\begin{aligned}
&Bg(\widetilde{C}(U,W)Y,Z) + \mu g(\widetilde{C}(U,W)hY,Z) + a\mu B(\eta(W)\eta(Z)g(Y,hU)\\
&-\eta(U)\eta(Z)g(Y,hW)) + a\mu^2(1+k)(\eta(W)\eta(Z)g(Y,U) - \eta(U)\eta(Z)g(Y,W))\\
&+b\mu(\eta(W)\eta(Z)S(Y,hU) - \eta(U)\eta(Z)S(Y,hW)) + AB(g(Y,U)g(W,Z)\\
&-g(Y,W)g(U,Z)) + A\mu(g(hY,U)g(W,Z) - g(hY,W)g(U,Z))\\
&+a\mu B(g(Y,U)g(hW,Z) - g(Y,W)g(hU,Z)) + a\mu^2(g(hY,U)g(hW,Z)\\
&-g(hY,W)g(hU,Z)) + Bb(g(Y,U)S(W,Z) - g(Y,W)S(U,Z))\\
&+\mu b(g(hY,U)S(W,Z) - S(U,Z)g(hY,W)) + Bb(\eta(W)\eta(Z)S(Y,U)\\
&-\eta(U)\eta(Z)S(Y,W)) + 2nkBb(\eta(U)\eta(Z)g(Y,W) - \eta(W)\eta(Z)g(Y,U))\\
&+2nkb\mu(\eta(Z)\eta(U)g(hY,W) - \eta(W)\eta(Z)g(hY,U)) = 0. \qquad (3.11)
\end{aligned}$$

Using (2.1), (2.12) and (2.15) choosing $U = Z = e_i$, $\xi$ in (3.11), $1 \leq i \leq n$, for orthonormal basis of $\chi(M)$, we arrive

$$BS(W,Y) + \mu S(W,hY) - 2nkBg(W,Y) - 2nk\mu g(W,hY) = 0. \qquad (3.12)$$

Using (2.8) and replacing $hY$ of $Y$ in (3.12), we get

$$BS(W,hY) + \mu(1+k)S(W,Y) - 2nkBg(W,hY) - 2nk\mu(1+k)g(W,Y) = 0. \qquad (3.13)$$

From (3.12) and (3.13), we have

$$S(W,Y) = 2nkg(W,Y).$$

So, $M$ is an Einstein manifold. Conversely, let $M^{2n+1}(\phi,\xi,\eta,g)$ be an Einstein manifold, i.e., $S(W,Y) = 2nkg(W,Y)$, then from (3.13)-(3.9), we have $\widetilde{Z} \cdot \widetilde{C} = 0$. □

**Theorem 3.2.** *Let $M^{2n+1}(\phi,\xi,\eta,g)$ be a $(k,\mu)$-paracontact space. Then $P \cdot \widetilde{C} = 0$ if and only if $M$ is an $\eta$-Einstein manifold.*

*Proof.* Assume that $P \cdot \widetilde{C} = 0$. Then we have

$$\begin{aligned}(P(X,Y)\widetilde{C})(U,W)Z &= P(X,Y)\widetilde{C}(U,W)Z - \widetilde{C}(P(X,Y)U,W)Z \\ &\quad -\widetilde{C}(U,P(X,Y)W)Z - \widetilde{C}(U,W)P(X,Y)Z \\ &= 0, \end{aligned} \tag{3.14}$$

for any $X, Y, U, W, Z \in \chi(M)$. Taking $X = Z = \xi$ in (3.14) and using (3.3), (3.7), (3.8) setting $A = [ak + 2nkb - \frac{r}{(2n+1)}(\frac{a}{2n} + 2b)]$, we obtain

