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A New Generalization of the Half-Logistic Model using the Transformed-Transformer Approach with Application to Wind Speed Sample from Nigeria

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Abstract

In this paper, a new generalization of the half-logistic distribution was introduced and called the 'Odd Weibull Exponentiated Half-Logistic Distribution (OWEHLD)'. The OWEHLD was realized by using the Weibull distribution to transform the exponentiated half-logistic distribution using the quantile function of the log-logistic distribution or the odd ratio as link. This approach for generalizing the half-logistic distribution is the so-called 'transformed-transformer approach'. Several mathematical properties of the OWEHLD were derived and studied and the maximum likelihood estimation method was employed in estimating its parameters. Real life data sets including a wind speed sample from Nigeria were further used to test the applicability of the new distribution.

1. Introduction

A random variable X is said to follow the half-logistic distribution if it has the cumulative distribution function (cdf) and probability density function (pdf) given by

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$$F(x) = \frac{1 - e^{-\theta x}}{1 + e^{-\theta x}}, \qquad x > 0, \ \theta > 0,$$

$$f(x) = \frac{2\theta e^{-\theta x}}{(1 + e^{-\theta x})^2}, \qquad x > 0, \ \theta > 0.$$

The quantile function (inverse of the cdf) corresponding to the cdf can be expressed as

$$Q(p) = \frac{1}{\theta} \log\left(\frac{p+1}{1-p}\right), \qquad \theta > 0, \ 0 < p < 1.$$

The half-logistic distribution was developed by Balakrishnan [1] as the distribution of the absolute value of a logistic random variable. This distribution has been widely studied by several researchers. Balakrishnan [1] developed some recurrence relation for the moments and product moment of order statistics for the half-logistic distribution. Using the linear functions of order statistics, Balakrishnan and Puthenpura [2] obtained the best linear unbiased estimators (BLUEs) of the parameter of the half-logistic distribution. Oliveira [3] studied the standardized version of the half-logistic distribution which proved to be an attractive model with simplified mathematical structures.

Several generalizations of the half-logistic distribution have appeared in the literature and they include: the exponentiated half-logistic family of distribution proposed by Cordeiro et al. [4], the exponentiated half-logistic distribution due to Gui [5], the Type I half-logistic family of distributions proposed by Cordeiro et al. [6], the Type II half-logistic family of distribution due to Soliman et al. [7], the exponentiated Generalized standardized half-logistic distribution due to Cordeiro et al. [8], the Kumaraswamy half logistic distribution proposed by Usman et al. [9], the Kumaraswamy Type I half-logistic family of distribution proposed by El-sayed and Mahmoud [10], the type II half-logistic Weibull distribution due to Hassan et al. [11], the exponentiated power half-logistic distribution proposed by Okereke et al. [12] and the exponentiated half-logistic odd Lindley-G family of distribution developed by Sengweni et al. [13].

In this paper, we employ the transformed - transformer framework proposed by Alzaatreh [14] to generalize the half-logistic distribution. We call the new distribution the odd Weibull exponentiated half-logistic distribution (OWEHLD). The remaining sections of this paper are organized as follows: Section 2 contains the formulation of the proposed distribution. Statistical properties of the proposed distribution are contained in Section 3. In Section 4 the maximum likelihood estimation of the parameters of the distribution is carried out. Section 5 contains the application of the proposed distribution to real life data sets. The paper closes in Section 6 with summary and conclusion.

2. The Odd Weibull Exponentiated Half Logistic Distribution (OWEHLD)

Let T, R and Y be Weibull, exponentiated standard half-logistic and standard log logistic random variables respectively. Denote their respective cdf by $F_T(x) = P(T \leq x)$, $F_R(x) = P(R \leq x)$, and $F_Y(x) = P(Y \leq x)$, where

$$F_T(x) = 1 - \mathrm{e}^{-\left(\frac{x}{\lambda}\right)^{\beta}}, \qquad x > 0,\ \beta, \lambda > 0,$$

$$F_R(x) = \left(\frac{1 - \mathrm{e}^{-x}}{1 + \mathrm{e}^{-x}}\right)^{\theta}, \qquad x > 0,\ \theta > 0,$$

$$F_Y(x) = 1 - (1 + x)^{-1}, \qquad x > 0.$$

Suppose the quantile functions corresponding to the cdfs of T, R and Y are $Q_T(p)$, $Q_R(p)$ and $Q_Y(p)$, such that $Q_W(p) = \inf\{w: F_W(w) \geq p\}, 0 < p < 1$, where

$$Q_T(p) = \lambda(-\log(1 - \mathrm{p}))^{1/\beta}, \qquad \beta, \lambda > 0,\ 0 < p < 1,$$

$$Q_R(p) = \log\left(\frac{1 + p^{1/\theta}}{1 - p^{1/\theta}}\right), \qquad \theta > 0,\ 0 < p < 1,$$

$$Q_Y(p) = \frac{p}{1 - p}, \qquad 0 < p < 1.$$

Let the corresponding densities of the random variables T, R and Y be denoted by $f_T(x), f_R(x)$ and $f_Y(x)$ where

$$f_T(x) = \frac{\beta}{\lambda}\left(\frac{x}{\lambda}\right)^{\beta - 1} \mathrm{e}^{-\left(\frac{x}{\lambda}\right)^{\beta}}, \qquad x > 0,\ \beta, \lambda > 0,$$

$$f_R(x) = \frac{2\theta \mathrm{e}^{-x}}{(1 + \mathrm{e}^{-x})^2}\left(\frac{1 - \mathrm{e}^{-x}}{1 + \mathrm{e}^{-x}}\right)^{\theta - 1}, \qquad x > 0,\ \theta > 0,$$

$$f_Y(x) = (1 + x)^{-2}, \qquad x > 0.$$

Using the transformed-transformer approach, we define the cdf of the OWEHLD by

$$F_X(x) = \int_0^{Q_Y(F_R(x))} f_T(t)dt = P[T \leq Q_Y(F_R(x))] = F_T\left(Q_Y(F_R(x))\right), \ x > 0, \quad (1)$$

where the pdf corresponding to the cdf in (1) can be expressed as

$$f_X(x) = F_R(x)\frac{f_T\left(Q_Y\left(F_R(x)\right)\right)}{f_Y\left(Q_Y\left(F_R(x)\right)\right)}, \qquad x > 0. \tag{2}$$

