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Multiple Split Equality Problem and Convergence Result

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Abstract

Let H_i (where $i = 1, ..., m$) be Hilbert spaces, and let H be another Hilbert space. Let $A_i : H_i \to H$ be bounded linear operators for $i = 1, ..., m$. We introduce the Multiple Split Equality Problem (MSEP), provide an algorithm for constructing a solution to MSEP, and establish the strong convergence of the algorithm to such a solution.

1 Introduction

Let D_1 and D_2 be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively. The split feasibility problem is formulated as finding a point x satisfying

$$x \in D_1 \text{ such that } Ax \in D_2, \tag{1}$$

where A is a bounded linear operator from H_1 into H_2. The split feasibility problem in finite-dimensional Hilbert spaces was first studied by Censor

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and Elfving [3] for modeling inverse problems that arise in medical image reconstruction, image restoration, and radiation therapy treatment planning (see, e.g., [1], [2], [3]). It is clear that $x \in D_1$ is a solution to the split feasibility problem (1) if and only if $Ax - P_{D_2}Ax = 0$, where P_{D_2} is the metric projection from H_2 onto D_2.

Let H_1, H_2, and H be Hilbert spaces, D_1, D_2 be nonempty closed convex subsets of H_1, H_2 respectively, and let $A : H_1 \to H$ and $B : H_2 \to H$ be bounded linear operators. The Split Equality Problem (SEP) is to find

$$x \in D_1 \text{ and } y \in D_2 \text{ such that } Ax = By. \tag{2}$$

This problem has been studied by several researchers; see, for example, Censor and Segal [4], Moudafi [7], Zhao [17], and references therein. Clearly, the SEP is a special case of the SFP. Since the introduction of the Split Feasibility Problem above, many authors have modified it to solve common fixed-point problems; see, for example, Hojo and Takahashi [14], Wang and Kim [15], Shehu *et al.* [13], Zegeye [16], Ofoedu and Araka [9], Nnubia *et al.* [8], and references therein.

Let X_i (where $i = 1, \ldots, m$), X be Banach spaces, $D_i \subset X_i$, $D \subset X$ be closed convex nonempty subsets of the respective Banach spaces, and let $A_i : X_i \to X$ (for $i = 1, 2, \ldots, m$) be bounded linear operators. The Multiple Split Feasibility Problem (MSFP) consists in finding

$$x_i \in D_i \text{ such that } A_i x_i \in D \quad (i = 1, \ldots, m). \tag{3}$$

Let $A = (A_1, A_2, \ldots, A_m)$ and $\bar{x} = (x_1, x_2, \ldots, x_m)$. Then $A : \prod_{i=1}^m X_i \to X^m$ is multilinear and multibounded (i.e., bounded and linear in each argument). The MSFP becomes the problem of finding

$$\bar{x} \in \prod_{i=1}^{m} D_i \text{ such that } A\bar{x} \in D^m. \tag{4}$$

Thus, the MSFP is an extension and generalization of SFP, since $m = 2$ yields the SFP.

Similarly, the Multiple Split Equality Problem (MSEP) also becomes the problem of finding

$$\bar{x} \in \prod_{i=1}^{m} D_i \text{ such that } A\bar{x} \in D_0^m, \tag{5}$$

where $D_0^m = \{(x, x, \ldots, x) \mid x \in D\} = \{(x_1, \ldots, x_m) \in D^m \mid x_i = x_j,\ i, j = 1, \ldots, m\} \subset D^m$.

In the above setting, the Multiple Split Equality Problem (MSEP) consists in finding

$$x_i \in D_i \quad (i = 1, \ldots, m) \text{ such that } A_i X_i = A_j X_j \tag{6}$$

(all $i, j = 1, \ldots, m$), so that the MSEP is an extension and generalization of SEP.

2 Preliminaries

It is our purpose in this work to introduce the Multiple Split Equality Problem (MSEP), provide an algorithm for constructing a solution to MSEP, and establish the strong convergence of the algorithm to such a solution.