$$\begin{aligned}(P(\xi,Y)\widetilde{C})(U,W)\xi &= P(\xi,Y)(A(\eta(W)U - \eta(U)W) + a\mu(\eta(W)hU - \eta(U)hW) \\ &\quad +b((\eta(W)QU - \eta(U)QW)) - \widetilde{C}(kg(Y,U)\xi + \mu(g(hY,U)\xi \\ &\quad -\eta(U)hY) - \frac{1}{2n}S(Y,U)\xi, W)\xi - \widetilde{C}(U, k(g(Y,W)\xi \\ &\quad +\mu(g(hY,W)\xi - \eta(W)hY) - \frac{1}{2n}S(Y,W)\xi)\xi \\ &\quad +\widetilde{C}(U,W)\mu hY = 0. \end{aligned} \tag{3.15}$$

Taking into account that (3.3), (3.4), (3.7) and setting $U = \xi$, inner product both sides of in (3.15) by $\xi \in \chi(M)$, we get

$$\begin{aligned}&a\mu S(Y,hW) - 2na\mu k g(Y,hW) - 2nbkS(Y,W) \\ &+bS(Y,QW) - 2nkAg(Y,W) + AS(Y,W) = 0. \end{aligned} \tag{3.16}$$

Using (2.1) and (2.15) in (3.16), we get

$$\begin{aligned}&(A + b[2(1-n) + n\mu] - 2nkb)S(Y,W) \\ &+(a\mu + b[2(n-1) + \mu])S(Y,hW) \\ &+(-2nkA)g(Y,W) + (-2nka\mu)g(Y,hW) \\ &+(2nkb[2(n-1) + n(2k-\mu)])\eta(Y)\eta(W) = 0. \end{aligned} \tag{3.17}$$

Replacing $hZ$ of $Z$ in (3.17) and making use of (2.8), we get

$$\begin{aligned}&(A+b[2(1-n)+n\mu]-2nkb)S(Y,hW)\\&+(1+k)(a\mu+b[2(n-1)+\mu])S(Y,W)\\&-2nk(1+k)(a\mu+b[2(n-1)+\mu])\eta(Y)\eta(W)\\&-2nkAg(Y,hW)+(1+k)(-2nka\mu)g(Y,W)\\&-(1+k)(-2nka\mu)\eta(Y)\eta(W)=0.\end{aligned} \tag{3.18}$$

From (3.17), (3.18) and using (2.10), we obtain

$$DS(Y,W)=Eg(Y,W)+F\eta(Y)\eta(W),$$

where,

$$\begin{aligned}c&=(A+b[2(1-n)+n\mu]-2nkb),\\d&=(a\mu+b[2(n-1)+\mu]),\\e&=-2nkA,\\f&=-2nka\mu,\\t&=(2nkb[2(n-1)+n(2k-\mu)])\end{aligned}$$

and

$$\begin{aligned}E&=(fd(1+k)-ec)[2(n-1)+\mu]+(fc-ed)[2(1-n)+n\mu],\\D&=(c^2-d^2(1+k))[2(n-1)+\mu]+(fc-de),\\F&=(fc-de)[2(n-1)+n(2k-\mu)]\\&\quad-(ct+2nkd^2(1+k)+fd(1+k))[2(n-1)+\mu]\end{aligned}$$

Thus, $M$ is an $\eta$-Einstein manifold. Conversely, let $M^{2n+1}(\phi,\xi,\eta,g)$ be an $\eta$-Einstein manifold i.e. $DS(Y,W)=Eg(Y,W)+F\eta(Y)\eta(W)$, then from (3.18)-(3.14) we obtain $P\cdot\widetilde{C}=0$. □

**Theorem 3.3.** *Let $M^{2n+1}(\phi,\xi,\eta,g)$ be a $(k,\mu)$-paracontact space. Then $M$ is a quasi-conformal semi-symmetric if and only if $M$ is an Einstein manifold.*