It follows from the definition of $F_T(x)$, $F_R(x)$, $F_Y(x)$, $f_T(x)$, $f_R(x)$, $f_Y(x)$, $Q_T(p)$, $Q_R(p)$ and $Q_Y(p)$ that

$$F_X(x) = 1 - \exp\left\{-\left[\frac{(1-\mathrm{e}^{-x})^\theta}{\lambda((1+\mathrm{e}^{-x})^\theta - (1-\mathrm{e}^{-x})^\theta)}\right]^\beta\right\}, \; x > 0, \; \beta, \lambda, \theta > 0, \tag{3}$$

$$f_X(x) = \frac{2\beta\theta\mathrm{e}^{-x}(1+\mathrm{e}^{-x})(1-\mathrm{e}^{-x})^{\beta\theta}}{\lambda^\beta(1-\mathrm{e}^{-2x})((1+\mathrm{e}^{-x})^\theta - (1-\mathrm{e}^{-x})^\theta)^{\beta+1}} \exp\left\{-\left[\frac{(1-\mathrm{e}^{-x})^\theta}{\lambda((1+\mathrm{e}^{-x})^\theta - (1-\mathrm{e}^{-x})^\theta)}\right]^\beta\right\}, \tag{4}$$

$$x > 0, \; \beta, \lambda, \theta > 0.$$

The shapes of the OWEHLD density in (4) for various combinations of parameter values are given in Figures 1-3.

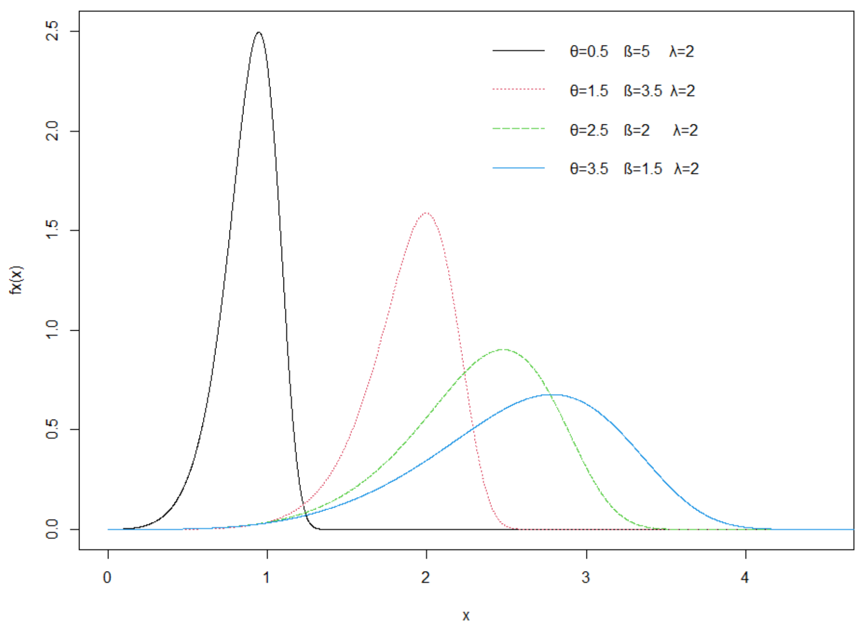

Figure 1: The OWEHLD density for fixed λ.

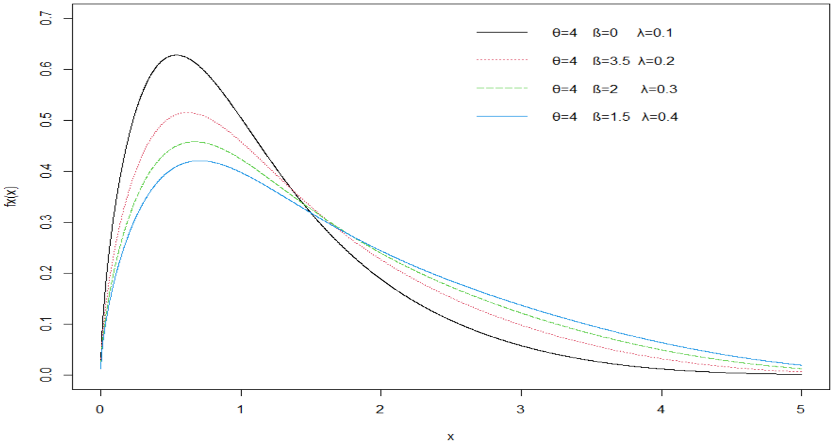

Figure 2: The OWEHLD density for fixed θ.

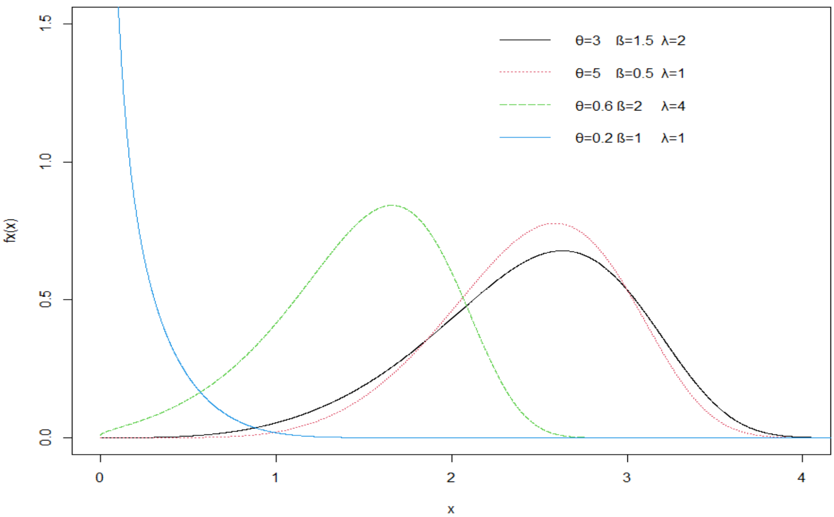

Figure 3: The OWEHLD density for varying parameters values.

From the shapes of the OWEHLD density in Figures 1-3, it can be observed that the distribution can be right skewed, left skewed and almost symmetric, properties which points out the flexibility of the distribution over the classical half logistic distribution and the exponentiated half-logistic distribution.

Lemma 1. *If X is a random variable following the OWEHLD, then X converges in distribution to a function of the random variable T following the Weibull distribution by the expression of the form*

$$X \xrightarrow{d} \log\left[\frac{1+(1-(1+T)^{-1})^{1/\theta}}{1-(1-(1+T)^{-1})^{1/\theta}}\right]. \tag{5}$$

Proof. The result follows from the fact that in the cdf in (1) the random variable T can be expressed as

$$T = Q_Y\big(\,F_R(X)\big).$$

Evaluating and using the definition of $F_R(.), F_Y(.), Q_R(.)$ and $Q_Y(.)$ one easily obtain the result.