We shall make use of the following lemmas.

Lemma 2.1. *Let E be a real normed linear space with a single-valued generalized duality mapping, and let $1 < p < \infty$. Then, for all $x, y \in E$, the following inequality holds:*

$$\|x + y\|^p \leq \|x\|^p + p\langle y, J_E^p(x + y)\rangle.$$

For $E = H$ and $x, y, z \in H$, the following also hold:

1. $\|x - y + z\|^2 - 2\langle z, x - y\rangle \geq \|x - y\|^2$,
2. $\|x + y\|^2 = \|x\|^2 + 2\langle y, x\rangle + \|y\|^2$.

Lemma 2.2. *For any x, y, z in a real Hilbert space H and a real number $\lambda \in [0, 1]$,*

$$\|\lambda x + (1 - \lambda)y - z\|^2 = \lambda\|x - z\|^2 + (1 - \lambda)\|y - z\|^2 - \lambda(1 - \lambda)\|x - y\|^2.$$

Lemma 2.3. *[14] Let K be a closed convex nonempty subset of a real Hilbert space H. Let $x \in H$, then $x_0 = P_K x$ if and only if*

$$\langle z - x_0, x - x_0 \rangle \leq 0 \quad \forall\, z \in K.$$

Lemma 2.4. *Let $\{\Gamma_n\}$ be a sequence of real numbers that does not decrease at infinity in the sense that there exists a subsequence $\{\Gamma_{n_j}\}$ of $\{\Gamma_n\}$ which satisfies $\Gamma_{n_j} < \Gamma_{n_j+1}$ for all $j \in \mathbb{N}$. Define the sequence $\{\tau(n)\}_{n \geq n_0}$ of integers as follows:*

$$\tau(n) = \max\{k \leq n_0 : \Gamma_k < \Gamma_{k+1}\},$$

where $n_0 \in \mathbb{N}$ and the set $\{k \leq n_0 : \Gamma_k < \Gamma_{k+1}\}$ is not empty. Then, the following hold:

1. *$\tau(n_0) \leq \tau(n_0 + 1)$ and $\tau(n) \to \infty$ as $n \to \infty$,*
2. *$\Gamma_{\tau(n)} \leq \Gamma_{\tau(n+1)}$ and $\Gamma_n \leq \Gamma_{\tau(n+1)}$ for all $n \in \mathbb{N}$.*

Lemma 2.5. *Let $\{x_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$x_{n+1} \leq x_n - \alpha_n x_n + \delta_n,\ n \geq n_0,$$

where $\{\alpha_n\}_{n\geq 1} \subset (0,1)$ and $\{\delta_n\}_{n \geq 1} \subset \mathbb{R}$ satisfy the following conditions: $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$, and $\limsup_{n\to\infty} \delta_n \leq 0$, then $\lim_{n\to\infty} x_n = 0$.

3 Main Result

Theorem 3.1. *Let H_i $(i = 1, ..., m)$ be Hilbert spaces, H another Hilbert space, and $A_i : H_i \to H$ be bounded linear operators with adjoint operators A_i^* $(i = 1, ..., m)$.*

Define $\Omega = \{(x_1, x_2, ..., x_m) \in \Pi_{i=1}^m H_i : A_i x_i = A_j x_j$ for $i, j = 1, ..., m\}$.

Starting with arbitrary $u_i \in H_i$ $(i = 1, ..., m)$, *define the iterative sequence* $\{x_{i,n}\}$ *by*

$$\begin{aligned} x_{i,n+1} &= \alpha_n u_i + (1-\alpha_n) y_{i,n} \\ y_{i,n} &= x_{i,n} - \beta_n A_i^*(A_i x_{i,n} - A_j x_{j,n}) \ : i \neq j. \end{aligned} \tag{7}$$

Suppose that $\Omega \neq \emptyset$, *then* $\{x_{i,n}\}$ *is bounded for all* $i \in \{1, 2, ..., m\}$ *provided that*

1. $\{\alpha_n\}_{n\geq 0} \subset (0,1)$ *such that* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$.

