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Some Basic Properties of $QIFSN(G)$

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Abstract

In this paper, as using norms (T and C), we introduce the concepts of strongest relations, cosets and middle cosets of Q-intuitionistic fuzzy subgroups and prove some simple but elegant results about them. Also we discuss few results of them under homomorphism as well as anti homomorphism.

1 Introduction

After the introduction of fuzzy sets by Zadeh [27], various notions of higher-order fuzzy sets have been proposed. Among them, intuitionistic fuzzy sets, introduced by Atanassov [3, 4], have drawn the attention of many researchers in the last decades. This is mainly due to the fact that intuitionistic fuzzy sets are consistent with human behavior, by ecting and modeling the hesitancy present in real-life situations. In fact, the fuzzy sets give the degree of membership of an element in a given set, while intuitionistic fuzzy sets give both a degree of membership and a degree of non-membership. As for fuzzy sets, the degree of membership

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is a real number between 0 and 1. This is also the case for the degree of nonmembership, and furthermore the sum of these two degrees is not greater than 1. Yuan and Lee [26] defined the fuzzy subgroup and fuzzy subring based on the theory of falling shadows. Solairaju and Nagarajan [25] introduced the notion of Q- fuzzy groups. Anthony and Sherwood [2] gave the definition of fuzzy subgroup based on t-norm. In previous works [6-24], by using norms, the second author investigated some properties of fuzzy algebraic structures, specially, we defined and investigated Q-fuzzy subgroups, anti Q-fuzzy subgroups, Q-intuitionistic fuzzy subgroups with respect to norms [6, 7, 8, 9, 15]. In this study, we consider the concepts of strongest relations, cosets and middle cosets of Q-intuitionistic fuzzy subgroups with respect to norms (T and C) and investigate some of their properties. Moreover, we investigate some related properties of them under homomorphism and anti homomorphism.

2 Preliminaries

This section contains some basic definitions and preliminary results which will be needed in the sequal. For more details we refer to [1, 2, 3, 4, 5, 6, 7, 8, 10, 15].

Definition 2.1. A fuzzy subset of X is a function from X into $[0, 1]$. The set of all fuzzy subsets of X is called the fuzzy power set of X and is denoted by $FP(X)$.

Definition 2.2. Let G be an arbitrary group with a multiplicative binary operation and identity e. A fuzzy subset of G, we mean a function from G into $[0, 1]$. The set of all fuzzy subsets of G is called the $[0, 1]$-power set of G and is denoted $FP(G)$.

Definition 2.3. Let $\mu \in FP(G)$. Then μ is called a fuzzy subgroup of G if
(1) $\mu(xy) \geq \mu(x) \wedge \mu(y)$ for all $x, y \in G$ and
(2) $\mu(x^{-1}) \geq \mu(x)$ for all $x \in G$.
Denote by $F(G)$, the set of all fuzzy subgroups of G.

Definition 2.4. For sets X, Y and Z, $f = (f_1, f_2) : X \to Y \times Z$ is called a complex mapping if $f_1 : X \to Y$ and $f_2 : X \to Z$ are mappings.

Definition 2.5. Let X be a non-empty set. A complex mapping $A = (\mu_A, \nu_A) : X \to [0,1] \times [0,1]$ is called an intuitionistic fuzzy set (in short, IFS) in X, where functions $\mu_A \in FP(X)$ and $\nu_A \in FP(X)$ define the degree of membership and the degree of non-membership of the element $x \in X$ to the set A, respectively, and for every $x \in X$

$$0 \leq (\mu_A(x) + \nu_A(x)) \leq 1.$$

In particular $\emptyset_X$ and U_X denote the intuitionistic fuzzy empty set and intuitionistic fuzzy whole set in X defined by $\emptyset_X(x) = (0,1)$ and $U_X(x) = (1,0)$, respectively. We will denote the set of all $IFSs$ in X as $IFS(X)$.

Definition 2.6. Let X be a non-empty set and let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be $IFSs$ in X. Then
(1) Inclusion: $A \subseteq B$ iff $\mu_A \leq \mu_B$ and $\nu_A \geq \nu_B$.
(2) Equality: $A = B$ iff $A \subseteq B$ and $B \subseteq A$.