Remark 1. The result in (5) is very useful in simulating random variates from the OWEHLD by first simulating Weibull random variates and then applying the transformation accordingly. The result can also be employed in computing the moments of the OWEHLD.

Lemma 2. *If X is a random variable following the OWEHLD, then the quantile function corresponding to the cdf of X is a function of the quantile function of the Weibull random variable T by the expression of the form*

$$Q_X(p) = \log\left[\frac{1+\left(1-\left(1+\ Q_T(p)\right)^{-1}\right)^{1/\theta}}{1-\left(1-\left(1+\ Q_T(p)\right)^{-1}\right)^{1/\theta}}\right]. \tag{6}$$

Proof. The proof follows from Lemma 1.

Remark 2. From the result in (6), one easily obtain the quantile function of the OWEHLD as

$$Q_X(p) = \log\left(\frac{1+A^{1/\theta}}{1-A^{1/\theta}}\right), \tag{7}$$

where

$$A = \frac{\lambda(-\log(1-p))^{1/\beta}}{1+\lambda(-\log(1-p))^{1/\beta}}, \qquad \beta, \lambda, \theta > 0,\ 0 < p < 1.$$

Consequently, the median of the OWEHLD is given by

$$Q_2 = Q_X(0.5) = \log\left(\frac{1+A^{1/\theta}}{1-A^{1/\theta}}\right), \tag{8}$$

where

$$A = \frac{\lambda(-\log(0.5))^{1/\beta}}{1+\lambda(-\log(0.5))^{1/\beta}}, \qquad \beta, \lambda, \theta > 0.$$

Also, if U is a uniform random variable defined on the interval $(0,1)$ then random variate from the OWEHLD can be simulated using the relation

$$Q_X(U) = \log\left(\frac{1+A^{1/\theta}}{1-A^{1/\theta}}\right), \tag{9}$$

where

$$A = \frac{\lambda(-\log(1-U))^{1/\beta}}{1+\lambda(-\log(1-U))^{1/\beta}}, \qquad \beta, \lambda, \theta > 0,\ 0 < U < 1.$$

3. Statistical Properties of the Odd Weibull Exponentiated Half Logistic Distribution (OWEHLD)

Here we present and discuss statistical properties of the OWEHLD. The hazard function of the distribution is used to start the section.

3.1. Hazard function

The hazard function of a probability distribution is the ratio of its density function to the complement of the its cumulative distribution function. For the OWEHLD, the hazard function is given by

$$h_X(x) = \frac{2\beta\theta \mathrm{e}^{-x}(1+\mathrm{e}^{-x})(1-\mathrm{e}^{-x})^{\beta\theta}}{\lambda^{\beta}(1-\mathrm{e}^{-2x})((1+\mathrm{e}^{-x})^{\theta}-(1-\mathrm{e}^{-x})^{\theta})^{\beta+1}}, \qquad x > 0,\ \beta, \lambda, \theta > 0. \tag{10}$$

The shapes of the OWEHLD for various combinations of parameters values are given in Figure 4-6. The graphs in Figures 4-6 clearly reveal that the OWEHLD hazard can be increasing, decreasing, bathtub and upside-down bathtub. These results further buttress its flexibility and adaptability in lifetime data analysis.

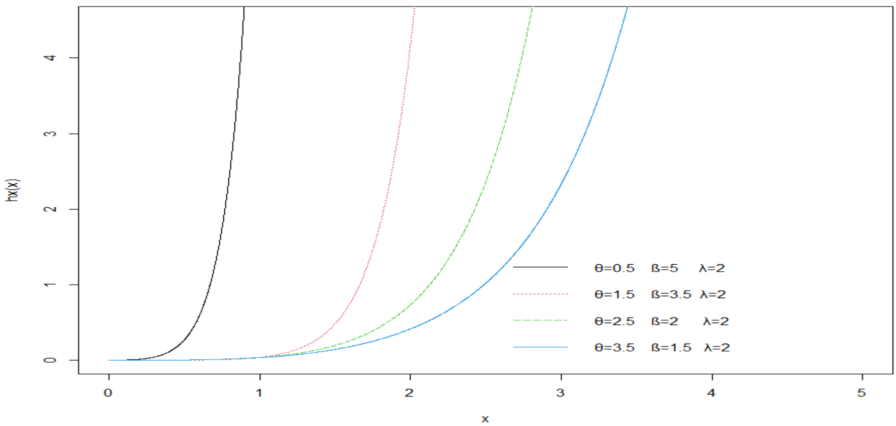

Figure 4: The OWEHLD hazard for fixed λ.

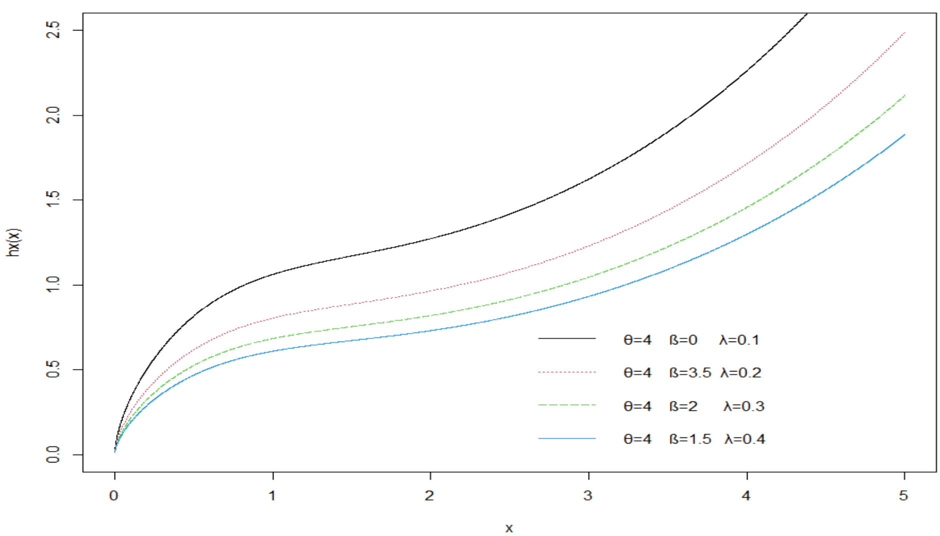

Figure 5: The OWEHLD hazard for fixed θ.