2. $2m - \beta_n \sum_{i=1}^{m} \|A_i\|^2 > 0$.

Proof. Let $(x_1^*, x_2^*, , ..., x_m^*) \in \Omega$, from (7), Lemma 2.1 we have

$$\begin{aligned} \|y_{i,n} - x_i^*\|^2 &= \|x_{i,n} - x_i^* - \beta_n A_i^*(A_i x_{i,n} - A_j x_{j,n})\|^2 \\ &= \|x_{i,n} - x_i^*\|^2 - 2\beta_n \langle A_i x_{i,n} - A_j x_{j,n}, A_i^* x_{i,n} - A_i^* x_i^* \rangle \\ &\quad + \beta_n^2 \|A_i\|^2 \|A_i x_{i,n} - A_j x_{j,n}\|^2. \end{aligned} \tag{8}$$

Moreover, with Lemma 2.3 and the hypothesis, we have

$$\begin{aligned} \|x_{i,n+1} - x_i^*\|^2 &= \|\alpha_n(u_i - x_i^*) - (1-\alpha_n)(y_{i,n} - x_i^*)\|^2 \\ &= \alpha_n \|u_i - x_i^*\|^2 + (1-\alpha_n)\|y_{i,n} - x_i^*\|^2 - \alpha_n(1-\alpha_n)\|u_i - y_{i,n}\|^2 \\ &= \alpha_n \|u_i - x_i^*\|^2 + (1-\alpha_n)\|x_{i,n} - x_i^*\|^2 \\ &\quad + (1-\alpha_n)\beta_n^2 \|A_i\|^2 \|A_i x_{i,n} - A_j x_{j,n}\|^2 \\ &\quad - 2(1-\alpha_n)\beta_n \langle A_i x_{i,n} - A_j x_{j,n}, A_i^* x_{i,n} - A_i^* x_i^* \rangle \\ &\quad - \alpha_n(1-\alpha_n)\|u_i - y_{i,n}\|^2. \end{aligned} \tag{9}$$

Define $D_n(x_1^*, x_2^*, ..., x_m^*) = \sum_{i=1}^{m} \|x_{i,n} - x_i^*\|^2$.

Then,

$$
\begin{aligned}
& D_{n+1}(x_1^*, x_2^*, , ..., x_m^*) \\
& \leq \alpha_n \sum_{i=1}^{m} \|u_i - x_i^*\|^2 + (1-\alpha_n) \sum_{i=1}^{m} \|x_{i,n} - x_i^*\|^2 \\
& -(1-\alpha_n)\beta_n(2m - \beta_n \sum_{i=1}^{m} \|A_i\|^2) \sum_{i=1}^{m} \|A_{i,n}x_{i,n} - A_j x_{j,n})\|^2 \\
& -\alpha_n(1-\alpha_n) \sum_{i=1}^{m} \|u_{i,n} - y_{i,n}\|^2.
\end{aligned} \tag{10}
$$

Since $2m - \beta_n \sum_{i=1}^{m} \|A_i\|^2 \geq 0$, we have that

$$
D_{n+1}(x_1^*, ..., x_m^*) \leq (1-\alpha_n) D_n(x_1^*, ..., x_m^*) + \alpha_n \sum_{i=1}^{m} \|u_i - x_i^*\|^2. \tag{11}
$$

Using mathematical induction, we show that the sequence $\{D_{n+1}(x_1^*, ..., x_k^*)\}_{n \geq 1}$ is bounded.

Let $d = \max\{\sum_{i=1}^{m} \|x_{i,0} - x_i^*\|^2, \sum_{i=1}^{m} \|u_i - x_i^*\|^2\}$.