Definition 2.7. A t-norm T is a function $T : [0,1] \times [0,1] \to [0,1]$ having the following four properties:
(T1) $T(x,1) = x$ (neutral element)
(T2) $T(x,y) \leq T(x,z)$ if $y \leq z$ (monotonicity)
(T3) $T(x,y) = T(y,x)$ (commutativity)
(T4) $T(x, T(y,z)) = T(T(x,y), z)$ (associativity),
for all $x, y, z \in [0,1]$.

Corollary 2.8. *Let T be a t-norm. Then for all $x \in [0,1]$*
(1) $T(x,0) = 0$,
(2) $T(0,0) = 0$.

Example 2.9. (1) Standard intersection t-norm

$$T_m(x,y) = \min\{x,y\}.$$

(2) Bounded sum t-norm

$$T_b(x,y) = \max\{0, x+y-1\}.$$

(3) Algebraic product t-norm

$$T_p(x, y) = xy.$$

Lemma 2.10. *Let T be a t-norm. Then*

$$T(T(x, y), T(w, z)) = T(T(x, w), T(y, z)),$$

for all $x, y, w, z \in [0, 1]$.

Definition 2.11. A t-conorm C is a function $C : [0, 1] \times [0, 1] \to [0, 1]$ having the following four properties:
(C1) $C(x, 0) = x$
(C2) $C(x, y) \leq C(x, z)$ if $y \leq z$
(C3) $C(x, y) = C(y, x)$
(C4) $C(x, C(y, z)) = C(C(x, y), z)$,
for all $x, y, z \in [0, 1]$.

Corollary 2.12. *Let C be a C-conorm. Then for all $x \in [0, 1]$*
(1) $C(x, 1) = 1$.
(2) $C(0, 0) = 0$.

Example 2.13. (1) Standard union t-conorm

$$C_m(x, y) = \max\{x, y\}.$$

(2) Bounded sum t-conorm

$$C_b(x, y) = \min\{1, x + y\}.$$

(3) Algebraic sum t-conorm

$$C_p(x, y) = x + y - xy.$$

Recall that t-norm T (t-conorm C) is idempotent if for all $x \in [0, 1]$, $T(x, x) = x(C(x, x) = x)$.

Lemma 2.14. *Let C be a t-conorm. Then*

$$C(C(x,y),C(w,z)) = C(C(x,w),C(y,z)),$$

for all $x,y,w,z \in [0,1]$.

Definition 2.15. Let $(G,.)$ be a group and Q be a non-empty set. An intuitionistic fuzzy set $A = (\mu_A, \nu_A) \in IFS(G \times Q)$ is said to be a Q-intuitionistic fuzzy subgroup of G with respect to norms (t-norm T and t-conorm C) if the following conditions are satisfied:
(1)

$$A(xy,q) = (\mu_A(xy,q), \nu_A(xy,q)) \supseteq A(T(\mu_A(x,q), \mu_A(y,q)), C(\nu_A(x,q), \nu_A(y,q))),$$

(2)

$$A(x^{-1},q) = (\mu_A(x^{-1},q), \nu_A(x^{-1},q)) \supseteq A(x,q) = (\mu_A(x,q), \nu_A(x,q))$$

which mean:
(a) $\mu_A(xy,q) \geq T(\mu_A(x,q), \mu_A(y,q))$,
(b) $\nu_A(xy,q) \leq C(\nu_A(x,q), \nu_A(y,q))$,
(c) $\mu_A(x^{-1},q) \geq \mu_A(x,q)$,
(d) $\nu_A(x^{-1},q) \leq \nu_A(x,q)$,
for all $x,y \in G$ and $q \in Q$.

Throughout this paper the set of all Q-intuitionistic fuzzy subgroups of G with respect to norms (t-norm T and t-conorm C) will be denoted by $QIFSN(G)$.

Proposition 2.16. *Let $A = (\mu_A, \nu_A) \in QIFSN(G)$ such that T and C be idempotent. Then*

$$A(e_G,q) \supseteq A(x,q)$$

for all $x \in G$ and $q \in Q$.