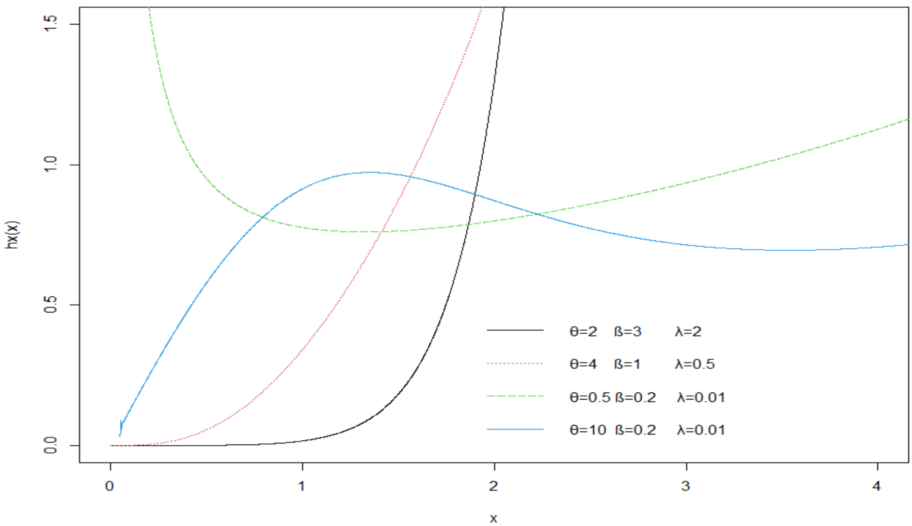

Figure 6: The OWEHLD hazard for varying parameters values.

3.2. Mode

Lemma 3. *The mode of the OWEHDLD density in (2) is either at* $x = 0$ *or it will satisfy the equation*

$$f_Y\left(Q_Y\left(F_R(x)\right)\right) f_R'(x) f_T\left(Q_Y\left(F_R(x)\right)\right)$$

$$= f_X(x)\left[f_Y'\left(Q_Y\left(F_R(x)\right)\right) f_R(x) - \frac{f_Y\left(Q_Y\left(F_R(x)\right)\right) f_T'\left(Q_Y\left(F_R(x)\right)\right)}{f_T\left(Q_Y\left(F_R(x)\right)\right)}\right]. \tag{11}$$

Proof. As observed from the graphs of the OWEHLD density, the distribution is always unimodal. On differentiating the OWEHLD density in (2) w.r.t. x, one obtains

$$f_X'(x) = \frac{f_Y\left(Q_Y\left(F_R(x)\right)\right)\left[f_T'\left(Q_Y\left(F_R(x)\right)\right) Q_Y'\left(F_R(x)\right) f_R(x) + f_R'(x) f_T\left(Q_Y\left(F_R(x)\right)\right)\right]}{\left[f_Y\left(Q_Y\left(F_R(x)\right)\right)\right]^2}$$

$$- \frac{f_R(x) f_T\left(Q_Y\left(F_R(x)\right)\right) f_Y'\left(Q_Y\left(F_R(x)\right)\right) Q_Y'\left(F_R(x)\right) f_R(x)}{\left[f_Y\left(Q_Y\left(F_R(x)\right)\right)\right]^2}.$$

Equating $f_X'(x) = 0$ and evaluating further one obtains

$$f_Y\left(Q_Y(F_R(x))\right) f_R'(x) f_T\left(Q_Y(F_R(x))\right)$$

$$= f_X(x)\left[f_Y'\left(Q_Y(F_R(x))\right) f_R(x) - \frac{f_Y\left(Q_Y(F_R(x))\right) f_T'\left(Q_Y(F_R(x))\right)}{f_T\left(Q_Y(F_R(x))\right)}\right].$$

The derivative $f_X'(x)$ does not exist when $x = 0$. Other critical point satisfies $f_X'(x) = 0$, hence the OWEHLD mode will either be at $x = 0$ or it will satisfy equation (11).

Remark 3. Observe that $\frac{f_T\left(Q_Y(F_R(x))\right)}{\left[f_Y\left(Q_Y(F_R(x))\right)\right]^2}$ is a factor of $f'(x)$ and has the same sign as $f'(x)$. Analytical solution of (11) for x is not possible. However, (11) can be solved numerically in order to obtain the desired mode.

3.3. Moments

A formula for computing the r^{th} non-central moments of the OWEHLD can be obtained by making using of the relationship between the OWEHLD random variable X and Weibull random variable T as specified in (5). In particular, the relation

$$X \xrightarrow{d} \log\left[\frac{1 + (1 - (1 + T)^{-1})^{1/\theta}}{1 - (1 - (1 + T)^{-1})^{1/\theta}}\right]$$

implies that

$$\mu_r' = E(X^r) = E\left\{\left[\log\left(\frac{1 + (1 - (1 + T)^{-1})^{1/\theta}}{1 - (1 - (1 + T)^{-1})^{1/\theta}}\right)\right]^r\right\}.$$

It follows that

$$\mu_r' = \frac{\beta}{\lambda}\int_0^\infty \left[\log\left(\frac{1 + (1 - (1 + t)^{-1})^{1/\theta}}{1 - (1 - (1 + t)^{-1})^{1/\theta}}\right)\right]^r \left(\frac{t}{\lambda}\right)^{\beta - 1} e^{-\left(\frac{t}{\lambda}\right)^\beta} dt. \quad (12)$$

The r^{th} non-central moments of the OWEHLD are computed from the relation in (12) numerically since a simple analytic algebraic structure is not possible for μ_r'.

The mean (μ), variance (σ^2), skewness (S) and kurtosis (K) of the OWEHLD are given respectively as

$$\mu = \mu_1',$$

$$\sigma^2 = \mu^2 + \mu_2' - 2\mu^2,$$

$$S = \frac{\mu_3' - 3\mu\mu_2' + 2\mu^2}{\left(\mu_2' - \mu^2\right)^{3/2}},$$

$$K = \frac{\mu_4' - 4\mu\mu_3' + 6\mu^2\mu_2' - 3\mu^4}{\left(\mu_2' - \mu^2\right)^2}.$$

The quantile function can also be used in computing the skewness and kurtosis of OWEHLD since the quantile function exists in a simple analytic form. Galton [15] proposed a quantile measure based approach for evaluating skewness while Moor [16] did the same for Kurtosis. Galton's skewness and Moor's kurtosis for the OWEHLD are evaluated using the relations

$$S = \frac{Q_X(6/8) - 2\,Q_X(4/8) + Q_X(2/8)}{Q_X(6/8) - Q_X(2/8)},$$

$$K = \frac{Q_X(7/8) - Q_X(5/8) + Q_X(3/8) - Q_X(1/8)}{Q_X(6/8) - Q_X(2/8)}.$$

3-D plots of the Galton's skewness and the Moore's kurtosis of the OWEHLD are presented in Figures 7.

