Earthline Journal of Mathematical Sciences
E-ISSN: 2581-8147
Volume 14, Number 1, 2024, Pages 151-159
https://doi.org/10.34198/ejms.14124.151159

The (ψ, φ)-Generalized Weakly Reich Contraction Mapping Theorem

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Abstract

In this paper we introduce a concept of (ψ, φ)-generalized weakly Reich contraction mapping and obtain a fixed point theorem. Some corollaries are consequences of the main result.

1 Introduction and Preliminaries

Theorem 1.1 ([1], [2]). *Let (X, d) be a complete metric space. If $T : X \mapsto X$ satisfies*

$$d(Tx, Ty) \leq k[d(x, Tx) + d(y, Ty)]$$

where $0 < k < \frac{1}{2}$ and $x, y \in X$, then T has a unique fixed point.

Theorem 1.2 ([3]). *Let (X, d) be a complete metric space. If $T : X \mapsto X$ satisfies*

$$d(Tx, Ty) \leq k[d(x, Ty) + d(y, Tx)]$$

where $0 < k < \frac{1}{2}$ and $x, y \in X$, then T has a unique fixed point.

Received: October 9, 2023; Accepted: November 14, 2023; Published: November 27, 2023
2020 Mathematics Subject Classification: 54H25, 47H10, 54E50.
Keywords and phrases: metric space, fixed point theorems, (ψ, φ)-generalized weakly Reich contraction mapping.
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Definition 1.3 ([4]). *A self mapping T on a metric space X is called weak ψ-contraction if there exists a function $\psi : [0,\infty) \mapsto [0,\infty)$ such that for each $x, y \in X$*

$$d(Tx, Ty) \leq d(x, y) - \psi(d(x, y)).$$

Definition 1.4 ([5]). *Let (X, d) be a metric space. A mapping $T : X \mapsto X$ is said to be weakly C-contractive (or a weak C-contraction) if for all $x, y \in X$*

$$d(Tx, Ty) \leq \frac{1}{2}[d(x, Ty) + d(y, Tx)] - \varphi(d(x, Ty), d(y, Tx))$$

where $\varphi : [0,\infty)^2 \mapsto [0,\infty)$ is a continuous function such that $\varphi(x, y) = 0$ if and only if $x = y = 0$.

Definition 1.5 ([6]). *Let (X, d) be a metric space. A mapping $T : X \mapsto X$ is said to be generalized f-weakly contractive if for all $x, y \in X$*

$$d(Tx, Ty) \leq \frac{1}{2}[d(fx, Ty) + d(fy, Tx)] - \varphi(d(fx, Ty), d(fy, Tx))$$

where $\varphi : [0,\infty)^2 \mapsto [0,\infty)$ is a continuous function such that $\varphi(x, y) = 0$ if and only if $x = y = 0$.

Remark 1.6. *If f is the identity mapping in the above definition then generalized f-weakly contractive mapping is weakly C-contractive.*

Definition 1.7 ([7]). *The function $\psi : [0,\infty) \mapsto [0,\infty)$ is called an altering distance function if the following properties are satisfied*

(a) *ψ is continuous and non-decreasing.*

(b) *$\psi(t) = 0$ if and only if $t = 0$.*

Definition 1.8 ([8]). *A map $T : X \mapsto X$ is called (ψ, φ)-generalized f-weakly contractive if for each $x, y \in X$*

$$\psi(d(Tx, Ty)) \leq \psi\left(\frac{1}{2}[d(fx, Ty) + d(fy, Tx)]\right) - \varphi(d(fx, Ty), d(fy, Tx))$$

where

(a) $\psi : [0, \infty) \mapsto [0, \infty)$ *is an altering distance function;*

(b) $\varphi : [0, \infty)^2 \mapsto [0, \infty)$ *is a continuous function with* $\varphi(t, s) = 0$ *if and only if* $t = s = 0$.

Remark 1.9. *If in the above definition we take* $\psi(t) = t$, *then* (ψ, φ)*-generalized f-weakly contractive mapping is generalized f-weakly contractive.*

Definition 1.10 ([8]). *Let* X *be a nonempty set. A point* $x \in X$ *is a coincidence point (common fixed point) of* $f : X \mapsto X$ *and* $T : X \mapsto X$ *if* $fx = Tx$ ($x = fx = Tx$).

