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Coefficient Bounds for a New Families of m-Fold Symmetric Bi-Univalent Functions Defined by Bazilevic Convex Functions

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Abstract

In this paper, we find upper bounds for the first two Taylor-Maclaurin $|a_{m+1}|$ and $|a_{2m+1}|$ for two new families $L_{\Sigma_m}(\delta,\gamma;\alpha)$ and $L^*_{\Sigma_m}(\delta,\gamma;\beta)$ of holomorphic and m-fold symmetric bi-univalent functions associated with the Bazilevic convex functions defined in the open unit disk U. Further, we point out several certain special cases for our results.

1. Introduction

Let $\mathcal{A}$ be the family of functions f that are holomorphic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by the conditions $f(0) = f'(0) - 1 = 0$ and having the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{1.1}$$

We also denote by S the subfamily of $\mathcal{A}$ consisting of functions satisfying (1.1) which are also univalent in U.

A function $f \in \mathcal{A}$ is called starlike of order δ $(0 \leq \delta < 1)$, if

$$\text{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \delta, \quad (z \in U).$$

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Singh [10] introduced and studied Bazilevic function that is the function f such that

$$\operatorname{Re}\left\{\frac{z^{1-\gamma} f'(z)}{(f(z))^{1-\gamma}}\right\} > 0, \quad (z \in U, \gamma \geq 0).$$

According to the Koebe one-quarter theorem (see [8]), every function $f \in S$ has an inverse f^{-1} which satisfies

$$f^{-1}(f(z)) = z, \qquad (z \in U)$$

and

$$f(f^{-1}(w)) = w, \qquad \left(|w| < r_0(f), r_0(f) \geq \frac{1}{4}\right),$$

where

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots. \quad (1.2)$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in U if both f and f^{-1} are univalent in U. We denote by Σ the family of bi-univalent functions in U given by (1.1). For a brief history and interesting examples in the family Σ see the pioneering work on this subject by Srivastava et al. [25], which actually revived the study of bi-univalent functions in recent years. In a considerably large number of sequels to the aforementioned work of Srivastava et al. [25], several different sub families of the bi-univalent function family Σ were introduced and studied analogously by the many authors (see, for example, [1,2,3,5,9,10,11,15,19,23,28,29,30,33]).

For each function $f \in S$, the function $h(z) = \sqrt[m]{f(z^m)}$, $(z \in U, m \in \mathbb{N})$ is univalent and maps the unit disk U into a region with m-fold symmetry. A function is said to be m-fold symmetric (see [12]) if it has the following normalized form:

$$f(z) = z + \sum_{k=1}^{\infty} a_{mk+1} z^{mk+1}, \quad (z \in U, m \in \mathbb{N}). \quad (1.3)$$

We denote by S_m the family of m-fold symmetric univalent functions in U, which are normalized by the series expansion (1.3). In fact, the functions in the family S are one-fold symmetric.

In [26] Srivastava et al. defined m-fold symmetric bi-univalent functions analogues to the concept of m-fold symmetric univalent functions. They gave some important results, such as each function $f \in \Sigma$ generates an m-fold symmetric bi-univalent function for each $m \in \mathbb{N}$. Furthermore, for the normalized form of f given by (1.3), they obtained the series expansion for f^{-1} as follows:

$$g(w) = w - a_{m+1}w^{m+1} + [(m+1)a_{m+1}^2 - a_{2m+1}]w^{2m+1}$$
$$-\left[\frac{1}{2}(m+1)(3m+2)a_{m+1}^3 - (3m+2)a_{m+1}a_{2m+1} + a_{3m+1}\right]w^{3m+1} + \cdots, \quad (1.4)$$

where $f^{-1} = g$. We denote by Σ_m the family of m-fold symmetric bi-univalent functions in U. It is easily seen that for $m = 1$, the formula (1.4) coincides with the formula (1.2) of the family Σ. Some examples of m-fold symmetric bi-univalent functions are given as follows:

$$\left(\frac{z^m}{1-z^m}\right)^{\frac{1}{m}}, \ \left[\frac{1}{2}\log\left(\frac{1+z^m}{1-z^m}\right)\right]^{\frac{1}{m}} \text{ and } [-\log(1-z^m)]^{\frac{1}{m}}$$

with the corresponding inverse functions

$$\left(\frac{w^m}{1+w^m}\right)^{\frac{1}{m}}, \ \left(\frac{e^{2w^m}-1}{e^{2w^m}+1}\right)^{\frac{1}{m}} \text{ and } \left(\frac{e^{w^m}-1}{e^{w^m}}\right)^{\frac{1}{m}},$$

respectively.