*Proof.* Assume that $M$ is a quasi-conformal semi-symmetric. This implies that

$$\begin{aligned}(R(X,Y)\widetilde{C})(U,W)Z &= R(X,Y)\widetilde{C}(U,W)Z - \widetilde{C}(R(X,Y)U,W)Z \\ &\quad -\widetilde{C}(U,R(X,Y)W)Z - \widetilde{C}(U,W)R(X,Y)Z \\ &= 0, \end{aligned} \tag{3.19}$$

for any $X,Y,U,W,Z \in \chi(M)$. Setting $X = Z = \xi$ in (3.19) and making use of (3.1), (3.2), (3.3), for $A = [ak + 2nkb - \frac{r}{(2n+1)}(\frac{a}{2n} + 2b)]$, we obtain

$$\begin{aligned}(R(\xi,Y)\widetilde{C})(U,W)\xi &= R(\xi,Y)(A(\eta(W)U - \eta(U)W) + a\mu(\eta(W)hU \\ &\quad -\eta(U)hW) + b(\eta(W)QU - \eta(U)QW)) \\ &\quad -\widetilde{C}(k(g(Y,U)\xi - \eta(U)Y) + \mu(g(hY,U)\xi \\ &\quad -\eta(U)hY), W)\xi - \widetilde{C}(U, k(g(Y,W)\xi \\ &\quad -\eta(W)Y) + \mu(g(hY,W)\xi - \eta(W)hY))\xi \\ &\quad -\widetilde{C}(U,W)(k(\eta(Y)\xi - Y) - \mu hY = 0. \end{aligned} \tag{3.20}$$

Inner product both sides of (3.20) by $Z \in \chi(M)$ and putting (3.1), (3.3) and (3.4), we get

$$\begin{aligned}&kg(\widetilde{C}(U,W)Y,Z) + \mu g(\widetilde{C}(U,W)hY,Z) + A\mu(\eta(W)\eta(Z)g(Y,hU) \\ &-\eta(U)\eta(Z)g(Y,hW)) + a\mu^2(1+k)(\eta(W)\eta(Z)g(Y,U) - \eta(U)\eta(Z)g(Y,W)) \\ &+b\mu(\eta(W)\eta(Z)S(Y,hU) - \eta(U)\eta(Z)S(Y,hW)) + Ak(g(Y,U)g(W,Z) \\ &-g(Y,W)g(U,Z)) + a\mu k(\eta(W)\eta(Z)g(Y,hU) - \eta(U)\eta(Z)g(Y,hW)) \\ &+bk(\eta(W)\eta(Z)S(Y,U) - \eta(U)\eta(Z)S(Y,W)) + b\mu(g(hY,U)S(W,Z) \\ &-g(hY,W)S(U,Z)) + a\mu^2(g(hY,U)g(hW,Z) - g(hY,W)g(hU,Z)) \\ &+a\mu k(g(Y,U)g(hW,Z) - g(Y,W)g(hU,Z)) + bk(g(Y,U)S(W,Z) \\ &-g(Y,W)S(U,Z)) + 2nkb\mu(\eta(U)\eta(Z)g(hY,W) - \eta(W)\eta(Z)g(hY,U)) \\ &+2nk^2b(\eta(U)\eta(Z)g(Y,W) - \eta(W)\eta(Z)g(Y,U)) = 0. \end{aligned} \tag{3.21}$$

Making use of (2.8), (2.15) and choosing $U = Z = e_i$, $\xi$ $1 \leq i \leq n$, for orthonormal basis of $\chi(M)$ in (3.21), we have

$$kS(W,Y) + \mu S(W,hY) - 2nk^2 g(W,Y) - 2nk\mu g(W,hY) = 0. \tag{3.22}$$

Replacing $hY$ of $Y$ in (3.22) and taking into account (2.8), we get

$$kS(W,hY)+\mu(1+k)S(W,Y)-2nk^2(1+k)g(W,hY)-2nk\mu(1+k)g(W,Y)=0. \tag{3.23}$$

From (3.22) and (3.23), we arrive

$$S(W,Y)=2nkg(W,Y).$$

This tell us, $M$ is an Einstein manifold. Conversely, let $M^{2n+1}(\phi,\xi,\eta,g)$ be an Einstein manifold, i.e., $S(Y,W)=2nkg(Y,W)$, then from (3.23)-(3.19), we get $M$ is a quasi-conformal semi-symmetric. □