Definition 1.11 ([8]). *Let* X *be a nonempty set. The pair* $\{f, T\}$ *is called commuting if* $Tfx = fTx$ *for all* $x \in X$.

In this paper we study some common fixed point theorem for (ψ, φ)-generalized weakly Reich contraction in metric spaces.

2 Main Result

Definition 2.1. *A map* $T : X \mapsto X$ *will be called* (ψ, φ)*-generalized weakly Reich contractive if for each* $x, y \in X$

$$\psi(d(Tx, Ty)) \leq \psi\left(\frac{1}{3}[d(fx, Tx) + d(fy, Ty) + d(fx, fy)]\right) - \varphi(d(fx, Tx), d(fy, Ty), d(fx, fy))$$

where

(a) $\psi : [0, \infty) \mapsto [0, \infty)$ *is an altering distance function*

(b) $\varphi : [0, \infty)^3 \mapsto [0, \infty)$ *is a continuous function with* $\varphi(x, y, z) = 0$ *if and only if* $x = y = z = 0$.

Theorem 2.2. *Let (X,d) be a metric space. Let $f, T : X \mapsto X$ satisfy $T(X) \subseteq f(X)$, $(f(X), d)$ is complete and for each $x, y \in X$*

$$\psi(d(Tx,Ty)) \leq \psi\left(\frac{1}{3}[d(fx,Tx)+d(fy,Ty)+d(fx,fy)]\right) - \varphi(d(fx,Tx),d(fy,Ty),d(fx,fy))$$

where

(a) $\psi : [0,\infty) \mapsto [0,\infty)$ *is an altering distance function,*

(b) $\varphi : [0,\infty)^3 \mapsto [0,\infty)$ *is a continuous function with $\varphi(x,y,z) = 0$ if and only if $x = y = z = 0$,*

then T and f have a coincidence point in X. Further, if T and f commute at their coincidence points, then T and f have a common fixed point.

Proof. Let $x_0 \in X$. Since $T(X) \subseteq f(X)$, we can choose $x_1 \in X$ so that $fx_1 = Tx_0$. Since $Tx_1 \in f(X)$, there exist $x_2 \in X$ such that $fx_2 = Tx_1$. By induction we construct a sequence $\{x_n\}$ in X such that $fx_{n+1} = Tx_n$ for every $n \geq 0$. Since T is a (ψ,φ)-generalized weakly Reich contraction, we deduce the following

$$\begin{aligned}
\psi(d(Tx_{n+1},Tx_n)) &\leq \psi\left(\frac{1}{3}[d(fx_{n+1},Tx_{n+1})+d(fx_n,Tx_n)+d(fx_{n+1},fx_n)]\right)\\
&- \varphi(d(fx_{n+1},Tx_{n+1}),d(fx_n,Tx_n),d(fx_{n+1},fx_n))\\
&\leq \psi\left(\frac{1}{3}[d(Tx_n,Tx_{n+1})+d(Tx_{n-1},Tx_n)+d(Tx_n,Tx_{n-1})\right)\\
&- \varphi(d(Tx_n,Tx_{n+1}),d(Tx_{n-1},Tx_n),d(Tx_n,Tx_{n-1}))\\
&\leq \psi\left(\frac{1}{3}[d(Tx_n,Tx_{n+1})+d(Tx_{n-1},Tx_n)+d(Tx_n,Tx_{n-1})\right)\\
&\leq \psi\left(\frac{1}{3}[3d(Tx_n,Tx_{n-1})\right)\\
&= \psi(d(Tx_n,Tx_{n-1})).
\end{aligned}$$

Hence for all $n = 1, 2, \cdots$, we have $d(Tx_{n+1}, Tx_n) \leq d(Tx_n, Tx_{n-1})$. Since ψ is a non-decreasing function. Thus, $\{d(Tx_{n+1}, Tx_n)\}$ is a monotone decreasing sequence of non-negative real numbers and hence is convergent. Let $\lim_{n\to\infty} d(Tx_{n+1}, Tx_n) = r$. Now, since

$$\begin{aligned}\psi(d(Tx_{n+1}, Tx_n)) \leq \psi\Big(\frac{1}{3}[d(Tx_n, Tx_{n+1}) + d(Tx_{n-1}, Tx_n) + d(Tx_n, Tx_{n-1})\Big)\\ - \varphi(d(Tx_n, Tx_{n+1}), d(Tx_{n-1}, Tx_n), d(Tx_n, Tx_{n-1})).\end{aligned}$$