Recently, many authors investigated bounds for various subfamilies of m-fold bi-univalent functions (see [4,7,13,18,20,21,24,26,27,31,32]).

In order to prove our main results, we require the following lemma.

Lemma 1.1 [3]**.** *If $h \in \mathcal{P}$, then $|c_k| \leq 2$ for each $k \in \mathbb{N}$, where $\mathcal{P}$ is the family of all f all functions h holomorphic in U for which*

$$Re(h(z)) > 0, \qquad (z \in U),$$

where

$$h(z) = 1 + c_1 z + c_2 z^2 + \cdots, \qquad (z \in U).$$

2. Coefficient Estimates for the Function Family $L_{\Sigma_m}(\delta, \gamma; \alpha)$

Definition 2.1. A function $f \in \Sigma_m$ given by (1.3) is said to be in the family $L_{\Sigma_m}(\delta, \gamma; \alpha)$ $(0 < \alpha \leq 1, 0 \leq \delta \leq 1, 0 \leq \gamma \leq 1)$ if it satisfies the following conditions:

$$\left|\arg\left((1-\delta)\frac{z^{1-\gamma}f'(z)}{\left(f(z)\right)^{1-\gamma}}+\delta(1+\frac{z^{2-\gamma}f''(z)}{\left(zf'(z)\right)^{1-\gamma}})\right)\right|<\frac{\alpha\pi}{2}, \qquad (z\in U) \tag{2.1}$$

and

$$\left|\arg\left((1-\delta)\frac{w^{1-\gamma}g'(w)}{\left(g(w)\right)^{1-\gamma}}+\delta(1+\frac{w^{2-\gamma}g''(w)}{\left(wg'(w)\right)^{1-\gamma}})\right)\right|<\frac{\alpha\pi}{2}, \qquad (w\in U), \tag{2.2}$$

where the function $g=f^{-1}$ is given by (1.4).

In particular, for one-fold symmetric bi-univalent functions, we denote the family $L_{\Sigma_1}(\delta,\gamma;\alpha)=L_{\Sigma}(\delta,\gamma;\alpha)$.

Remark 2.1. It should be remarked that the family $L_{\Sigma}(\delta,\gamma;\alpha)$ is a generalization of well-known families consider earlier. These families are:

(1) For $\delta=0$, the family $L_{\Sigma}(\delta,\gamma;\alpha)$ reduce to the family $P_{\Sigma}(\alpha,\gamma)$ which was introduced by Prema and Keerthi [17];

(2) For $\delta=1$, the family $L_{\Sigma}(\delta,\gamma;\alpha)$ reduce to the family $B_{\Sigma}(\gamma,\alpha)$ which was introduced by Sakar and Wanas [22];

(3) For $\delta=\gamma=0$, the family $L_{\Sigma}(\delta,\gamma;\alpha)$ reduce to the family $S^{*}_{\Sigma}(\alpha)$ which was given by Brannan and Taha [6];

(4) $\delta=0$ and $\gamma=1$, the family $L_{\Sigma}(\delta,\gamma;\alpha)$ reduce to the family H^{α}_{Σ} which was investigated by Srivastava et al. [25].

Theorem 2.1. *Let* $f\in L_{\Sigma_m}(\delta,\gamma;\alpha)$ $(0<\alpha\le 1, 0\le\ \delta\le 1,\ 0\le\gamma\le 1,\ m\in\mathbb{N})$ *be given by* (1.3). *Then*

$$|a_{m+1}|\le\frac{2\alpha}{\sqrt{\left|\begin{array}{c}2\alpha\left[(1-\delta)(2m+\gamma)+2m\delta(2m+1)+\frac{1}{2}(1-\delta)(\gamma-1)(2(m+\gamma)+\gamma)\right]\\+(1-\alpha)[(1-\delta)(m+\gamma)+\delta m(m+1)]^2\end{array}\right|}} \tag{2.3}$$

and

$$|a_{2m+1}|\le\frac{4\alpha^2}{[(1-\delta)(m+\gamma)+\delta m(m+1)]^2}+\frac{2\alpha}{(1-\delta)(2m+\gamma)+2m\delta(2m+1)}. \tag{2.4}$$