So, $D_0(x_1^*, ..., x_k^*) \leq d$, observe that

$$
\begin{aligned}
D_1(x_1^*, ..., x_m^*) & \leq (1-\alpha_0) D_0(x_1^*, ..., x_m^*) + \alpha_0 \sum_{i=1}^{m} \|u_i - x_i^*\|^2 \\
& = (1-\alpha_0)d + \alpha_0 d = d.
\end{aligned}
$$

Suppose $D_k(x_1^*, ..., x_m^*) \leq d$, then,

$$
\begin{aligned}
D_{k+1}(x_1^*, ..., x_m^*) & \leq (1-\alpha_k) D_k(x_1^*, ..., x_m^*) + \alpha_k \sum_{i=1}^{m} \|u_i - x_i^*\|^2 \\
& \leq (1-\alpha_k)d + \alpha_k d = d
\end{aligned}
$$

so that $\{D_n(x_1^*, ..., x_m^*)\}_{n\geq 1}$ is bounded and hence $\{x_{i,n}\}_{n\geq 1}$ is bounded $\forall\, i, ..., m$.

□

Theorem 3.2. *Let H_i $(i = 1, ..., m)$ be Hilbert spaces, H another Hilbert space, and $A_i : H_i \to H$ be bounded linear operators with adjoint operators A_i^* $(i = 1, ..., m)$. Let Ω be as defined in Theorem 3.1. Consider the sequence $\{x_{i,n}\}$ defined by (7). Then, $\{x_{i,n}\}_{n\geq 0}$, where $i \in \{1, 2, ..., m\}$, converges strongly to an element $(\hat{x}_1, \hat{x}_2, ..., \hat{x}_k) \in P_\Omega(u_1, u_2, ..., u_m)$.*

Proof. From Theorem 3.1, we know that $\{x_{i,n}\}_{n\geq 1}$ is bounded for all $i, ..., m$.

Let $(\hat{x}_1, \hat{x}_2, ..., \hat{x}_m) \in P_\Omega(u_1, u_2, ..., u_m)$ for all $(y_1, y_2, ..., y_k) \in \Omega$.

From Lemma 2.3:

$$\begin{aligned}&\langle (y_1, y_2, ..., y_m) - (\hat{x}_1, \hat{x}_2, ..., \hat{x}_m), (u_1, u_2, ..., u_m) - (\hat{x}_1, \hat{x}_2, ..., \hat{x}_m)\rangle \\ &= \langle (y_1 - \hat{x}_1, y_2 - \hat{x}_2, ..., y_m - \hat{x}_m), (u_1 - \hat{x}_1, u_2 - \hat{x}_2, ..., u_m - \hat{x}_m)\rangle \leq 0.\end{aligned}$$

From Lemma 2.1, Theorem 3.1, and the hypothesis, we conclude that

$$\begin{aligned}
\|x_{i,n+1} - \hat{x}_i\|^2 &= \|\alpha_n u_i + (1-\alpha_n) y_{i,n} - \hat{x}_i\|^2 \\
&= \|\alpha_n u_i + (1-\alpha_n) y_{i,n} - \hat{x}_i - \alpha_n(u_i - \hat{x}_i) + \alpha_n(u_i - \hat{x}_i)\|^2 \\
&\leq \|\alpha_n u_i + (1-\alpha_n) y_{i,n} - \hat{x}_i - \alpha_n(u_i - \hat{x}_i)\|^2 \\
&\quad +2\alpha_n \langle u_i - \hat{x}_i, x_{i,n+1} - \hat{x}_i\rangle \\
&= \|\alpha_n \hat{x}_i + (1-\alpha_n) y_{i,n} - \hat{x}_i\|^2 + 2\alpha_n \langle u_i - \hat{x}_i, x_{i,n+1} - \hat{x}_i\rangle \\
&= (1-\alpha_n)^2 \|y_{i,n} - \hat{x}_i\|^2 + 2\alpha_n \langle u_i - \hat{x}_i, x_{i,n+1} - \hat{x}_i\rangle \\
&\leq (1-\alpha_n) \|y_{i,n} - \hat{x}_i\|^2 + 2\alpha_n \langle u_i - \hat{x}_i, x_{i,n+1} - \hat{x}_i\rangle \\
&\leq (1-\alpha_n) \|x_{i,n} - x_i^*\|^2 + 2\alpha_n \langle u_i - \hat{x}_i, x_{i,n+1} - \hat{x}_i\rangle \\
&\quad +(1-\alpha_n)\beta_n^2 \|A_i\|^2 \|A_i x_{i,n} - A_j x_{j,n}\|^2 \\
&\quad -2(1-\alpha_n)\beta_n \langle A_i x_{i,n} - A_j x_{j,n}, A_i^* x_{i,n} - A_i^* x_i^*\rangle
\end{aligned}$$