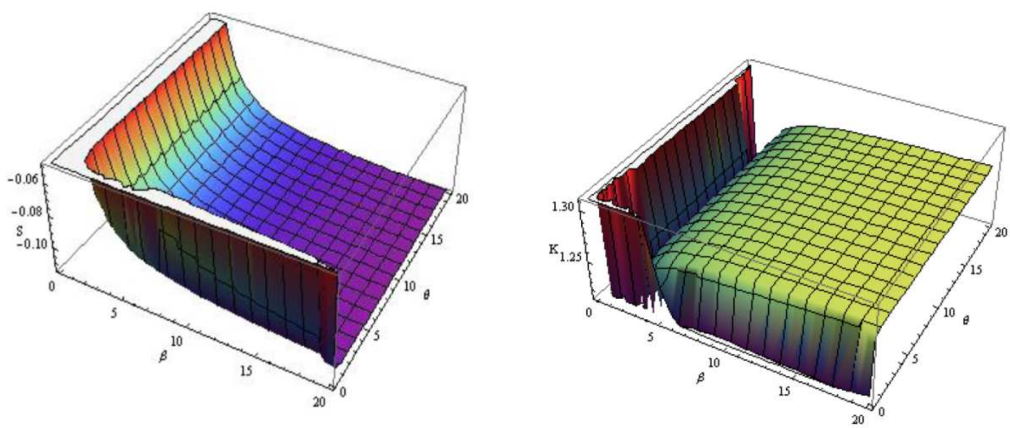

Figure 7: Galton's skewness (S) and Moore's kurtosis (K) for the EGuWL distribution ($\lambda = 1$).

3.4. Entropy

Shannon [17] gave a probabilistic definition of entropy. The Shannon entropy η_X of a random variable X following a known probability distribution is a measure of variation of uncertainty. For a continuous random variable X with density function $f_X(x)$, the Shannon entropy is defined as

$$\eta_X = E\left[-\log\left(f_X(X)\right)\right].$$

Lemma 3. *The Shannon entropy of a random variable X following the OWEHLD can be expressed as*

$$\eta_X = \xi(1 - 1/\beta) + \log(\lambda/\beta) + 1 + \mu - \log(\theta/2) - W(\beta, \lambda, \theta), \tag{13}$$

where $\xi = 0.5772$ *is the Euler-Mascheroni constant,* $\mu = E(X)$ *is the mean of the random variable X following the OWEHLD and the function* $W(\beta, \lambda, \theta)$ *is given by*

$$W(\beta, \lambda, \theta) = \frac{-2\beta}{\lambda}\int_0^\infty \left\{\log(1+t) + \log\left(1 + (1-(1+t)^{-1})^{\frac{1}{\theta}}\right)\right.$$
$$\left. + \frac{\theta - 1}{2\theta}\log(1-(1+t)^{-1})\right\} \times \left(\frac{t}{\lambda}\right)^{\beta - 1} e^{-\left(\frac{t}{\lambda}\right)^{\beta}}\ dt.$$

Proof. Given that

$$f_X(x) \ = \ f_R(x)\frac{f_T\left(Q_Y\left(F_R(x)\right)\right)}{f_Y\left(Q_Y\left(F_R(x)\right)\right)},$$

It follows that

$$f(X) = \ f_R(X)\frac{f_T\left(Q_Y\left(F_R(X)\right)\right)}{f_Y\left(Q_Y\left(F_R(X)\right)\right)}.$$

From Lemma 1, $T = Q_Y\left(F_R(X)\right)$. It follows that

$$f(X) = \ f_R(X)\frac{f_T(T)}{f_Y(T)}$$

and

$$\eta_X = E\left[-\log\left(f(X)\right)\right] = E\left[-\log\left(f_T(T)\right)\right] - E\left[\log\left(f_R(X)\right)\right] + E\left[\log\left(f_Y(T)\right)\right].$$

It follows that

$$\eta_X = \eta_T - 2E[\log(1+T)] - E\left[\log\left(f_R(X)\right)\right].$$

Now,

$$\log\left(f_R(X)\right) = \log(2\theta) - X - (\theta+1)\log(1+e^{-X}) + (\theta-1)\log(1-e^{-X}),$$

and

$$E\left[\log\left(f_R(X)\right)\right] = \log(2\theta) - E(X) - E[(\theta+1)\log(1+e^{-X}) - (\theta-1)\log(1-e^{-X})].$$

From Lemma 1, $X = \log\left[\frac{1+\left(1-(1+T)^{-1}\right)^{1/\theta}}{1-(1-(1+T)^{-1})^{1/\theta}}\right]$ and thus

$$\eta_X = \eta_T + E(X) - \log\left(\frac{\theta}{2}\right) - 2E[\log(1+T)]$$

$$-2E\left[\log\left(1+(1-(1+T)^{-1})^{\frac{1}{\theta}}\right)\right]$$

$$-\frac{\theta-1}{\theta}E[\log(1-(1+T)^{-1})].$$

Given that T is a Weibull random variable, we have from the result in Song [18] that

$$\eta_T = \xi(1-1/\beta) + \log(\lambda/\beta) + 1,\ \xi = 0.5772,$$

and $E(X) = \mu$ the mean of the OWEHLD. On evaluating the expectations, the result in (13) is achieved and that completes the proof.