Proof. It follows from conditions (2.1) and (2.2) that

$$\left((1-\delta)\frac{z^{1-\gamma}f'(z)}{\big(f(z)\big)^{1-\gamma}}+\delta(1+\frac{z^{2-\gamma}f''(z)}{(zf'(z))^{1-\gamma}}\right)=[p(z)]^{\alpha} \tag{2.5}$$

and

$$\left((1-\delta)\frac{w^{1-\gamma}g'(w)}{\big(g(w)\big)^{1-\gamma}}+\delta\,(1+\frac{w^{2-\gamma}g''(w)}{(wg'(w))^{1-\gamma}}\right)=[q(w)]^{\alpha}, \tag{2.6}$$

where $g=f^{-1}$ and p,q in $\mathcal{P}$ have the following series representations:

$$p(z)=1+p_m z^m+p_{2m}z^{2m}+p_{3m}z^{3m}+\cdots \tag{2.7}$$

and

$$q(w)=1+q_m w^m+q_{2m}w^{2m}+q_{3m}w^{3m}+\cdots. \tag{2.8}$$

Comparing the corresponding coefficients of (2.5) and (2.6) yields

$$[(1-\delta)(m+\gamma)+\delta m(m+1)]\,a_{m+1}=\alpha p_m \tag{2.9}$$

$$[(1-\delta)(2m+\gamma)+2m\delta(2m+1)]a_{2m+1}+\frac{1}{2}(1-\delta)(\gamma-1)(2(m+\gamma)+\gamma)a_{m+1}^2$$
$$=\alpha p_{2m}+\frac{\alpha(\alpha-1)}{2}p_m^2, \tag{2.10}$$

$$-[(1-\delta)(m+\gamma)+\delta m(m+1)]a_{m+1}=\alpha q_m \tag{2.11}$$

and

$$[(1-\delta)(2m+\gamma)+2m\delta(2m+1)](2a_{m+1}^2-a_{2m+1})$$
$$+\frac{1}{2}(1-\delta)(\gamma-1)(2(m+\gamma)+\gamma)a_{m+1}^2=\alpha q_{2m}+\frac{\alpha(\alpha-1)}{2}q_m^2. \tag{2.12}$$

Making use of (2.9) and (2.11), we obtain

$$p_m=-q_m \tag{2.13}$$

and

$$2[(1-\delta)(m+\gamma)+\delta m(m+1)]^2a_{m+1}^2=\alpha^2(p_m^2+q_m^2). \tag{2.14}$$

Also, from (2.10), (2.12) and (2.14), we find that

$$2\left[(1-\delta)(2m+\gamma)+2m\delta(2m+1)+\frac{1}{2}(1-\delta)(\gamma-1)(2(m+\gamma)+\gamma)\right]a_{m+1}^2$$

$$= \alpha(p_{2m} + q_{2m}) + \frac{\alpha(\alpha-1)}{2}(p_m^2 + q_m^2)$$

$$= \alpha(p_{2m} + q_{2m}) + \frac{(\alpha-1)[(1-\delta)(m+\gamma)+\delta m(m+1)]^2}{\alpha} a_{m+1}^2.$$

Therefore, we have

$$a_{m+1}^2 = \frac{\alpha^2(p_{2m}+q_{2m})}{2\alpha\left[(1-\delta)(2m+\gamma)+2m\delta(2m+1)+\frac{1}{2}(1-\delta)(\gamma-1)(2(m+\gamma)+\gamma)\right] + (1-\alpha)[(1-\delta)(m+\gamma)+\delta m(m+1)]^2}. \quad (2.15)$$

Now, taking the absolute value of (2.15) and applying Lemma 1.1 for the coefficients p_{2m} and q_{2m}, we obtain

$$|a_{m+1}| \leq \frac{2\alpha}{\sqrt{\left|2\alpha\left[(1-\delta)(2m+\gamma)+2m\delta(2m+1)+\frac{1}{2}(1-\delta)(\gamma-1)(2(m+\gamma)+\gamma)\right] + (1-\alpha)[(1-\delta)(m+\gamma)+\delta m(m+1)]^2\right|}}.$$

This gives the desired estimate for $|a_{m+1}|$ as asserted in (2.3).