so,

$$\begin{aligned} D_{n+1}(\hat{x}_1,...,\hat{x}_m) \leq & \ (1-\alpha_n)D_n(\hat{x}_1,...,\hat{x}_m) + 2\alpha_n \sum_{i=1}^{m} \langle u_i - \hat{x}_i, x_{i,n+1} - \hat{x}_i \rangle \\ & -(1-\alpha_n)\beta_n(2m - \beta_n \sum_{i=1}^{m} \|A_i\|^2) \sum_{i=1, i\neq j}^{m} \|A_i x_{i,n} - A_j x_{j,n}\|. \end{aligned} \tag{12}$$

Hence

$$\begin{aligned} D_{n+1}(\hat{x}_1,...,\hat{x}_m) \leq & \ (1-\alpha_n)D_n(\hat{x}_1,...,\hat{x}_m) + 2\alpha_n \sum_{i=1}^{m} \|u_i - \hat{x}_i\|\|x_{i,n+1} - x_{i,n}\| \\ & +2\alpha_n \sum_{i=1}^{m} \langle u_i - \hat{x}_i, x_{i,n} - \hat{x}_i \rangle. \end{aligned} \tag{13}$$

Furthermore, we show that $\{D_n(\hat{x}_1,...,\hat{x}_m)\}$ converges strongly to zero. We discern two possible cases

Case 1: Suppose that the real sequence $\{D_n(\hat{x}_1,...,\hat{x}_m)\}_{n\geq 1}$ is non-increasing for $n \geq n_0$, for some $n_0 \in \mathbb{N}$. This implies that $\{D_n(\hat{x}_1,...,\hat{x}_m)\}_{n\geq 1}$ is monotonic and bounded, and hence converges.

Moreover, using (12) and the fact that $\alpha_n \to 0$ as $n \to \infty$, we have that for all $i, j \in \{1, 2, ..., m\}$, $j \neq i$:
$\lim_{n\to\infty} \|A_i x_{i,n} - A_j x_{j,n}\| = 0$,
$\lim_{n\to\infty} \|y_{i,n} - x_{i,n}\| = \lim_{n\to\infty} \beta_n \|A_i x_{i,n} - A_j x_{j,n}\| = 0$,
$\lim_{n\to\infty} \|x_{i,n+1} - y_{i,n}\| = \lim_{n\to\infty} \alpha_n \|u_i - y_{i,n}\| = 0 \quad \forall\, i$,
and
$\lim_{n\to\infty} \|x_{i,n+1} - x_{i,n}\| \leq \lim_{n\to\infty}(\|x_{i,n+1} - y_{i,n}\| + \|y_{i,n} - x_{i,n}\|) = 0 \quad \forall\, i.$

Claim:

$$\limsup_{n\to\infty} \sum_{i=1}^{m} \langle u_i - \hat{x}_i, x_{i,n} - \hat{x}_i \rangle \leq 0.$$

Proof of Claim: Let $\{x_{1,n_l}, x_{2,n_l}, ..., x_{m,n_l}\}_{l\geq 1}$ be a subsequence of