Remark 4. The integral

$$\int_0^\infty \left\{\log(1+t) + \log\left(1+(1-(1+t)^{-1})^{\frac{1}{\theta}}\right) + \frac{\theta-1}{2\theta}\log(1-(1+t)^{-1})\right\}$$

$$\times\left(\frac{t}{\lambda}\right)^{\beta-1} e^{-\left(\frac{t}{\lambda}\right)^\beta}\ dt.$$

in (13) exist because

$$|\log(1+t)| \le |\log(1+\mathrm{e}^t)| \le \log2 + t \ \text{ when } \ t > 0,$$

$$|\log(1+t)| \le |\log(1+\mathrm{e}^t)| \le \log2 \ \text{ when } \ t < 0,$$

$$\left|\log\left(1+(1-(1+t)^{-1})^{\frac{1}{\theta}}\right)\right| \le |\log(1+\mathrm{e}^t)| \le \log2 + t \ \text{ when } \ t > 0,$$

$$\left|\log\left(1+(1-(1+t)^{-1})^{\frac{1}{\theta}}\right)\right| \leq |\log(1+\mathrm{e}^{t})| \leq \log 2 \ \text{ when } \ t<0,$$

$$|\log(1-(1+t)^{-1})| \leq |\log(1+\mathrm{e}^{t})| \leq \log 2+t \ \text{ when } \ t>0,$$

$$|\log(1-(1+t)^{-1})| \leq |\log(1+\mathrm{e}^{t})| \leq \log 2 \ \text{ when } \ t<0.$$

4. Estimation

In this section, the maximum likelihood method of estimation of parameters is adopted for the estimation of the parameters of the OWEHLD.

4.1. Maximum likelihood method of estimation of the parameters of the OWEHLD

Given a non-censored random sample $x_1, x_2, \dots, x_n$ of size n, the log-likelihood function of the OWEHLD is

$$\mathcal{L} = n\log(2\beta\theta) - n\log\lambda - \sum_{i=1}^{n} x_i + \sum_{i=1}^{n}\log(1+\mathrm{e}^{-x_i}) - \sum_{i=1}^{n}\log(1-\mathrm{e}^{-2x_i})$$

$$+\beta\theta\sum_{i=1}^{n}\log(1+\mathrm{e}^{-x_i}) - \sum_{i=1}^{n}\left[\frac{(1-\mathrm{e}^{-x_i})^{\theta}}{\lambda((1+\mathrm{e}^{-x_i})^{\theta}-(1-\mathrm{e}^{-x_i})^{\theta})}\right]^{\beta}$$

$$-(\beta+1)\sum_{i=1}^{n}\log\left((1+\mathrm{e}^{-x_i})^{\theta}-(1-\mathrm{e}^{-x_i})^{\theta}\right). \quad (14)$$

Suppose $\Theta = (\,\beta\ \theta\ \ \lambda\,)^{T}$ be the unknown parameter vector, the associated score function is given by

$$\boldsymbol{U}(\Theta) = \left(\frac{\partial\mathcal{L}}{\partial\beta}\frac{\partial\mathcal{L}}{\partial\theta}\frac{\partial\mathcal{L}}{\partial\lambda}\right)^{T},$$

where $\frac{\partial\mathcal{L}}{\partial\beta}$, $\frac{\partial\mathcal{L}}{\partial\theta}$ and $\frac{\partial\mathcal{L}}{\partial\lambda}$ are the partial derivatives of the log-likelihood function w.r.t. to each parameter and are given by

$$\frac{\partial\mathcal{L}}{\partial\beta} = \frac{n}{\beta} + \theta\sum_{i=1}^{n}\log(1+\mathrm{e}^{-x_i})$$

$$-\sum_{i=1}^{n}\left[\frac{(1-\mathrm{e}^{-x_i})^{\theta}}{\lambda((1+\mathrm{e}^{-x_i})^{\theta}-(1-\mathrm{e}^{-x_i})^{\theta})}\right]^{\beta}\log\left[\frac{(1-\mathrm{e}^{-x_i})^{\theta}}{\lambda((1+\mathrm{e}^{-x_i})^{\theta}-(1-\mathrm{e}^{-x_i})^{\theta})}\right]$$

$$-\sum_{i=1}^{n}\log\left((1+\mathrm{e}^{-x_i})^{\theta}-(1-\mathrm{e}^{-x_i})^{\theta}\right),$$

$$\frac{\partial\mathcal{L}}{\partial\theta}=\frac{n}{\theta}+\beta\sum_{i=1}^{n}\log(1+\mathrm{e}^{-x_i})$$

$$-\frac{\beta}{\lambda}\sum_{i=1}^{n}\left[\frac{(1-\mathrm{e}^{-x_i})^{\theta}}{\lambda((1+\mathrm{e}^{-x_i})^{\theta}-(1-\mathrm{e}^{-x_i})^{\theta})}\right]^{\beta-1}\frac{(1-\mathrm{e}^{-2x_i})^{\theta}\log\left(\frac{(1-\mathrm{e}^{-x_i})}{(1+\mathrm{e}^{-x_i})}\right)}{((1+\mathrm{e}^{-x_i})^{\theta}-(1-\mathrm{e}^{-x_i})^{\theta})^{2}}$$

$$-(\beta+1)\sum_{i=1}^{n}\frac{(1+\mathrm{e}^{-x_i})^{\theta}\log(1+\mathrm{e}^{-x_i})-(1-\mathrm{e}^{-x_i})^{\theta}\log(1+\mathrm{e}^{-x_i})}{(1+\mathrm{e}^{-x_i})^{\theta}-(1-\mathrm{e}^{-x_i})^{\theta}},$$

$$\frac{\partial\mathcal{L}}{\partial\lambda}=-\frac{n}{\lambda}+\frac{\beta}{\lambda}\left[\frac{(1-\mathrm{e}^{-x_i})^{\theta}}{\lambda((1+\mathrm{e}^{-x_i})^{\theta}-(1-\mathrm{e}^{-x_i})^{\theta})}\right]^{\beta}.$$

The maximum likelihood estimate of Θ is obtained by solving the non-linear systems of equations $\boldsymbol{U}(\Theta)=0$. Observe that the resulting systems of equations are not in closed form, the solutions can only be found numerically using any numerical optimization scheme such as the Newton type algorithms.

The Fisher information matrix (FIM) of the OWEHLD is the 3×3 symmetric matrix given by

$$\boldsymbol{I}(\Theta)=-E_{\Theta}\begin{pmatrix}\Delta_{\beta\beta} & \Delta_{\beta\theta} & \Delta_{\beta\lambda}\\ \Delta_{\theta\beta} & \Delta_{\theta\theta} & \Delta_{\theta\lambda}\\ \Delta_{\lambda\beta} & \Delta_{\lambda\theta} & \Delta_{\lambda\lambda}\end{pmatrix},$$

where the elements $\Delta_{ij}(\Theta)=\left[\frac{\partial^2\mathcal{L}}{\partial\Theta_i\partial\Theta_j}\right]$. Thus, the elements of the FIM can be obtained by realizing the second order partial derivatives of the log-likelihood function w.r.t. to the parameters. These elements can be numerically obtained through computation using a good computing software. The total FIM, $\boldsymbol{I}(\Theta)$, can be approximated by

$$\boldsymbol{J}(\widehat{\Theta})\approx\left[-\frac{\partial^2\mathcal{L}}{\partial\Theta_i\partial\Theta_j}\bigg|_{\Theta=\widehat{\Theta}}\right]_{\mathbf{3\times 3}}.$$

For real data, $\boldsymbol{J}(\widehat{\Theta})$ is obtained after the maximum likelihood estimate of Θ is obtained, which implies the convergence of the iterative numerical procedure involved in finding such estimate.