In order to find the bound on $|a_{2m+1}|$, by subtracting (2.12) from (2.10), we get

$$2[(1-\delta)(2m+\gamma)+2m\delta(2m+1)](a_{2m+1}-a_{m+1}^2)$$

$$= \alpha(p_{2m}-q_{2m}) + \frac{\alpha(\alpha-1)}{2}(p_m^2-q_m^2). \quad (2.16)$$

It follows from (2.13), (2.14) and (2.16) that

$$a_{2m+1} = \frac{\alpha^2(p_m^2+q_m^2)}{2[(1-\delta)(m+\gamma)+\delta m(m+1)]^2} + \frac{\alpha(p_{2m}-q_{2m})}{2[(1-\delta)(2m+\gamma)+2m\delta(2m+1)]}. \quad (2.17)$$

Taking the absolute value of (2.17) and applying Lemma 1.1 once again for the coefficients p_m, p_{2m}, q_m and q_{2m}, we obtain

$$|a_{2m+1}| \leq \frac{4\alpha^2}{[(1-\delta)(m+\gamma)+\delta m(m+1)]^2} + \frac{2\alpha}{(1-\delta)(2m+\gamma)+2m\delta(2m+1)}.$$

For one-fold symmetric bi-univalent functions, Theorem 2.1 reduces to the following corollary:

Corollary 2.1. *Let* $f \in L_\Sigma(\delta,\gamma;\alpha)$ $(0 < \alpha \leq 1, 0 \leq \delta \leq 1, 0 \leq \gamma \leq 1)$ *be given by* (1.1). *Then*

$$|a_2| \leq \frac{2\alpha}{\sqrt{\left|2\alpha[(1-\delta)(2+\gamma)+6\delta+\frac{1}{2}(1-\delta)(\gamma-1)(2(1+\gamma)+\gamma)+(1-\alpha)[(1-\delta)(1+\gamma)+2\delta]^2\right|}}$$

and

$$|a_3| \leq \frac{4\alpha^2}{[(1-\delta)(1+\gamma)+2\delta]^2} + \frac{2\alpha}{(1-\delta)(2+\gamma)+6\delta}.$$

Remark 2.2. In Corollary 2.1, if we choose

(1) $\delta = 0$, then we have the results obtained by Prema and Keerthi [17, Theorem 2.2];

(2) $\delta = 1$, then we have the results obtained by Sakar and Wanas [22, Theorem 2.2];

(3) $\delta = \gamma = 0$, then we have the results obtained by Murugusundaramoorthy et al. [16, Corollary 6];

(4) $\delta = 0$ and $\gamma = 1$, then we have the results obtained by Srivastava et al. [25, Theorem 1].

3. Coefficient Estimates for the Function Family $L^*_{\Sigma_m}(\delta, \gamma; \beta)$

Definition 3.1. A function $f \in \Sigma_m$ given by (1.3) is said to be in the family $L^*_{\Sigma_m}(\delta, \gamma; \beta)$ $(0 \leq \beta < 1, 0 \leq \delta \leq 1, 0 \leq \gamma \leq 1)$ if it satisfies the following conditions:

$$Re\left\{(1-\delta)\frac{z^{1-\gamma}f'(z)}{\left(f(z)\right)^{1-\gamma}} + \delta(1+\frac{z^{2-\gamma}f''(z)}{\left(zf'(z)\right)^{1-\gamma}})\right\} > \beta, \tag{3.1}$$

and

$$Re\left\{(1-\delta)\frac{w^{1-\gamma}g'(w)}{\left(g(w)\right)^{1-\gamma}} + \delta(1+\frac{w^{2-\gamma}g''(w)}{\left(wg'(w)\right)^{1-\gamma}})\right\} > \beta, \tag{3.2}$$

where the function $g = f^{-1}$ is given by (1.4).

In particular, for one-fold symmetric bi-univalent functions, we denote the family $L^*_{\Sigma_1}(\delta, \gamma; \beta) = L^*_{\Sigma}(\delta, \gamma; \beta)$.