If we take limits in the above inequality as $n \to \infty$ we get that

$$\begin{aligned}\psi(r) &\leq \psi\Big(\frac{1}{3}(r + r + r)\Big) - \varphi(r, r, r)\\ &= \psi(r) - \varphi(r, r, r)\end{aligned}$$

which implies that $\varphi(r, r, r) \leq 0$. Thus $r = 0$, and hence $\lim_{n\to\infty} d(Tx_{n+1}, Tx_n) = 0$. Now we show that $\{Tx_n\}$ is a Cauchy sequence. If otherwise, then there exist $\epsilon > 0$ for which we can find subsequences $\{Tx_{m(k)}\}$ and $\{Tx_{n(k)}\}$ of $\{Tx_n\}$ with $n(k) > m(k) > k$ such that for every k, $d(Tx_{m(k)}, Tx_{n(k)}) \geq \epsilon$,$d(Tx_{m(k)}, Tx_{n(k)-1}) < \epsilon$. So we have,

$$\begin{aligned}\epsilon &\leq d(Tx_{m(k)}, Tx_{n(k)})\\ &\leq d(Tx_{m(k)}, Tx_{n(k)-1}) + d(Tx_{n(k)-1}, Tx_{n(k)})\\ &\leq \epsilon + d(Tx_{n(k)-1}, Tx_{n(k)}).\end{aligned}$$

Letting $n \to \infty$ and using $d(Tx_{n-1}, Tx_n) \to 0$, we have, $\lim d(Tx_{m(k)}, Tx_{n(k)}) = \epsilon = \lim d(Tx_{m(k)}, Tx_{n(k)-1})$. Now we have

$$\begin{aligned}\psi(\epsilon) &\leq \psi(d(Tx_{m(k)}, Tx_{n(k)}))\\ &\leq \psi\Big(\frac{1}{3}[d(fx_{m(k)}, Tx_{m(k)}) + d(fx_{n(k)}, Tx_{n(k)}) + d(fx_{m(k)}, fx_{n(k)})\Big)\\ &- \varphi(d(fx_{m(k)}, Tx_{m(k)}), d(fx_{n(k)}, Tx_{n(k)}), d(fx_{m(k)}, fx_{n(k)}))\end{aligned}$$

which implies that

$$\begin{aligned}\psi(\epsilon) &\leq \psi(d(Tx_{m(k)}, Tx_{n(k)})) \\ &\leq \psi\left(\frac{1}{3}[d(Tx_{m(k)-1}, Tx_{m(k)}) + d(Tx_{n(k)-1}, Tx_{n(k)}) + d(Tx_{m(k)-1}, Tx_{n(k)-1})\right) \\ &- \varphi(d(Tx_{m(k)-1}, Tx_{m(k)}), d(Tx_{n(k)-1}, Tx_{n(k)}), d(Tx_{m(k)-1}, Tx_{n(k)-1})).\end{aligned}$$

If we take limits in the above inequality and using the fact that $\lim_{n\to\infty} d(Tx_{n+1}, Tx_n) = 0$ we deduce the following

$$\begin{aligned}\psi(\epsilon) &\leq \psi\left(\frac{1}{3}[0 + 0 + \epsilon]\right) - \varphi(0, 0, \epsilon) \\ &= \psi\left(\frac{1}{3}\epsilon\right) - \varphi(0, 0, \epsilon) \\ &\leq \psi(\epsilon) - \varphi(0, 0, \epsilon)\end{aligned}$$

which implies that $\varphi(0, 0, \epsilon) \leq 0$, which is a contradiction since $\epsilon > 0$. Hence $\{Tx_n\}$ is a Cauchy sequence and therefore is convergent in the complete metric space (X, d). Since $fx_n = Tx_{n-1}$, $\{fx_n\}$ is a Cauchy sequence in $(f(X), d)$ which is complete. Thus there is $z \in X$ such that $\lim_{n\to\infty} fx_n = fz$. Now we have

$$\begin{aligned}\psi(d(Tz, fx_{n+1})) &= \psi(d(Tz, Tx_n)) \\ &\leq \psi\left(\frac{1}{3}[d(fz, Tz) + d(fx_n, Tx_n) + d(fz, fx_n)]\right) \\ &- \varphi(d(fz, Tz), d(fx_n, Tx_n), d(fz, fx_n)) \\ &= \psi\left(\frac{1}{3}[d(fz, Tz) + d(fx_n, fx_{n+1}) + d(fz, fx_n)]\right) \\ &- \varphi(d(fz, Tz), d(fx_n, fx_{n+1}), d(fz, fx_n)).\end{aligned}$$