Suppose $\widehat{\Theta}$ is the maximum likelihood estimate of Θ. Under the usual regularity conditions and that the parameters are in the interior of the parameter space, but not on the boundary, we have: $\sqrt{n}(\widehat{\Theta}-\Theta) \overset{d}{\to} N_3\left(\mathbf{0}, \boldsymbol{I}^{-1}(\Theta)\right)$, where $\boldsymbol{I}^{-1}(\Theta)$ is the inverse of the expected FIM, which also corresponds to the variance-covariance matrix of the parameters. The asymptotic behavior is still valid if $\boldsymbol{I}^{-1}(\Theta)$ is replaced by the inverse of the observed information matrix evaluated at $\widehat{\Theta}$, that is $\boldsymbol{J}^{-1}(\widehat{\Theta})$. The multivariate normal distribution with mean vector $\mathbf{0} = (0\ 0\ 0)^T$ and covariance matrix $\boldsymbol{I}^{-1}(\Theta)$ can be used to construct confidence intervals for the OWEHLD parameters. The approximate $100(1-\alpha)\%$ two-sided confidence interval for the parameters β, θ and λ are given by

$$\hat{\beta} \pm Z_{\alpha/2}\sqrt{\boldsymbol{I}_{\beta\beta}^{-1}(\widehat{\Theta})}, \qquad \hat{\theta} \pm Z_{\alpha/2}\sqrt{\boldsymbol{I}_{\theta\theta}^{-1}(\widehat{\Theta})}, \qquad \hat{\lambda} \pm Z_{\alpha/2}\sqrt{\boldsymbol{I}_{\lambda\lambda}^{-1}(\widehat{\Theta})},$$

respectively, where $\boldsymbol{I}_{\beta\beta}^{-1}(\widehat{\Theta})$, $\boldsymbol{I}_{\theta\theta}^{-1}(\widehat{\Theta})$, and $\boldsymbol{I}_{\lambda\lambda}^{-1}(\widehat{\Theta})$ are diagonal elements of $\boldsymbol{I}^{-1}(\widehat{\Theta})$ and $Z_{\alpha/2}$ is the upper $(\alpha/2)^{th}$ percentile of a standard normal distribution.

5. Applications

The OWEHLD will be applied to fit the monthly average wind speed of Southeastern Nigeria and one other data set to test its applicability and flexibility in modeling data sets with different shape and tail properties. The fit of the OWEHLD will be compared with that of the half-logistic distribution (HLD) and two other generalizations of the HLD namely: the exponentiated half-logistic distribution (EHLD) by Seo and Kang [19] and the exponentiated half-logistic exponential distribution (EHLED) by Almarashi et al. [20]. The pdf of the EHLD and the EHLED are given respectively by

$$f_{EHLD}(x) = \frac{2\theta\beta e^{-\beta x}}{(1+e^{-\beta x})^2}\left(\frac{1-e^{-\beta x}}{1+e^{-\beta x}}\right)^{\theta-1}, \qquad x > 0,\ \beta, \theta > 0,$$

$$f_{EHLED}(x) = \frac{2\alpha\lambda\beta e^{-\beta\lambda x}\left(1-e^{-\beta\lambda x}\right)^{\alpha-1}}{(1+e^{-\beta\lambda x})^{\alpha+1}}, \qquad x > 0,\ \alpha, \lambda, \beta > 0,$$

The pdf of the HLD is already defined in Section 1.

(i) Application to wind speed data from Southeastern Nigeria

For the first application, the OWEHLD is used to fit the monthly average wind speed of Southeastern Nigeria for a 32 years period (1987 - 2019). The wind speed recordings were carried out at a height of 10 meters and there are 384 observations. The highest and lowest wind speed observations are $5.8m/s$ and $0.7m/s$ respectively and this indicates that the region lies on the low wind speed zone in Nigeria. The mean wind speed is $3.59m/s$, while the median wind speed is $3.6m/s$. The coefficient of skewness is -0.12 which clearly indicates that the distribution of the wind speed observation is slightly skewed to the left and almost symmetric. Also, the coefficient of excess kurtosis is -0.36 which implies that the distribution of the wind speed is light-tailed. The wind speed sample can be made available upon request from the corresponding author.

The HLD, EHLD and EHLED are also used to fit the data and the results of the fits which include the estimate of the parameters, the standard errors of these estimated parameters, the loglikelihood (loglik) values, the Akaike Information Criterion (AIC) values and the Kolmogorov – Smirnov (K-S) statistic values (the corresponding p-values are also reported) of all the fitted distributions are reported in Table 1.

Table 1: Maximum likelihood fit of the wind speed data.

Distribution	OWEHLD	HLD	EHLD	EHLED
Parameter estimates	$\hat{\theta} = 8.9512$ (1.8894) $\hat{\beta} = 0.8914$ (0.0579) $\hat{\lambda} = 2.4707$ (0.6830)	$\hat{\theta} = 0.4236$ (0.0168)	$\hat{\theta} = 13.5979$ (1.5777) $\hat{\beta} = 1.0629$ (0.0392)	$\hat{\alpha} = 13.602$ (1.5794) $\hat{\beta} = 0.2201$ (0.2649) $\hat{\lambda} = 4.8250$ (5.8080)
Log Likelihood	-518.94	-808.34	-554.60	-554.60
AIC	1043.88	1618.65	1113.19	1115.19
K-S p-value	0.0440 0.4347	0.3634 $7.1e-15$	0.1003 0.0008	0.1002 0.0008

(Standard error of estimates in parenthesis)

Figure 8 shows the graph of all the fitted densities alongside the histogram of the data. The results in Table 1 clearly show that the OWEHLD provided the best fit to the data. This result points to the fact that the OWEHLD can be considered as a good wind speed model for carrying out wind speed analysis for the region. Also, observe that the fit of the EHLD and EHLED are almost the same.