Remark 3.1. It should be remarked that the family $L^*_{\Sigma}(\delta, \gamma; \beta)$ is a generalization of well-known families consider earlier. These families are:

(1) For $\delta = 0$, the family $L^*_\Sigma(\delta, \gamma; \beta)$ reduce to the family $P_\Sigma(\beta, \gamma)$ which was introduced by Prema and Keerthi [17];

(2) For $\delta = 1$, the family $L^*_\Sigma(\delta, \gamma; \beta)$ reduce to the family $\beta^*_{\Sigma(\gamma;\beta)}$ which was introduced by Sakar and Wanas [22];

(3) For $\delta = \gamma = 0$, the family $L^*_\Sigma(\delta, \gamma; \beta)$ reduce to the family $S^*_\Sigma(\beta)$ which was given by Brannan and Taha [6];

(4) For $\delta = 0$ and $\gamma = 1$, the family $L^*_\Sigma(\delta, \gamma; \beta)$ reduce to the family $H_\Sigma(\beta)$ which was investigated by Srivastava et al. [25].

Theorem 3.1. *Let* $f \in L^*_{\Sigma_m}(\delta, \gamma; \beta)$ $(0 \le \beta < 1, 0 \le \delta \le 1, 0 \le \gamma \le 1, m \in \mathbb{N})$ *be given by* (1.3). *Then*

$$|a_{m+1}| \le \sqrt{\frac{2(1-\beta)}{\left|(1-\delta)(2m+\gamma) + 2m\delta(2m+1) + \frac{1}{2}(1-\delta)(\gamma-1)(2(m+\gamma)+\gamma)\right|}} \tag{3.3}$$

and

$$|a_{2m+1}| \le \frac{4(1-\beta)^2}{[(1-\delta)(m+\gamma) + \delta m(m+1)]^2} + \frac{4(1-\beta)}{(1-\delta)(2m+\gamma) + 2m\delta(2m+1)}. \tag{3.4}$$

Proof. It follows from conditions (3.1) and (3.2) that there exist $p, q \in \mathcal{P}$ such that

$$\left((1-\delta)\frac{z^{1-\gamma} f'(z)}{\left(f(z)\right)^{1-\gamma}} + \delta(1 + \frac{z^{2-\gamma} f''(z)}{\left(zf'(z)\right)^{1-\gamma}})\right) = \beta + (1-\beta)p(z) \tag{3.5}$$

and

$$\left((1-\delta)\frac{w^{1-\gamma} g'(w)}{\left(g(w)\right)^{1-\gamma}} + \delta(1 + \frac{w^{2-\gamma} g''(w)}{\left(wg'(w)\right)^{1-\gamma}})\right) = \beta + (1-\beta)q(w), \tag{3.6}$$

where $p(z)$ and $q(w)$ have the forms (2.7) and (2.8), respectively. Equating coefficients (3.5) and (3.6) yields

$$[(1-\delta)(m+\gamma) + \delta m(m+1)]a_{m+1} = (1-\beta)p_m , \tag{3.7}$$

$$[(1-\delta)(2m+\gamma) + 2m\delta(2m+1)]a_{2m+1} + \frac{1}{2}(1-\delta)(\gamma-1)(2(m+\gamma)+\gamma)a_{m+1}^2$$

$$= (1-\beta)p_{2m} , \tag{3.8}$$

$$-[(1-\delta)(m+\gamma)+\delta m(m+1)]a_{m+1}=(1-\beta)q_m \tag{3.9}$$

and

$$[(1-\delta)(2m+\gamma)+2m\delta(2m+1)](2a_{m+1}^2-a_{2m+1})$$
$$+\frac{1}{2}(1-\delta)(\gamma-1)(2(m+\gamma)+\gamma)a_{m+1}^2=(1-\beta)q_{2m}. \tag{3.10}$$

From (3.7) and (3.9), we get

$$p_m=-q_m \tag{3.11}$$

and

$$2[(1-\delta)(m+\gamma)+m\delta(m+1)]^2a_{m+1}^2=(1-\beta)^2(p_m^2+q_m^2). \tag{3.12}$$