$\{x_{1,n}, x_{2,n}, ..., x_{m,n}\}_{n\geq 1}$ such that

$$\limsup_{n\to\infty} \sum_{i=1}^{m} \langle u_i - \hat{x}_i, x_{i,n} - \hat{x}_i\rangle = \lim_{l\to\infty} \sum_{i=1}^{m} \langle u_i - \hat{x}_i, x_{i,n_l} - \hat{x}_i\rangle$$

so, $\forall\ i$:

$$\limsup_{n\to\infty} \langle u_i - \hat{x}_i, x_{i,n} - \hat{x}_i\rangle = \lim_{l\to\infty} \langle u_i - \hat{x}_i, x_{i,n_l} - \hat{x}_i\rangle.$$

Furthermore, since $H_0 = \Pi_{i=1}^{m} H_i$ is a Hilbert space and reflexive, and $\{x_{1,n_l}, x_{2,n_l}, ..., x_{m,n_l}\}_{l\geq 1}$ is a bounded sequence in H_0, there exists a subsequence $\{x_{1,n_{l_t}}, x_{2,n_{l_t}}, ..., x_{m,n_{l_t}}\}_{t\geq 1}$ of $\{x_{1,n_l}, x_{2,n_l}, ..., x_{m,n_l}\}_{l\geq 1}$ that converges weakly to $(\hat{x}_1, ..., \hat{x}_m) \in H$; that is, $x_{i,n_{l_t}} \to^{w} \hat{x}_i$ as $t \to \infty$. Hence, the subsequences $\{y_{i,n_{l_t}}\} \subset \{y_{i,n_l}\}$ converge weakly to $\hat{x}_i$ for all i.

Moreover, for all i and $j \neq i$:

$$\begin{aligned}\|A_i x_i^* - A_j x_j^*\|^2 \leq \|&A_i x_{i,n_{l_t}} - A_j x_{j,n_{l_t}}\|^2 \\ &+ 2\langle A_i x_i^* - A_i x_{i,n_{l_t}} + A_j x_{j,n_{l_t}} - A_j x_j^*, A_i x_{i,n_{l_t}} - A_j x_{i,n_{l_t}}\rangle.\end{aligned}$$

Since $x_{i,n_{l_t}} \to^{w} x_i^*$ as $t \to \infty$ for all i, we have that $A_i x_{i,n_{l_t}} \to^{w} A_i x_i^*$ as $t \to \infty$. Thus, taking limits on both sides and using the fact that $A_i x_i^* = A_j x_j^*$ for all $i, j \in \{1, 2, ..., m\}$, we conclude that $(x_1^*, x_2^*, ..., x_m^*) \in \Omega$.

$$\begin{aligned}&\limsup_{t\to\infty} \sum_{i=1}^{m} \langle u_i - x_i^*, x_{i,n} - x_i^*\rangle \\ &= \lim_{t\to\infty} \langle (x_{1,n_l}, x_{2,n_l}, ..., x_{m,n_l}) - (x_1^*, x_2^*, ..., x_m^*), (x_{1,0}, x_{2,0}, ..., x_{m,0}) - (x_1^*, x_2^*, ..., x_m^*)\rangle \\ &= \langle (x_1^*, x_2^*, ..., x_m^*) - (x_1^*, x_2^*, ..., x_m^*), (u_1, x_2, ..., x_m) - (x_1^*, x_2^*, ..., x_m^*)\rangle \leq 0.\end{aligned}$$

Now, from (12) and (13), we have

$$\begin{aligned}D_{n+1}(\hat{x}_1, ..., \hat{x}_m) \leq (1 - \alpha_n)&D_n(\hat{x}_1, ..., \hat{x}_m) \\ &+ 2\alpha_n \sum_{i=1}^{m} \langle u_i - \hat{x}_i, x_{i,n} - \hat{x}_i\rangle \\ &+ 2\alpha_n \sum_{i=1}^{m} \|u_i - \hat{x}_i\| \|x_{i,n+1} - x_{i,n}\|.\end{aligned}$$

So, by Lemma 2.5, $\{D_n(\hat{x}_1, ..., \hat{x}_m)\}_{n\geq 0}$ converges strongly to zero as $n \to \infty$. Hence, $(x_{1,n}, x_{2,n}, ..., x_{m,n})$ converges strongly to $(\hat{x}_1, ..., \hat{x}_m) \in \Omega$. This completes the proof of this Case.