3 Strongest Relations, Cosets and Middle Cosets of $QIFSN(G)$

Definition 3.1. Let $A = (\mu_A, \nu_A) \in IFS(G \times Q)$ and $B = (\mu_B, \nu_B) \in IFS((G \times G) \times Q)$. We say that B is the strongest relation of G with respect to A if $B((x, y), q) = (\mu_B((x, y), q), \nu_B((x, y), q))$ such that $\mu_B((x, y), q) = T(\mu_A(x, q), \mu_A(y, q))$ and $\nu_B((x, y), q) = C(\nu_A(x, q), \nu_A(y, q))$ for all $(x, y) \in G \times G$ and $q \in Q$.

Proposition 3.2. *Let $A = (\mu_A, \nu_A) \in IFS(G \times Q)$ and $B = (\mu_B, \nu_B) \in IFS((G \times G) \times Q)$ such that B be the strongest relation of G with respect to A. Then $A \in QIFSN(G)$ if and only if $B \in QIFSN(G \times G)$.*

Proof. Let $A = (\mu_A, \nu_A) \in QIFSN(G)$. First we show that

$$\mu_B((x_1, x_2)(y_1, y_2), q) \geq T(\mu_B((x_1, y_1), q), \mu_B((x_2, y_2), q))$$

for all $(x_1, x_2), (y_1, y_2) \in G \times G$ and $q \in Q$.

$$\begin{aligned}
\mu_B((x_1, x_2)(y_1, y_2), q) &= \mu_B((x_1y_1, x_2y_2), q) \\
&= T(\mu_A(x_1y_1, q), \mu_A(x_2y_2, q)) \\
&\geq T(T(\mu_A(x_1, q), \mu_A(x_2, q)), T(\mu_A(y_1, q), \mu_A(y_2, q))) \\
&= T(T(\mu_A(x_1, q), \mu_A(y_1, q)), T(\mu_A(x_2, q), \mu_A(y_2, q))) \\
&= T(\mu_B((x_1, y_1), q), \mu_B((x_2, y_2), q))
\end{aligned}$$

Similarly we can see that

$$\nu_B((x_1, x_2)(y_1, y_2), q) \leq C(\nu_B((x_1, y_1), q), \nu_B((x_2, y_2), q))$$

for all $(x_1, x_2), (y_1, y_2) \in G \times G$ and $q \in Q$.

We now show that

$$\mu_B((x, y)^{-1}, q) \geq \mu_B((x, y), q)$$

for all $(x, y) \in G \times G$ and $q \in Q$.

$$\begin{aligned}
\mu_B((x,y)^{-1}, q) &= \mu_B((x^{-1}, y^{-1}), q) \\
&= T(\mu_A(x^{-1}, q), \mu_A(y^{-1}, q)) \\
&\geq T(\mu_A(x, q), \mu_A(y, q)) \\
&= \mu_B((x, y), q)
\end{aligned}$$

Similarly

$$\nu_B((x,y)^{-1}, q) \leq \nu_B((x, y), q)$$

for all $(x, y) \in G \times G$ and $q \in Q$. Hence, by Definition 2.15, $B = (\mu_B, \nu_B) \in QIFSN(G \times G)$.

Conversely, suppose that $B = (\mu_B, \nu_B) \in QIFSN(G \times G)$.

(1) Let $x_1, x_2, y_1, y_2 \in G$ with $x_2 = y_2 = e_G$ and $q \in Q$. Then Proposition 2.16 give us that $A(e, q) \supseteq A(x_1 y_1, q)$. Then

$$\begin{aligned}
\mu_A(x_1 y_1, q) &= T(\mu_A(x_1 y_1, q), \mu_A(e_G, q)) \\
&= T(\mu_A(x_1 y_1, q), \mu_A(x_2 y_2, q)) \\
&= \mu_B((x_1 y_1, x_2 y_2), q) \\
&= \mu_B((x_1, x_2)(y_1, y_2), q) \\
&\geq T(\mu_B((x_1, x_2), q), \mu_B((y_1, y_2), q)) \\
&= T(T(\mu_A(x_1, q), \mu_A(x_2, q)), T(\mu_A(y_1, q), \mu_A(y_2, q))) \\
&= T(T(\mu_A(x_1, q), \mu_A(e, q)), T(\mu_A(y_1, q), \mu_A(e, q))) \\
&\geq T(T(\mu_A(x_1, q), \mu_A(x_1, q)), T(\mu_A(y_1, q), \mu_A(y_1, q))) \\
&= T(\mu_A(x_1, q), \mu_A(y_1, q)))
\end{aligned}$$