Adding (3.8) and (3.10), we obtain

$$2\left[(1-\delta)(2m+\gamma)+2m\delta(2m+1)+\frac{1}{2}(1-\delta)(\gamma-1)(2(m+\gamma)+\gamma)\right]a_{m+1}^2$$
$$=(1-\beta)(p_{2m}+q_{2m}). \tag{3.13}$$

Therefore, we have

$$a_{m+1}^2=\frac{(1-\beta)(p_{2m}+q_{2m})}{2\left[(1-\delta)(2m+\gamma)+2m\delta(2m+1)+\frac{1}{2}(1-\delta)(\gamma-1)(2(m+\gamma)+\gamma)\right]}.$$

Applying Lemma 1.1 for the coefficients p_{2m} and q_{2m}, we obtain

$$|a_{m+1}|\leq\sqrt{\frac{2(1-\beta)}{\left|(1-\delta)(2m+\gamma)(2m\delta(2m+1)+\frac{1}{2}(1-\delta)(\gamma-1)(2(m+\gamma)+\gamma)\right|}}.$$

This gives the desired estimate for $|a_{m+1}|$ as asserted in (3.3).

In order to find the bound on $|a_{2m+1}|$, by subtracting (3.10) from (3.8), we get

$$[(1-\delta)(2m+\gamma)+2m\delta(2m+1)](a_{2m+1}-a_{m+1}^2)=(1-\beta)(p_{2m}-q_{2m}),$$

or equivalently

$$a_{2m+1}=a_{m+1}^2+\frac{(1-\beta)(p_{2m}-q_{2m})}{(1-\delta)(2m+\gamma)+2m\delta(2m+1)}.$$

Upon substituting the value of a_{m+1}^2 from (3.12), it follows that

$$a_{2m+1}=\frac{(1-\beta)^2(p_m^2+q_m^2)}{2[(1-\delta)(m+\gamma)+\delta m(m+1)]^2}+\frac{(1-\beta)(p_{2m}-q_{2m})}{(1-\delta)(2m+\gamma)+2m\delta(2m+1)}.$$

Applying Lemma 1.1 once again for the coefficients p_m, p_{2m}, q_m and q_{2m}, we obtain

$$|a_{2m+1}|\leq\frac{4(1-\beta)^2}{[(1-\delta)(m+\gamma)+\delta m(m+1)]^2}+\frac{4(1-\beta)}{(1-\delta)(2m+\gamma)+2m\delta(2m+1)},$$

which completes the proof of Theorem 3.1.

For one-fold symmetric bi-univalent functions, Theorem 3.1 reduces to the following corollary:

Corollary 3.1. *Let* $f\in L_\Sigma^*(\delta,\gamma;\beta)$ $(0\leq\beta<1, 0\leq\delta\leq 1, 0\leq\gamma\leq 1)$ *be given by* (1.1). *Then*

$$|a_2|\leq\sqrt{\frac{2(1-\beta)}{\left|(1-\delta)(2+\gamma)+6\delta+\frac{1}{2}(1-\delta)(\gamma-1)(2(1+\gamma)+\gamma)\right|}}$$

and

$$|a_3|\leq\frac{4(1-\beta)^2}{[(1-\delta)(1+\gamma)+2\delta]^2}+\frac{4(1-\beta)}{(1-\delta)(2+\gamma)+6\delta}.$$

Remark 3.1. In Corollary 3.1, if we choose

(1) $\delta=0$, then we have the results obtained by Prema and Keerthi [17, Theorem 3.2];

(2) $\delta=1$, then we have the results obtained by Sakar and Wanas [22,Theorem 3.2];

(3) $\delta=\gamma=0$, then we have the results obtained by Murugusundaramoorthy et al. [16, Corollary 7];

(4) $\delta=0$ and $\gamma=1$, then we have the results obtained by Srivastava et al. [25, Theorem 2].

4. Conclusion

The present study has introduced a new subfamilies $L_{\Sigma_m}(\delta,\gamma;\alpha)$ and $L_{\Sigma_m}^*(\delta,\gamma;\beta)$ of Σ_m for normalized holomorphic and m-fold symmetric bi-univalent functions defined by the Bazilevic convex functions and investigated the initial coefficient bounds $|a_{m+1}|$ and $|a_{2m+1}|$ for functions in each of these new subfamilies.

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