Now taking limits as $n \to \infty$ we deduce the following

$$\begin{aligned}\psi(d(Tz, fz)) &\leq \psi\left(\frac{1}{3}d(fz, Tz)\right) - \varphi(d(fz, Tz), 0, 0) \\ &\leq \psi\Big(d(fz, Tz)\Big) - \varphi(d(fz, Tz), 0, 0)\end{aligned}$$

which implies $\varphi(d(fz,Tz),0,0)\leq 0$. Hence, $d(fz,Tz)=0$, thus, $fz=Tz$, and hence z is a coincidence point of T and f. Now suppose that T and f commute at z. Let $w=Tz=fz$, then $T(w)=T(f(z))=f(T(z))=f(w)$. Now we have

$$\begin{aligned}\psi(d(Tz,Tw)) &\leq \psi\Big(\frac{1}{3}[d(fz,Tz)+d(fw,Tw)+d(fz,fw)]\Big)\\ &- \varphi(d(fz,Tz),d(fw,Tw),d(fz,fw))\\ &= \psi\Big(\frac{1}{3}[d(Tz,Tz)+d(Tw,Tw)+d(Tz,Tw)]\Big)\\ &- \varphi(d(Tz,Tz),d(Tw,Tw),d(Tz,Tw))\\ &= \psi\Big(\frac{1}{3}d(Tz,Tw)\Big)-\varphi(0,0,d(Tz,Tw))\\ &\leq \psi\Big(d(Tz,Tw)\Big)-\varphi(0,0,d(Tz,Tw))\end{aligned}$$

which implies that $\varphi(0,0,d(Tz,Tw))\leq 0$. Hence, $d(Tz,Tw)=0$. Therefore, $Tw=fw=w$ and the proof is finished. □

Corollary 2.3. *Let (X,d) be a complete metric space. If $T:X\mapsto X$ satisfies*

$$\psi(d(Tx,Ty))\leq \psi\Big(\frac{1}{3}[d(x,Tx)+d(y,Ty)+d(x,y)]\Big)-\varphi(d(x,Tx),d(y,Ty),d(x,y))$$

where

(a) $\psi:[0,\infty)\mapsto[0,\infty)$ *is an altering distance function*

(b) $\varphi:[0,\infty)^3\mapsto[0,\infty)$ *is a continuous function with $\varphi(x,y,z)=0$ if and only if $x=y=z=0$*

then T has a unique fixed point.

Proof. It follows by taking f to be the identity mapping in the above theorem. The uniqueness of the fixed point follows from the contractive condition of the Corollary. □

Corollary 2.4. *Let (X, d) be a metric space. Let $f, T : X \mapsto X$ satisfy $T(X) \subseteq f(X)$, $(f(X), d)$ is complete and for each $x, y \in X$*

$$d(Tx, Ty) \leq \frac{1}{3}[d(fx, Tx)+d(fy, Ty)+d(fx, fy)]-\varphi(d(fx, Tx), d(fy, Ty), d(fx, fy))$$

where $\varphi : [0, \infty)^3 \mapsto [0, \infty)$ is a continuous function with $\varphi(x, y, z) = 0$ if and only if $x = y = z = 0$, then T and f have a coincidence point in X. Further, if T and f commute at their coincidence points, then T and f have a common fixed point.

Proof. Let $\psi(t) = t$ in the above theorem. □

Corollary 2.5. *Let (X, d) be a complete metric space. If $T : X \mapsto X$ satisfies for all $x, y \in X$,*

$$d(Tx, Ty) \leq \frac{1}{3}[d(x, Tx) + d(y, Ty) + d(x, y)] - \varphi(d(x, Tx), d(y, Ty), d(x, y))$$

where $\varphi : [0, \infty)^3 \mapsto [0, \infty)$ is a continuous function with $\varphi(x, y, z) = 0$ if and only if $x = y = z = 0$, then T has a unique fixed point.

Proof. It follows by taking f to be the identity mapping in the above Corollary. The uniqueness of the fixed point follows from the contractive condition of the Corollary. □

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