and thus

$$\mu_A(x_1 y_1, q) \geq T(\mu_A(x_1, q), \mu_A(y_1, q))).$$

Also

$$\begin{aligned}
\nu_A(x_1y_1,q) &= C(\nu_A(x_1y_1,q),\nu_A(e_G,q))\\
&= C(\nu_A(x_1y_1,q),\nu_A(x_2y_2,q))\\
&= \nu_B((x_1y_1,x_2y_2),q)\\
&= \nu_B((x_1,x_2)(y_1,y_2),q)\\
&\leq C(\nu_B((x_1,x_2),q),\nu_B((y_1,y_2),q))\\
&= C(C(\nu_A(x_1,q),\nu_A(x_2,q)),C(\nu_A(y_1,q),\nu_A(y_2,q)))\\
&= C(C(\nu_A(x_1,q),\nu_A(e,q)),C(\nu_A(y_1,q),\nu_A(e,q)))\\
&\leq C(C(\nu_A(x_1,q),\nu_A(x_1,q)),C(\nu_A(y_1,q),\nu_A(y_1,q)))\\
&= C(\nu_A(x_1,q),\nu_A(y_1,q)))
\end{aligned}$$

thus

$$\nu_A(x_1y_1,q) \leq C(\nu_A(x_1,q),\nu_A(y_1,q))).$$

(2) Let $x,y\in G$ with $y=e_G$ and $q\in Q$. Then as Proposition 2.16 we obtain

$$\begin{aligned}
\mu_A(x^{-1},q) &= T(\mu_A(x^{-1},q),\mu_A(e_G,q))\\
&\geq T(\mu_A(x^{-1},q),\mu_A(y^{-1},q))\\
&= \mu_B((x^{-1},y^{-1}),q)\\
&= \mu_B((x,y)^{-1},q)\\
&\geq \mu_B((x,y),q)\\
&= T(\mu_A(x,q),\mu_A(y,q))\\
&= T(\mu_A(x,q),\mu_A(e,q))\\
&\geq T(\mu_A(x,q),\mu_A(x,q))\\
&= \mu_A(x,q)
\end{aligned}$$

and then

$$\mu_A(x^{-1},q) \geq \mu_A(x,q).$$

Also

$$\begin{aligned}
\nu_A(x^{-1},q) &= C(\nu_A(x^{-1},q),\nu_A(e_G,q))\\
&\le C(\nu_A(x^{-1},q),\nu_A(y^{-1},q))\\
&= \nu_B((x^{-1},y^{-1}),q)\\
&= \nu_B((x,y)^{-1},q)\\
&\le \nu_B((x,y),q)\\
&= C(\nu_A(x,q),\nu_A(y,q))\\
&= C(\nu_A(x,q),\nu_A(e,q))\\
&\le C(\nu_A(x,q),\nu_A(x,q))\\
&= \nu_A(x,q)
\end{aligned}$$

thus

$$\nu_A(x^{-1},q) \le \nu_A(x,q).$$

Therefore from (1)-(2) we get that $A=(\mu_A,\nu_A)\in QIFSN(G)$. This completes the proof. □

Definition 3.3. Let $A=(\mu_A,\nu_A)\in QIFSN(G)$. Then middle coset $aAb: G\times Q\to[0,1]$ is defined by

$$(aAb)(x,q)=(a\mu_A b, a\nu_A b)(x,q)=(\mu_A(a^{-1}xb^{-1},q),\nu_A(a^{-1}xb^{-1},q))$$

for all $x\in G, q\in Q$ and $a,b\in G$.