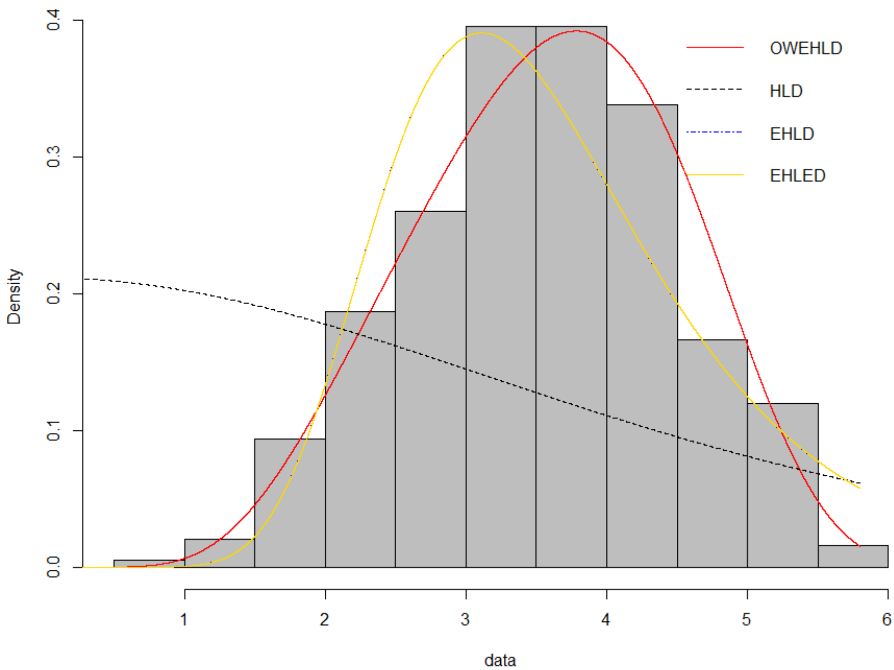

Figure 8: Fitted densities of the wind speed data.

(ii) Failure times data

For the second application, the OWEHLD is used to fit the time to failure (10^3h) of turbo charger of one type of engine reported in Xu et al. [21] The data set is left-skewed with coefficient of skewness being -0.6379 and coefficient of kurtosis being 2.5106. The data is given in Table 2.

The HLD, EHLD and EHLED are also used to fit the data and the results of the fits which include the estimate of the parameters, the standard errors of these estimated parameters, the loglikelihood (loglik) values, the Akaike Information Criterion (AIC) values and the Kolmogorov – Smirnov (K-S) statistic values (the corresponding p-values are also reported) of all the fitted distributions are reported in Table 3. Figure 9 shows the graph of all the fitted densities alongside the histogram of the data. The results in Table 3 clearly show that the OWEHLD provided the best fit to the data and this demonstrates the applicability of the OWHELD in fitting left-skewed mesokurtic data. Again, observe that the fit of the EHLD and EHLED are almost the same.

Table 2: Failure times data

1.6, 2.0,2.6, 3.0,3.5, 3.9,4.5, 4.6,4.8, 5.0,5.1, 5.3,5.4, 5.6,5.8, 6.0, 6.0, 6.1,6.3, 6.5,6.5, 6.7,7.0, 7.1, 7.3, 7.3, 7.3, 7.7, 7.7, 7.8, 7.9, 8.0, 8.1, 8.3, 8.4, 8.4, 8.5, 8.7, 8.8, 9.0

Table 3: Maximum likelihood fit of the failure times data.

Distribution	OWEHLD	HLD	EHLD	EHLED
Parameter estimates	$\hat{\theta} = 7.5075$ (6.2817) $\hat{\beta} = 0.6218$ (0.0866) $\hat{\lambda} = 80.9273$ (74.1899)	$\hat{\theta} = 0.2409$ (0.0298)	$\hat{\theta} = 5.8671$ (1.6574) $\hat{\beta} = 0.4744$ (0.0552)	$\hat{\alpha} = 5.8736$ (1.6608) $\hat{\beta} = 0.5315$ (5.4902) $\hat{\lambda} = 0.8923$ (9.2235)
Log Likelihood	-74.19	-106.85	-89.23	-89.23
AIC	164.94	215.70	182.46	184.46
K-S p-value	0.0897 0.8761	0.3445 $9.6e-05$	0.1514 0.2881	0.1514 0.2881

(Standard error of estimates in parenthesis)

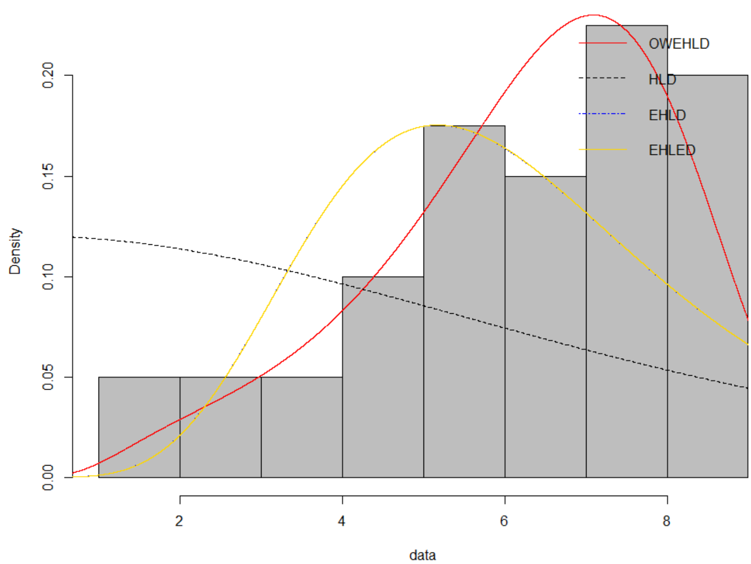

Figure 9: Fitted densities of the failure times data.

6. Summary and Conclusion

A new generalization of the half-logistic distribution has been introduced in this paper. Several statistical properties of the new distribution have been studied. The shapes of the density of the new generalized half-logistic distribution is observed to possessed flexibility in that it can be symmetric, right skewed and left skewed while also accounting for varying tail properties found in data. The distribution was also applied in fitting two data sets including a wind speed sample and it outperformed the classical half-logistic distribution and two other generalizations of the half-logistic distribution. We hope that the new model will attract applications in other areas of study.

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