**Theorem 3.4.** *Let $M^{2n+1}(\phi,\xi,\eta,g)$ be a $(k,\mu)$-paracontact space. Then $\widetilde{C}\cdot\widetilde{C}=0$ if and only if $M$ is an $\eta$-Einstein manifold.*

*Proof.* Suppose that $\widetilde{C}\cdot\widetilde{C}=0$. This means that

$$\begin{aligned}(\widetilde{C}(X,Y)\widetilde{C})(U,W,Z) &= \widetilde{C}(X,Y)\widetilde{C}(U,W)Z-\widetilde{C}(\widetilde{C}(X,Y)U,W)Z\\ &\quad -\widetilde{C}(U,\widetilde{C}(X,Y)W)Z-\widetilde{C}(U,W)\widetilde{C}(X,Y)Z\\ &= 0,\end{aligned} \tag{3.24}$$

for any $X,Y,U,W,Z\in\chi(M)$. Setting $X=Z=\xi$ in (3.24) and making use of (3.3), (3.4), for $A=[ak+2nkb-\frac{r}{(2n+1)}(\frac{a}{2n}+2b)]$, we obtain

$$\begin{aligned}(\widetilde{C}(\xi,Y)\widetilde{C})(U,W)\xi &= \widetilde{C}(\xi,Y)(A(\eta(W)U-\eta(U)W)+a\mu(\eta(W)hU\\ &\quad -\eta(U)hW)+b(\eta(W)QU-\eta(U)QW))\\ &\quad -\widetilde{C}(A(g(Y,U)\xi-\eta(U)Y)+a\mu(g(hY,U)\xi\\ &\quad -\eta(U)hY)+b(S(Y,U)\xi-\eta(U)QY,W))\xi\\ &\quad -\widetilde{C}(U,A(g(Y,W)\xi-\eta(W)Y)+a\mu(g(hY,W)\xi\\ &\quad -\eta(W)hY)+b(S(Y,W)\xi-\eta(W)QY))\xi\\ &\quad -\widetilde{C}(U,W)(A(\eta(Y)\xi-AY-a\mu hY)\\ &\quad +b(2nk\eta(Y)\xi-QY)=0.\end{aligned} \tag{3.25}$$

By using (2.12), (2.15) inner product both sides of (3.25) by $Z \in \chi(M)$ and choosing $W = Y = e_i$, $\xi$, $1 \leq i \leq n$, for orthonormal basis of $\chi(M)$ in (3.25), we have

$$\begin{aligned}
&(Aa + ba[2(1-n) + n\mu] - b(A - ak - 2nkb) - b^2[2(1-n) + n\mu])S(U,Z)\\
&+(Aa\mu + ba[2(n-1) + \mu])S(U,hZ) + (2nA(A - ak - 2nkb)\\
&+2na\mu b(1+k)[2(n-1) + \mu] + br(A - ak - 2nkb)\\
&+bak[2(n-1) + n(2k-\mu)] + b^2r^2[2(1-n) + n\mu])g(U,Z)\\
&+(a\mu(ak + 2nkb) + ba\mu[2(n-1) + n(2k-\mu)] - 2nAa\mu\\
&-abr\mu)g(U,hZ) + (-bak[2(n-1) + n(2k-\mu)] + 2nkb^2[2(1-n)\\
&+n\mu]) - b^2r[2(1-n) + n\mu] - 2nb^2(1+k)[2(n-1) + \mu]^2\\
&-(a\mu)^2(1+k)(2n+1) - 2nab\mu(1+k)[2(n-1) + \mu] - Abr\\
&-2nab\mu(1+k)[2(n-1) + \mu] - (2nkb)^2\\
&+2nkAb(2n+1) + 2nkrb^2)\eta(U)\eta(Z) = 0. \qquad (3.26)
\end{aligned}$$