Proposition 3.4. *Let $A=(\mu_A,\nu_A)\in QIFSN(G)$. Then $aAa^{-1}\in QIFSN(G)$ for any $a\in G$.*

Proof. Let $a,x,y\in G$ and $q\in Q$. Then

$$\begin{aligned}
(a\mu_A a^{-1})(xy,q) &= \mu_A(a^{-1}xya,q)\\
&= \mu_A(a^{-1}xaa^{-1}ya,q)\\
&\ge T(\mu_A(a^{-1}xa,q),\mu_A(a^{-1}ya,q))\\
&= T((a\mu_A a^{-1})(x,q),(a\mu_A a^{-1})(y,q))
\end{aligned}$$

then

$$(a\mu_A a^{-1})(xy,q) \geq T((a\mu_A a^{-1})(x,q),(a\mu_A a^{-1})(y,q)).$$

Also

$$\begin{aligned}(a\nu_A a^{-1})(xy,q) &= \nu_A(a^{-1}xya,q)\\ &= \nu_A(a^{-1}xaa^{-1}ya,q)\\ &\leq C(\nu_A(a^{-1}xa,q),\nu_A(a^{-1}ya,q))\\ &= C((a\nu_A a^{-1})(x,q),(a\nu_A a^{-1})(y,q))\end{aligned}$$

then

$$(a\nu_A a^{-1})(xy,q) \leq C((a\nu_A a^{-1})(x,q),(a\nu_A a^{-1})(y,q)).$$

And

$$\begin{aligned}(a\mu_A a^{-1})(x^{-1},q) &= \mu_A(a^{-1}x^{-1}a,q)\\ &= \mu_A((a^{-1}xa)^{-1},q)\\ &\geq \mu_A(a^{-1}xa,q)\\ &= (a\mu_A a^{-1})(x,q)\end{aligned}$$

thus

$$(a\mu_A a^{-1})(x^{-1},q) \geq (a\mu_A a^{-1})(x,q).$$

Moreover

$$\begin{aligned}(a\nu_A a^{-1})(x^{-1},q) &= \nu_A(a^{-1}x^{-1}a,q)\\ &= \nu_A((a^{-1}xa)^{-1},q)\\ &\leq \nu_A(a^{-1}xa,q)\\ &= (a\nu_A a^{-1})(x,q)\end{aligned}$$

thus

$$(a\nu_A a^{-1})(x^{-1},q) \leq (a\nu_A a^{-1})(x,q).$$

Then $aAa^{-1} \in QIFSN(G)$ for any $a \in G$. □

Definition 3.5. Let $A = (\mu_A, \nu_A) \in QIFSN(G)$. Then coset $aA = (a\mu_A, a\nu_A) : G \times Q \to [0,1]$ is defined by

$$(aA)(x,q) = (a\mu_A, a\nu_A)(x,q) = (\mu_A(a^{-1}x,q), \nu_A(a^{-1}x,q)) = A(a^{-1}x,q)$$

for all $x \in G, q \in Q$ and $a \in G$.

Proposition 3.6. *Let $A = (\mu_A, \nu_A) \in QIFSN(G)$ and T, C be idempotent norms. Then*

$$xA = yA$$

if and only if

$$A(x^{-1}y,q) = A(y^{-1}x,q) = A(e_G,q)$$

for all $x, y \in G$ and $q \in Q$.

Proof. Let $x, y, g \in G$. If $xA = yA$, then $xA(x,q) = yA(x,q)$, then $A(x^{-1}x,q) = A(y^{-1}x,q)$ and so $A(e_G,q) = A(y^{-1}x,q)$.

Also as $xA = yA$ so $xA(y,q) = yA(y,q)$ and then $A(x^{-1}y,q) = A(y^{-1}y,q)$ then $A(x^{-1}y,q) = A(e_G,q)$. Therefore $A(x^{-1}y,q) = A(y^{-1}x,q) = A(e_G,q)$.

Conversely, let $A(x^{-1}y,q) = A(y^{-1}x,q) = A(e_G,q)$. Then

$$\begin{aligned}
x\mu_A(g,q) &= \mu_A(x^{-1}g,q) \\
&= \mu_A(x^{-1}yy^{-1}g,q) \\
&\geq T(\mu_A(x^{-1}y,q), \mu_A(y^{-1}g,q)) \\
&= T(\mu_A(e_G,q), \mu_A(y^{-1}g,q)) \\
&\geq T(\mu_A(y^{-1}g,q), \mu_A(y^{-1}g,q)) \\
&= \mu_A(y^{-1}g,q) \\
&= y\mu_A(g,q)
\end{aligned}$$