Case 2: Suppose there exists a subsequence $\{D_{n_s}(\hat{x}_1, ..., \hat{x}_m)\}$ of $\{D_n(\hat{x}_1, ..., \hat{x}_m)\}$ such that $D_{n_s}(\hat{x}_1, ..., \hat{x}_m) \leq D_{n_s+1}(\hat{x}_1, ..., \hat{x}_m)$ for all $s \in \mathbb{N}$. Then, by Lemma 2.4, there exists a non-decreasing sequence $\{q_c\}_{c\geq 1} \subset \mathbb{N}$ such that:

(i) $\lim_{n\to\infty} q_c = \infty$

(ii) $D_{q_c}(\hat{x}_1, ..., \hat{x}_m) \leq D_{q_c+1}(\hat{x}_1, ..., \hat{x}_m)$ for all $c \in \mathbb{N}$.

Since the sequence $\{x_{i,q_c}\}_{c\geq 1}$ for $i = 1, ..., m$ is bounded, we obtain from (12) and using the arguments earlier:

$$\|y_{i,q_c} - x_{i,q_c}\| \to 0, \quad \text{as } c \to \infty \quad \forall\, i$$

and

$$\|x_{i,q_c+1} - x_{i,q_c}\| \to 0, \quad \text{as } c \to \infty \quad \forall\, i.$$

Also,

$$\limsup_{t\to\infty} \sum_{i=1}^{m} \langle u_i - \hat{x}_i, x_{i,q_c} - \hat{x}_i\rangle \leq 0.$$

Since $D_{q_c}(\hat{x}_1, ..., \hat{x}_m) \leq D_{q_c+1}(\hat{x}_1, ..., \hat{x}_m)$, we have:

$$\begin{aligned} \alpha_{q_c} D_{q_c}(\hat{x}_1, ..., \hat{x}_m) \leq{}& D_{q_c}(\hat{x}_1, ..., \hat{x}_m) - D_{q_c+1}(\hat{x}_1, ..., \hat{x}_m) \\ &+ 2\alpha_{q_c} \sum_{i=1}^{m} \langle u_i - \hat{x}_i, x_{i,q_c} - \hat{x}_i\rangle \\ &+ 2\alpha_{q_c} \sum_{i=1}^{m} \|u_i - \hat{x}_i\| \|x_{i,q_c+1} - xi, q_c\|. \end{aligned}$$

Dividing through by α_{q_c} and taking limits as $c \to \infty$, we get:

$$D_{q_c}(\hat{x}_1, ..., \hat{x}_m) \leq 2\sum_{i=1}^{m} \langle u_i - \hat{x}_i, x_{i,q_c} - \hat{x}_i\rangle + 2\sum_{i=1}^{m} \|u_i - \hat{x}_i\| \|x_{i,q_c+1} - x_{i,q_c}\|$$

and

$$D_{q_c+1}(\hat{x}_1, ..., \hat{x}_m) \to 0 \quad \text{as } c \to \infty.$$

Since $D_c(\hat{x}_1, ..., \hat{x}_m) \leq D_{q_c+1}(\hat{x}_1, ..., \hat{x}_m)$, then $D_c(\hat{x}_1, ..., \hat{x}_m) \to 0$ as $c \to \infty$. This implies that

$$\lim_{c\to\infty} \|x_{i,c} - \hat{x}_i\| = 0 \quad \forall\, i,$$

so that $x_{i,c} \to \hat{x}_i$ as $c \to \infty$. Thus, $(x_{1,n}, x_{2,n}, ..., x_{m,n}) \to (\hat{x}_1, ..., \hat{x}_m)$ as $n \to \infty$. This completes the proof of the second case and thus the proof of the Theorem. □

4 Concluding Remarks

The Multiple Split Equality Problem (MSEP) has been introduced in this work, and an algorithm for constructing its solution has been provided. Moreover, the strong convergence of the algorithm to such a solution has been established.

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