Replacing $hZ$ of $Z$ in (3.26) and taking into account (2.8), we get

$$\begin{aligned}
&(Aa + ba[2(1-n) + n\mu] - b(A - ak - 2nkb)\\
&-b^2[2(1-n) + n\mu])S(U,hZ)\\
&+(1+k)(Aa\mu + ba[2(n-1) + \mu])S(U,Z)\\
&-2nk(1+k)(Aa\mu + ba[2(n-1) + \mu])\eta(U)\eta(Z)\\
&+(2nA(A - ak - 2nkb) + 2na\mu b(1+k)[2(n-1) + \mu]\\
&+br(A - ak - 2nkb) + bak[2(n-1) + n(2k-\mu)]\\
&+b^2r^2[2(1-n) + n\mu])g(U,hZ)\\
&+(1+k)(a\mu(ak + 2nkb) + ba\mu[2(n-1)\\
&+n(2k-\mu)] - 2nAa\mu - abr\mu)g(U,Z)\\
&-(1+k)(a\mu(ak + 2nkb) + ba\mu[2(n-1)\\
&+n(2k-\mu)] - 2nAa\mu - abr\mu)\eta(U)\eta(Z) = 0. \qquad (3.27)
\end{aligned}$$

From (3.26), (3.27) and by using (2.10), for the sake of brevity, we set

$$
\begin{aligned}
c &= (Aa + ba[2(1-n) + n\mu] - b(A - ak - 2nkb) - b^2[2(1-n) + n\mu]), \\
d &= (Aa\mu + ba[2(n-1) + \mu]), \\
e &= (2nA(A - ak - 2nkb) + 2na\mu b(1+k)[2(n-1) + \mu] + br(A - ak - 2nkb) \\
&\quad + bak[2(n-1) + n(2k - \mu)] + b^2r^2[2(1-n) + n\mu]), \\
f &= (a\mu(ak + 2nkb) + ba\mu[2(n-1) + n(2k - \mu)] - 2nAa\mu - abr\mu), \\
t &= (-bak[2(n-1) + n(2k - \mu)] + 2nkb^2[2(1-n) + n\mu]) - b^2r[2(1-n) + n\mu] \\
&\quad - 2nb^2(1+k)[2(n-1) + \mu]^2 - (a\mu)^2(1+k)(2n+1) \\
&\quad - 2nab\mu(1+k)[2(n-1) + \mu] - Abr - 2nab\mu(1+k)[2(n-1) + \mu] \\
&\quad - (2nkb)^2 + 2nkAb(2n+1) + 2nkrb^2)
\end{aligned}
$$

and

$$
\begin{aligned}
E &= [fd(1+k) - ec][2(n-1) + \mu] + (fc - de)[2(1-n) + n\mu], \\
D &= (c^2 - d^2(1+k))[2(n-1) + \mu] + (fc - ed), \\
F &= (fc - de)[2(n-1) + n(2k - \mu)] \\
&\quad - (ct + 2nkd^2(1+k) + fd(1+k))[2(n-1) + \mu],
\end{aligned}
$$

we obtain

$$
DS(U, Z) = Eg(U, Z) + F\eta(U)\eta(Z).
$$

So, $M$ is an $\eta$-Einstein manifold. $M$ is an $\eta$-Einstein manifold. Conversely, let $M^{2n+1}(\phi, \xi, \eta, g)$ be an Einstein manifold, i.e., $DS(U, Z) = Eg(U, Z) + F\eta(U)\eta(Z)$, then from (3.27)-(3.24), we arrive $\widetilde{C} \cdot \widetilde{C} = 0$. □

**Conclusion and Recommendations** The tensors studied above can also be applied to other Manifolds.

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