$$
\begin{aligned}
&= \mu_A(y^{-1}g, q) \\
&= \mu_A(y^{-1}xx^{-1}g, q) \\
&\geq T(\mu_A(y^{-1}x, q), \mu_A(x^{-1}g, q)) \\
&= T(\mu_A(e_G, q), \mu_A(x^{-1}g, q)) \\
&\geq T(\mu_A(x^{-1}g, q), \mu_A(x^{-1}g, q)) \\
&= \mu_A(x^{-1}g, q) \\
&= x\mu_A(g, q)
\end{aligned}
$$

and then

$$x\mu_A(g, q) = y\mu_A(g, q).$$

Also

$$
\begin{aligned}
x\nu_A(g, q) &= \nu_A(x^{-1}g, q) \\
&= \nu_A(x^{-1}yy^{-1}g, q) \\
&\leq C(\nu_A(x^{-1}y, q), \nu_A(y^{-1}g, q)) \\
&= C(\nu_A(e_G, q), \nu_A(y^{-1}g, q)) \\
&\leq C(\nu_A(y^{-1}g, q), \nu_A(y^{-1}g, q)) \\
&= \nu_A(y^{-1}g, q) \\
&= y\nu_A(g, q) \\
&= \nu_A(y^{-1}g, q) \\
&= \nu_A(y^{-1}xx^{-1}g, q) \\
&\leq C(\nu_A(y^{-1}x, q), \nu_A(x^{-1}g, q)) \\
&= C(\nu_A(e_G, q), \nu_A(x^{-1}g, q)) \\
&\leq C(\nu_A(x^{-1}g, q), \nu_A(x^{-1}g, q)) \\
&= \nu_A(x^{-1}g, q) \\
&= x\nu_A(g, q)
\end{aligned}
$$

and then

$$x\nu_A(g,q) = y\nu_A(g,q).$$

Therefore $xA(g,q) = (x\mu_A(g,q), x\nu_A(g,q)) = (y\mu_A(g,q), y\nu_A(g,q)) = yA(g,q)$ then $xA = yA$. $\square$

Proposition 3.7. *Let $A = (\mu_A, \nu_A) \in QIFSN(G)$ and T, C be idempotent norms. If $xA = yA$, then $A(x,q) = A(y,q)$ for all $x, y \in G$ and $q \in Q$.*

Proof. As $xA = yA$, Proposition 3.6 gives us that $A(x^{-1}y,q) = A(y^{-1}x,q) = A(e_G,q)$ for all $x, y \in G$ and $q \in Q$. From

$$\begin{aligned}
\mu_A(x,q) &= \mu_A(yy^{-1}x,q) \\
&\geq T(\mu_A(y,q), \mu_A(y^{-1}x,q)) \\
&= T(\mu_A(y,q), \mu_A(e_G,q)) \\
&\geq T(\mu_A(y,q), \mu_A(y,q)) \\
&= \mu_A(y,q) \\
&= \mu_A(xx^{-1}y,q) \\
&\geq T(\mu_A(x,q), \mu_A(x^{-1}y,q)) \\
&= T(\mu_A(x,q), \mu_A(e_G,q)) \\
&\geq T(\mu_A(x,q), \mu_A(x,q)) \\
&= \mu_A(x,q)
\end{aligned}$$

we get that $\mu_A(x,q) = \mu_A(y,q)$. Also

$$\begin{aligned}
\nu_A(x,q) &= \nu_A(yy^{-1}x,q) \\
&\leq C(\nu_A(y,q), \nu_A(y^{-1}x,q)) \\
&= C(\nu_A(y,q), \nu_A(e_G,q)) \\
&\leq C(\nu_A(y,q), \nu_A(y,q)) \\
&= \nu_A(y,q)
\end{aligned}$$

$$\begin{aligned}
&= \nu_A(xx^{-1}y, q) \\
&\leq C(\nu_A(x,q), \nu_A(x^{-1}y, q)) \\
&= C(\nu_A(x,q), \nu_A(e_G, q)) \\
&\leq C(\nu_A(x,q), \nu_A(x,q)) \\
&= \nu_A(x,q)
\end{aligned}$$

then $\nu_A(x,q) = \nu_A(y,q)$. Thus $A(x,q) = (\mu_A(x,q), \nu_A(x,q)) = (\mu_A(y,q), \nu_A(y,q)) = A(y,q)$. $\square$

Definition 3.8. We say that $A = (\mu_A, \nu_A) \in QIFSN(G)$ is a normal if $\mu_A(xyx^{-1}, q) = \mu_A(y,q)$ and $\nu_A(xyx^{-1}, q) = \nu_A(y,q)$ for all $x, y \in G$ and $q \in Q$. We denote by $NQIFSN(G)$ the set of all normal Q-intuitionistic fuzzy subgroups of G with respect to norms (t-norm T and t-conorm C).

Proposition 3.9. *If $A = (\mu_A, \nu_A) \in NQIFSN(G)$, then the set $\frac{G}{A} = \{xA : x \in G\}$ is a group with the operation $(xA)(yA) = (xy)A$.*

Proof. This is straightforward. $\square$

Proposition 3.10. *Let $f : G \to H$ be a homomorphism of groups and let $B = (\mu_B, \nu_B) \in NQIFSN(H)$ and $A = (\mu_A, \nu_A) \in NQIFSN(G)$ be homomorphic pre-image of B. Then $\varphi : \frac{G}{A} \to \frac{H}{B}$ such that $\varphi(xA) = f(x)B$, for every $x \in G$, is an isomorphism of groups.*

Proof. Firstly, we prove that φ is a group homomorphism. Let $x, y \in G$ and $q \in Q$. Then

$$\varphi((xA)(yA)) = \varphi((xy)A) = f(xy)B = f(x)f(y)B = f(x)Bf(y)B = \varphi(xA)\varphi(yA)$$

and so φ is a group homomorphism. Clearly φ is onto and we prove that φ is one-one. If $\varphi(xA) = \varphi(yA)$, then $f(x)B = f(y)B$ and from Proposition 3.6 we get that

$$B(f(x)^{-1}f(y), q) = B(f(y)^{-1}f(x), q) = B(f(e_G), q)$$

and so

$$B(f(x^{-1})f(y),q) = B(f(y^{-1})f(x),q) = B(f(e_G),q)$$

and then

$$B(f(x^{-1}y),q) = B(f(y^{-1}x),q) = B(f(e_G),q)$$

which implies that

$$A(x^{-1}y,q) = A(y^{-1}x,q) = A(e_G,q)$$

and thus Proposition 3.6 gives us that $xA = yA$ which implies that φ is one-one. Therefore φ will be an isomorphism of groups. □

Proposition 3.11. *Let $f : G \to H$ be an anti homomorphism of groups and let $B = (\mu_B, \nu_B) \in NQIFSN(H)$ and $A = (\mu_A, \nu_A) \in NQIFSN(G)$ be anti homomorphic pre-image of B. Then $\varphi : \frac{G}{A} \to \frac{H}{B}$ such that $\varphi(xA) = f(x)B$, for every $x \in G$, is an anti isomorphism of groups.*

Proof. Firstly, we prove that φ is an anti group homomorphism. Let $x, y \in G$ and $q \in Q$. Then

$$\varphi((xA)(yA)) = \varphi((xy)A) = f(xy)B = f(y)f(x)B = f(y)Bf(x)B = \varphi(yA)\varphi(xA)$$

and so φ is a group homomorphism. Clearly φ is onto and we prove that φ is one-one. If $\varphi(xA) = \varphi(yA)$, then $f(x)B = f(y)B$ and from Proposition 3.6 we get that

$$B(f(x)^{-1}f(y),q) = B(f(y)^{-1}f(x),q) = B(f(e_G),q)$$

and so

$$B(f(x^{-1})f(y),q) = B(f(y^{-1})f(x),q) = B(f(e_G),q)$$

and then

$$B(f(x^{-1}y),q) = B(f(y^{-1}x),q) = B(f(e_G),q)$$

which implies that

$$A(x^{-1}y,q) = A(y^{-1}x,q) = A(e_G,q)$$

and thus Proposition 3.6 gives us that $xA = yA$ which implies that φ is one-one. Therefore φ will be an anti isomorphism of groups. □

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